Spaceability for Sets of Bandlimited Input Functions and Stable Linear Time-Invariant Systems with Divergence Behavior

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Abstract

The approximation of linear time-invariant systems by sampling series is studied for bandlimited input functions in the Paley-Wiener space $\mathcal{PW}_{\pi}^{1}$. It has been known that there exist functions and systems such that the approximation process diverges. In this paper we identify a signal set and a system set with divergent Shannon sampling expression. We analyze the structure of these sets and prove that they are jointly spaceable, i.e., that each of them contains an infinite dimensional closed subspace, such that for any pair of function and system from these subspaces, except the zero elements, we have divergence.

Keywords: Spaceability, Linear time-invariant system, Paley–Wiener space, Approximation process, Sampling series, Divergence

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1. Introduction

A central problem in signal processing is the approximation of linear time-invariant (LTI) systems, like the Hilbert transform or the derivative, by sampling series. For a given bandlimited input function f and stable LTI system T, the canonical approximation process is given by

$$\sum_{k=-\infty}^{\infty} f(k) h_T(t-k), \tag{1}$$

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where $h_T = T\,\mathrm{sinc}$ denotes the response of the system T to the sinc-function. The convergence of (1) is not guaranteed and has to be checked from case to case.

In [4, 5, 6] the convergence behavior of (1) was analyzed for functions f in the Paley–Wiener space $\mathcal{PW}_\pi^1$ of bandlimited functions with absolutely integrable Fourier transform. It was shown that for each $t \in \mathbb{R}$ there exists a stable LTI system T and a function $f \in \mathcal{PW}_\pi^1$ such that

$$\limsup_{N\to\infty} \left| \sum_{k=-N}^{N} f(k) h_T(t-k) \right| = \infty, \tag{2}$$

i.e., that the approximation error grows arbitrarily large. Using the Banach–Steinhaus theorem, it is easy to see that, for a fixed divergence-creating system T, the set of functions f for which (2) holds is a residual set. Equally, for a fixed divergence-creating function f, the set of systems T for which (2) holds is a residual set. However, it is not clear whether there exist a residual set of functions and a residual set of systems, such that for any pair of function and system from the two sets, (2) holds.

In this paper we study the structure of the sets of divergence creating functions and systems. It would be interesting to know whether these sets contain subsets which exhibit a linear structure, because in this case any linear combinations of functions or systems from those subsets, which do not result in the zero elements, would lead to divergence as well. We prove that both sets are spaceable, i.e., contain a closed infinite dimensional subspace with linear structure. It is even true that both sets are jointly spaceable in the sense that there exist two closed infinite dimensional subspaces D_{sig} and D_{sys}, such that for all pairs of functions and systems $(f, T) \in D_{\mathrm{sig}} \times D_{\mathrm{sys}}$, $f \not\equiv 0$, $T \not\equiv 0$, we have divergence as stated in (2).

This work was motivated by questions raised by Hans Feichtinger in early 2015 about the structure of divergence-creating signal sets.

2. General Notation

Let $\hat{f}$ denote the Fourier transform of a function f, where $\hat{f}$ is to be understood in the distributional sense. $L^p(\mathbb{R})$, $1 \leq p < \infty$, is the space of all measurable, pth-power Lebesgue integrable functions on $\mathbb{R}$, with the usual norm $\|\cdot\|_p$, and $L^\infty(\mathbb{R})$ the space of all functions for which the essential supremum norm $\|\cdot\|_\infty$ is finite. $L^p[t_1, t_2]$, $1 \leq p < \infty$, is the space of all measurable, pth-power Lebesgue integrable functions on $[t_1, t_2]$. $C[t_1, t_2]$ denotes the space of all continuous functions on $[t_1, t_2]$ For $1 \leq p \leq \infty$, $\mathcal{PW}_\pi^p$ denotes the Paley-Wiener space of functions f with a representation $f(z) = 1/(2\pi) \int_{-\pi}^{\pi} g(\omega) \, \mathrm{e}^{iz\omega} \, \mathrm{d}\omega$, $z \in \mathbb{C}$, for some $g \in L^p[-\pi, \pi]$. If $f \in \mathcal{PW}_\pi^p$ then $g(\omega) = \hat{f}(\omega)$. The norm for $\mathcal{PW}_\pi^p$, $1 \leq p < \infty$, is given by $\|f\|_{\mathcal{PW}_\pi^p} = (1/(2\pi) \int_{-\pi}^{\pi} |\hat{f}(\omega)|^p \, \mathrm{d}\omega)^{1/p}$.

We briefly review some definitions and facts about stable linear time-invariant (LTI) systems, which will be relevant. A linear system $T : \mathcal{PW}_\pi^p \to \mathcal{PW}_\pi^p$, $1 \leq p \leq \infty$, is called stable if the operator T is bounded, i.e., if $\|T\| := \sup_{\|f\|_{\mathcal{PW}_\pi^p} \leq 1} \|Tf\|_{\mathcal{PW}_\pi^p} < \infty$. Furthermore, it is called time-invariant if $(Tf(\cdot - a))(t) = (Tf)(t-a)$ for all $f \in \mathcal{PW}_\pi^p$ and $t, a \in \mathbb{R}$.

In this paper we are mainly interested in stable LTI systems operating on the space $\mathcal{PW}_\pi^1$, i.e., in the case $p = 1$. By $\mathcal{T}$ we denote the set of stable LTI systems $T\colon \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$. The operator norm of a stable LTI system T is given by $\|T\| = \|\hat{h}_T\|_{L^\infty[-\pi,\pi]}$. For every stable LTI system $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ there exists exactly one function $\hat{h}_T \in L^\infty[-\pi, \pi]$ such that

$$(Tf)(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) \hat{h}_T(\omega) \, \mathrm{e}^{i\omega t} \, \mathrm{d}\omega, \quad t \in \mathbb{R}, \tag{3}$$

for all $f \in \mathcal{PW}_\pi^1$. Conversely, every function $\hat{h}_T \in L^\infty[-\pi, \pi]$ defines a stable LTI system $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$. Hence, we can identify stable LTI systems with $L^\infty[-\pi, \pi]$ functions. By $Q\colon \mathcal{T} \to L^\infty[-\pi, \pi]$ we denote the isometric isomorphism that performs this mapping. We have $h_T = T \,\mathrm{sinc}$, where sinc denotes the usual sinc-function which is defined by $\mathrm{sinc}(t) = \sin(\pi t)/(\pi t)$ for $t \neq 0$ and $\mathrm{sinc}(t) = 1$ for $t = 0$.

Remark 1. Adding two functions in $\mathcal{PW}_\pi^1$ or multiplying a function in $\mathcal{PW}_\pi^1$ with a scalar gives again a function in $\mathcal{PW}_\pi^1$. The same linear structure also holds for stable LTI systems. This is important for signal processing applications, because it allows to compose complex systems out of simple ones.

3. Spaceability

Before we state the main result, we introduce the concept of spaceability. Spaceability, which has recently been used for example in [10, 13, 1, 3, 7], is a concept that describes the structure of some given subset of an ambient normed space or, more generally, topological space. A set S in a linear topological space X is said to be spaceable if $S \cup \{0\}$ contains a closed infinite dimensional subspace of X. A closely related concept is lineability. A set S in a linear topological space X is said to be lineable if $S \cup \{0\}$ contains an infinite dimensional subspace.

In [12] it was proved that the set of continuous nowhere differentiable functions on $\mathbb{R}$ is lineable. Later, it was shown that the set of continuous nowhere differentiable functions on $C[0, 1]$ is spaceable [10]. The divergence of Fourier series was analyzed in [3], where it was shown that the set of functions in $L^1(\partial\mathbb{D})$, whose Fourier series diverges everywhere on $\partial\mathbb{D}$ is spaceable. Spaceability and lineability in different setting was further analyzed in [11, 2].

Spaceablility of normed spaces is also interesting for signal processing, because the linear structure and the norm are both relevant concepts there.

Remark 2. Spaceability is a stronger property than lineability. Every spaceable set is lineable but not vice-versa.

4. Main Result

Now we are in the position to state our main result.

Theorem 1. *There exist an infinite dimensional closed subspace $D_{sig} \subset \mathcal{PW}_\pi^1$ and an infinite dimensional closed subspace $D_{sys} \subset \mathcal{T}$ such that for all $f \in D_{sig}$, $f \not\equiv 0$, and all $T \in D_{sys}$, $T \not\equiv 0$, we have*

$$\limsup_{N\to\infty} \left| \sum_{k=-N}^{N} f(k) h_T(-k) \right| = \infty.$$

All $T \in D_{sys}$ are such that $\hat{h}_T$ is continuous.

Theorem 1 shows that there exist a spaceable set of functions $D_{\mathrm{sig}} \subset \mathcal{PW}_\pi^1$ and a spaceable set of stable LTI systems $D_{\mathrm{sys}} \subset \mathcal{T}$ such that the system approximation process (1) diverges at $t = 0$ for any pair of function and system $(f, T) \in D_{\mathrm{sig}} \times D_{\mathrm{sys}}$, $f \not\equiv 0$, $T \not\equiv 0$, chosen from the two sets. In the previous expression, we denoted the zero element by 0. For the signal space it is the signal f that is identical zero, i.e., $f(t) = 0$ for all $t \in \mathbb{R}$, and for the system space it is the LTI system T with $\hat{h}_T(\omega) = 0$ for almost all $\omega \in [-\pi, \pi]$. From the context it will be always clear which zero element we refer to, when writing 0.

Remark 3. Note that is significantly more difficult to show a linear structure in the set of functions and systems with divergent system approximation process, compared to showing a linear structure in the set of functions and systems with convergent system approximation process. If we have two functions f_1 and f_2, for which (1) converges, it is clear that the sum of both functions, i.e., $f_1 + f_2$, is a function for which we have convergence as well. Hence, any finite linear combination of functions with convergent system approximation process will be a function with convergent system approximation process. However, for divergence this is not true. Given two functions w_1 and w_2 for which (1) diverges, we cannot conclude that the sum of both functions, i.e., $w_1 + w_2$, is a function for which (1) diverges. This can be easily seen by choosing $w_1 = f_1 + g$ and $w_2 = f_1 - g$, where f_1 is any function with convergent system approximation process and g any function with divergent system approximation process. Obviously, for the sum $w_1 + w_2 = 2f_1$ we do not have divergence. This shows that the sum of two functions, each of which leads to divergence, does not necessarily lead to divergence.

Remark 4. Theorem 1 is concerned with the sets of functions and system for which we have the divergence (2). As for convergence, we have the following situation. For all functions $f \in \mathcal{PW}_\pi^2$ and all systems $T \in \mathcal{T}$ we have

$$\lim_{N\to\infty} \max_{t\in\mathbb{R}} \left| (Tf)(t) - \sum_{k=-N}^{N} f(k) h_T(t-k) \right| = 0,$$

i.e. we have lineability of the set of input functions which lead to a convergent system approximation for all stable LTI systems. Further, for all stable LTI FIR-systems T_{FIR}, i.e., systems $T \in \mathcal{T}$ with $h_T(k) \neq 0$ for only finitely many $k \in \mathbb{Z}$, we have for all $f \in \mathcal{PW}_\pi^1$ that

$$(Tf)(0) = \sum_{k=-\infty}^{\infty} f(k) h_{T_{\mathrm{FIR}}}(-k),$$

because only finitely many summands are non-zero. Therefore, we also have lineability of the set of systems for which $(Tf)(0)$ can be represented by a finite sampling series for all functions in $\mathcal{PW}_\pi^1$.

The divergence for arbitrary $t \neq 0$ follows easily from Theorem 1 and is stated in the following corollary, the proof of which is given after the proof of Theorem 1.

Corollary 1. *Let $t \in \mathbb{R}$ be arbitrary but fixed. There exist an infinite dimensional closed subspace $D_{sig} \subset \mathcal{PW}_\pi^1$ and an infinite dimensional closed subspace $D_{sys2} \subset \mathcal{T}$ such that for all $f \in D_{sig}$, $f \not\equiv 0$, and all $T \in D_{sys2}$, $T \not\equiv 0$, we have*

$$\limsup_{N\to\infty} \left| \sum_{k=-N}^{N} f(k) h_T(t-k) \right| = \infty.$$

All $T \in D_{sys2}$ are such that $\hat{h}_T$ is continuous.

For the proof of Theorem 1 we need the following lemma.

Lemma 1. *There exist two sequences of functions $\{\phi_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ with:*

1. *The functions ϕ_n, $n \in \mathbb{N}$, are finitely linearly independent, $\{\phi_n\}_{n\in\mathbb{N}} \subset \mathcal{PW}_\pi^1$, and there exists a constant C_1 such that $\|\phi_n\|_{\mathcal{PW}_\pi^1} \leq C_1$ for all $n \in \mathbb{N}$.*
2. *The functions g_n, $n \in \mathbb{N}$, are finitely linearly independent, $\{\hat{g}_n\}_{n\in\mathbb{N}} \subset C[-\pi,\pi]$, and there exists a constant C_2 such that $\|\hat{g}_n\|_\infty \leq C_2$ for all $n \in \mathbb{N}$.*
3. *For all $n, m \in \mathbb{N}$ there exists a sequences $\{N_r(n,m)\}_{r\in\mathbb{N}}$ and a constant C_3 such that*
$$\limsup_{r\to\infty} \left| \sum_{k=0}^{N_r(n,m)} \phi_n(-k) g_m(k) \right| = \infty$$
and
$$\sup_{r\in\mathbb{N}} \left| \sum_{k=0}^{N_r(\hat{n},\hat{m})} \phi_n(-k) g_m(k) \right| \leq C_3$$
for all $\hat{n}, \hat{m} \in \mathbb{N}$ with $(\hat{n},\hat{m}) \neq (n,m)$.

Proof of Lemma 1. For $N \in \mathbb{N}$ and $k \in \mathbb{Z}$ let
$$p_N^*(k) = \begin{cases} 1 - \frac{|k|}{N}, & -N < k < N, \\ 0, & |k| \geq N, \end{cases}$$
and define
$$p_N(k) = p_N^*(k-N)$$
as well as
$$p_N(t) = \sum_{k=0}^{2N} p_N(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}, \quad t \in \mathbb{R}.$$
We have
$$\|p_N\|_{\mathcal{PW}_\pi^1} = 1 \tag{4}$$
for all $N \in \mathbb{N}$, which follows from the fact that the $L^1[-\pi,\pi]$-norm of the Fejér kernel is one. Further, for $N \in \mathbb{N}$ and $k \in \mathbb{Z}$, let
$$q_N^*(k) = \begin{cases} -\frac{1}{k}, & -N < k \leq -1, \\ -\frac{1}{k}, & 1 \leq k < N, \\ 0, & k = 0, |k| \geq N, \end{cases}$$

and define

$$q_N(k) = q_N^*(k-N)$$

as well as

$$q_N(t) = \sum_{k=0}^{2N} q_N(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}, \quad t \in \mathbb{R}.$$

We have

$$\|\hat{q}_N\|_\infty \le C_4 \tag{5}$$

for all $N \in \mathbb{N}$, which is a simple consequence of [14, p. 183, Remark (b)]. It follows that

$$\begin{aligned}
\sum_{k=0}^{N} p_N(k) q_N(k) &= \sum_{k=1}^{N-1} \frac{k}{N} \frac{1}{N-k} \\
&= \sum_{l=1}^{N-1} \frac{1}{l} \frac{N-l}{N} \\
&= \sum_{l=1}^{N-1} \frac{1}{l} - \frac{1}{N} \sum_{l=1}^{N-1} 1 \\
&> \sum_{l=1}^{N-1} \int_l^{l+1} \frac{1}{x} \, \mathrm{d}x + \frac{1-N}{N} \\
&= \int_1^N \frac{1}{x} \, \mathrm{d}x + \frac{1-N}{N} \\
&= \log(N) + \frac{1-N}{N}
\end{aligned} \tag{6}$$

for all $N \in \mathbb{N}$. Further, using Parseval's theorem, (5), and (4), we see that

$$\begin{aligned}
\left| \sum_{k=0}^{2N} p_N(k) q_N(k) \right| &= \left| \sum_{k=-\infty}^{\infty} p_N(k) q_N(k) \right| \\
&= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{p}_N(\omega) \overline{\hat{q}_N(\omega)} \, \mathrm{d}\omega \right| \\
&\le \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{p}_N(\omega)| |\overline{\hat{q}_N(\omega)}| \, \mathrm{d}\omega \\
&\le \frac{C_4}{2\pi} \int_{-\pi}^{\pi} |\hat{p}_N(\omega)| \, \mathrm{d}\omega \\
&= C_4,
\end{aligned} \tag{7}$$

for all $N \in \mathbb{N}$ with a constant C_4 that is independent of N.

Additionally, we need to define several matrices. We construct them iteratively. Let

$$R_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and

$$R_2 = \begin{pmatrix} 1 & 1 & 0 \\ \multicolumn{2}{c}{I_2} & R_1 \end{pmatrix},$$

where I_l denotes the $l \times l$ identity matrix. Having defined the $(r-1)$-th matrix, the r-th matrix is given by

$$R_r = \begin{pmatrix} \overbrace{1 \quad \cdots \quad 1}^{r \text{ times}} & 0 \quad \cdots \quad 0 \\ I_r & R_{r-1} \end{pmatrix}.$$

Elementary facts about R_r, $r \in \mathbb{N}$, are:

1. R_r has $(r+1)$ rows and $\frac{r(r+1)}{2}$ columns.
2. In each row of R_r we have exactly r elements that are 1, all other elements are 0.
3. In each column of R_r we have exactly 2 elements that are 1, all other elements are 0.

For $r \in \mathbb{N}$, we further set

$$B_r = \begin{pmatrix} w_1^{(r)}(1) & w_1^{(r)}(2) & \cdots & w_1^{(r)}(l_r) \\ w_2^{(r)}(1) & w_2^{(r)}(2) & \cdots & w_2^{(r)}(l_r) \\ \vdots & \vdots & & \vdots \\ w_{r+1}^{(r)}(1) & w_{r+1}^{(r)}(2) & \cdots & w_{r+1}^{(r)}(l_r) \end{pmatrix} = \begin{pmatrix} R_r & I_{r+1} \end{pmatrix},$$

where $\{w_m^{(r)}(l)\}_{m,l}$, $m = 1, \ldots, r+1$, $l = 1, \ldots, l_r$ denote the elements of the matrix B_r, and

$$l_r = \frac{r(r+1)}{2} + r + 1 = \frac{r(r+3)}{2} + 1.$$

Next, for $n \in \mathbb{N}$, we will construct two convergent sequences $\{\phi_{n,r}\}_{r\in\mathbb{N}}$ and $\{g_{n,r}\}_{r\in\mathbb{N}}$. The limit functions will be the desired functions ϕ_n and g_n. For $r \in \mathbb{N}$ let

$$N_r = 2^{(r^7)}. \tag{8}$$

For $r=1$ we set $M_1=0$,

$$\phi_{1,1}(t)=\sum_{l=1}^{l_1} w_1^{(1)}(l)p_{N_1}(t-(l-1)(2N_1+1)-M_1),$$

$$g_{1,1}(t)=\sum_{l=1}^{l_1} w_1^{(1)}(l)q_{N_1}(t-(l-1)(2N_1+1)-M_1),$$

$$\phi_{2,1}(t)=\sum_{l=1}^{l_1} w_2^{(1)}(l)p_{N_1}(t-(l-1)(2N_1+1)-M_1),$$

$$g_{2,1}(t)=\sum_{l=1}^{l_1} w_2^{(1)}(l)q_{N_1}(t-(l-1)(2N_1+1)-M_1),$$

$$N_1(n,n)=M_1+N_1+(2N_1+1)+(n-1)(2N_1+1)$$

for $1\le n\le 2$, and

$$N_1(n,m)=M_1+N_1+\sum_{l=1}^{l_1} w_n^{(1)}(l)w_m^{(1)}(l)(l-1)(2N_1+1)$$

for $1\le n,m\le 2$, $m\ne n$.

For $r=2$ we set $M_2=M_1+l_1(2N_1+1)$,

$$\phi_{1,2}(t)=\phi_{1,1}(t)+\frac{1}{2^3}\sum_{l=1}^{l_2} w_1^{(2)}(l)p_{N_2}(t-(l-1)(2N_2+1)-M_2),$$

$$g_{1,2}(t)=g_{1,1}(t)+\frac{1}{2^3}\sum_{l=1}^{l_2} w_1^{(2)}(l)q_{N_2}(t-(l-1)(2N_2+1)-M_2),$$

$$\phi_{2,2}(t)=\phi_{2,1}(t)+\frac{1}{2^3}\sum_{l=1}^{l_2} w_2^{(2)}(l)p_{N_2}(t-(l-1)(2N_2+1)-M_2),$$

$$g_{2,2}(t)=g_{2,1}(t)\frac{1}{2^3}\sum_{l=1}^{l_2} w_2^{(2)}(l)q_{N_2}(t-(l-1)(2N_2+1)-M_2),$$

$$\phi_{3,2}(t)=\frac{1}{2^3}\sum_{l=1}^{l_2} w_3^{(2)}(l)p_{N_2}(t-(l-1)(2N_2+1)-M_2),$$

$$g_{3,2}(t)=\frac{1}{2^3}\sum_{l=1}^{l_2} w_3^{(2)}(l)q_{N_2}(t-(l-1)(2N_2+1)-M_2),$$

$$N_2(n,n)=M_2+N_2+\frac{2(2+1)}{2}(2N_2+1)+(n-1)(2N_2+1),$$

for $1 \leq n \leq 3$, and

$$N_2(n,m) = M_2 + N_2 + \sum_{l=1}^{l_2} w_n^{(2)}(l) w_m^{(2)}(l)(l-1)(2N_2+1),$$

for $1 \leq m, n \leq 3$, $m \neq n$.

Suppose, for some r with $r \geq 2$ we have constructed the number M_r, the functions $\phi_{n,r}$ and $g_{n,r}$, $1 \leq n \leq r+1$, the numbers $N_r(n,m)$, $1 \leq n \leq r+1$, $1 \leq m \leq r+1$. Then we set

$$M_{r+1} = M_r + l_r(2N_r + 1),$$

$$\phi_{n,r+1}(t) = \phi_{n,r}(t) + \underbrace{\frac{1}{(r+1)^3} \sum_{l=1}^{l_{r+1}} w_n^{(r+1)}(l) p_{N_{r+1}}(t-(l-1)(2N_{r+1}+1) - M_{r+1})}_{=\Psi_{n,r+1}(t)},$$

for $1 \leq n \leq r+1$,

$$\phi_{r+2,r+1}(t) = \frac{1}{(r+1)^3} \sum_{l=1}^{l_{r+1}} w_{r+2}^{(r+1)}(l) p_{N_{r+1}}(t-(l-1)(2N_{r+1}+1) - M_{r+1}),$$

$$g_{n,r+1}(t) = g_{n,r}(t) + \underbrace{\frac{1}{(r+1)^3} \sum_{l=1}^{l_{r+1}} w_n^{(r+1)}(l) q_{N_{r+1}}(t-(l-1)(2N_{r+1}+1) - M_{r+1})}_{=\Gamma_{n,r+1}(t)},$$

for $1 \leq n \leq r+1$,

$$g_{r+2,r+1}(t) = \frac{1}{(r+1)^3} \sum_{l=1}^{l_{r+1}} w_{r+2}^{(r+1)}(l) q_{N_{r+1}}(t-(l-1)(2N_{r+1}+1) - M_{r+1}),$$

$$N_{r+1}(n,n) = M_{r+1} + N_{r+1} + \frac{(r+1)(r+2)}{2}(2N_{r+1}+1) + (n-1)(2N_{r+1}+1),$$

for $1 \leq n \leq r+2$, and

$$N_{r+1}(n,m) = M_{r+1} + N_{r+1} + \sum_{l=1}^{l_{r+1}} w_n^{(r+1)}(l) w_m^{(r+1)}(l)(l-1)(2N_{r+1}+1)$$

for $1 \leq n, m \leq r+2$, $n \neq m$.

Now, let $n \in \mathbb{N}$, $n \geq 1$, and $r \geq \max\{n+1, 3\}$ be arbitrary. Then we have

$$\begin{aligned}
\|\phi_{n,r}\|_{\mathcal{PW}_\pi^1} &= \left\| \phi_{n,r-1} + \frac{1}{r^3} \sum_{l=1}^{l_r} w_n^{(r)}(l) p_{N_r}(\,\cdot\, - (l-1)(2N_r+1) - M_r) \right\|_{\mathcal{PW}_\pi^1} \\
&= \left\| \sum_{s=\max\{n-1,1\}}^{r} \frac{1}{s^3} \sum_{l=1}^{l_s} w_n^{(s)}(l) p_{N_s}(\,\cdot\, - (l-1)(2N_s+1) - M_s) \right\|_{\mathcal{PW}_\pi^1} \\
&\leq \sum_{s=\max\{n-1,1\}}^{r} \frac{1}{s^3} \sum_{l=1}^{l_s} w_n^{(s)}(l) \left\| p_{N_s}(\,\cdot\, - (l-1)(2N_s+1) - M_s) \right\|_{\mathcal{PW}_\pi^1} \\
&= \sum_{s=\max\{n-1,1\}}^{r} \frac{s+1}{s^3} \leq 2 \sum_{s=\max\{n-1,1\}}^{\infty} \frac{1}{s^2} \leq \frac{\pi^2}{3} < \infty,
\end{aligned}$$

where we used that $\|p_N\|_{\mathcal{PW}_\pi^1} = 1$ for all $N \in \mathbb{N}$, and the fact that only $s+1$ coefficients are non-zero and equal to 1.

Hence, for $r \to \infty$, the sequence $\{\phi_{n,r}\}$ converges to a function in $\mathcal{PW}_\pi^1$. The convergence is in the $\mathcal{PW}_\pi^1$-norm, and consequently pointwise. The same argument is valid for $\hat{g}_{n,r}$, where we have convergence in the maximum-norm, which implies uniform convergence of $g_{n,r}$ on $\mathbb{Z}$.

Next, for $n, m \geq 1$, we will show that

$$\limsup_{N\to\infty} \left| \sum_{k=0}^{N} \phi_n(k) g_m(k) \right| = \infty. \tag{9}$$

Let $r \in \mathbb{N}$, $r \geq \max(m, n, 3)$ be arbitrary. We consider $N = N_r(n,m)$. For $k \in \mathbb{Z}$ with $0 \leq k \leq N_r(n,m)$ we have, according to the constructions of the function ϕ_n and the sequence $\{\phi_{n,\hat{r}}\}_{\hat{r}\in\mathbb{N}}$, that $\phi_n(k) = \phi_{n,r}(k)$. The same holds true for g_m, i.e., for $0 \leq k \leq N_r(n,m)$ we have $g_m(k) = g_{m,r}(k)$. Therefore, it follows that

$$\begin{aligned}
\sum_{k=0}^{N_r(n,m)} \phi_n(k) g_m(k) &= \sum_{k=0}^{N_r(n,m)} \phi_{n,r}(k) g_{m,r}(k) \\
&= \sum_{k=0}^{M_r-1} \phi_{n,r}(k) g_{m,r}(k) + \sum_{k=M_r}^{N_r(n,m)} \phi_{n,r}(k) g_{m,r}(k). \qquad (10)
\end{aligned}$$

For the first term in (10) we have

$$
\begin{aligned}
\left|\sum_{k=0}^{M_r-1}\phi_{n,r}(k)g_{m,r}(k)\right| &= \left|\sum_{s=\max\{m-1,n-1,1\}}^{r-1}\ \sum_{k=M_s}^{M_{s+1}-1}\phi_{n,r}(k)g_{m,r}(k)\right| \\
&= \left|\sum_{s=\max\{m-1,n-1,1\}}^{r-1}\ \sum_{k=M_s}^{M_{s+1}-1}\phi_{n,s}(k)g_{m,s}(k)\right| \\
&= \left|\sum_{s=\max\{m-1,n-1,1\}}^{r-1}\ \sum_{k=M_s}^{M_{s+1}-1}\Psi_{n,s}(k)\Gamma_{m,s}(k)\right|. \qquad (11)
\end{aligned}
$$

For each $s \in \mathbb{N}$ there exists exactly one natural number $l(n,m,s)$, $1 \leq l(n,m,s) \leq l_s$, such that $w_n^{(s)}(l(n,m,s))w_m^{(r)}(l(n,m,s)) > 0$. Hence, for k in the interval

$$
M_s \leq k < M_{s+1}, \qquad (12)
$$

the expression $\phi_{n,s}(k)g_{m,s}(k)$ can be non-zero only for those k satisfying

$$
M_s + (l(n,m,s)-1)(2N_s+1) \leq k < M_s + l(n,m,s)(2N_s+1).
$$

It follows that

$$
\begin{aligned}
&\left|\sum_{k=M_s}^{M_{s+1}-1}\Psi_{n,s}(k)\Gamma_{m,s}(k)\right| \\
&= \left|\sum_{k=M_s+(l(n,m,s)-1)(2N_s+1)}^{M_s+l(n,m,s)(2N_s+1)-1}\Psi_{n,s}(k)\Gamma_{m,s}(k)\right| \\
&= \Bigg|\frac{1}{s^6}\sum_{k=M_s+(l(n,m,s)-1)(2N_s+1)}^{M_s+l(n,m,s)(2N_s+1)-1} p_{N_s}(k-(l(n,m,s)-1)(2N_s+1)-M_s)\times \\
&\qquad\qquad \times q_{N_s}(k-(l(n,m,s)-1)(2N_s+1)-M_s)\Bigg| \\
&= \frac{1}{s^6}\left|\sum_{k=0}^{2N_s} p_{N_s}(k)q_{N_s}(k)\right| \\
&\leq \frac{1}{s^6}C_4, \qquad (13)
\end{aligned}
$$

where we used (7) in the last line. Combining (11) and (13), we see that

$$\begin{aligned}\left|\sum_{k=0}^{M_r-1} \phi_{n,r}(k) g_{m,r}(k)\right| &\le C_4 \sum_{s=\max\{m-1,n-1,1\}}^{r-1} \frac{1}{s^6} \\ &\le C_4 \sum_{s=1}^{\infty} \frac{1}{s^6} \\ &\le C_5, \end{aligned} \tag{14}$$

with a constant C_5 that is independent of r.

Next, we analyze the second term in (10). There exists exactly one natural number $l(n,m,r)$, $1 \le l(n,m,r) \le l_r$, such that $w_n^{(r)}(l(n,m,r)) w_m^{(r)}(l(n,m,r)) > 0$. Hence, for k in the interval

$$M_r \le k < M_{r+1},$$

the expression $\phi_{n,r}(k) g_{m,r}(k)$ can be non-zero only for those k satisfying

$$M_r + (l(n,m,r)-1)(2N_r+1) \le k < M_r + l(n,m,r)(2N_r+1). \tag{15}$$

We have $\phi_{n,r}(k) g_{m,r}(k) = 0$ for all $k \ge M_r$ that do not satisfy (15). Since $N_r(n,m) = M_r + (l(n,m,r)-1)(2N_r+1) + N_r$, it follows that

$$\sum_{k=M_r}^{N_r(m,n)} \phi_{n,r}(k) g_{m,r}(k) \tag{16}$$

$$\begin{aligned} &= \sum_{k=M_r+(l(n,m,r)-1)(2N_r+1)}^{M_r+(l(n,m,r)-1)(2N_r+1)+N_r} \phi_{n,r}(k) g_{m,r}(k) \\ &= \frac{1}{r^6} \sum_{k=M_r+(l(n,m,r)-1)(2N_r+1)}^{M_r+(l(n,m,r)-1)(2N_r+1)+N_r} p_{N_r}(k-(l(n,m,r)-1)(2N_r+1)-M_r) \times \\ &\qquad \times q_{N_r}(k-(l(n,m,r)-1)(2N_r+1)-M_r) \\ &= \frac{1}{r^6} \sum_{k=0}^{N_r} p_{N_r}(k) q_{N_r}(k) \\ &> \frac{1}{r^6}\left(\log(N_r) + \frac{1-N_r}{N_r}\right) \\ &> r\log(2) - 1. \end{aligned} \tag{17}$$

This inequality is valid for all $r \geq \max(m, n, 3)$. Form (10), (14), and (17), we see that

$$\left|\sum_{k=0}^{N_r(n,m)} \phi_n(k) g_m(k)\right| \geq \left|\sum_{k=M_r}^{N_r(n,m)} \phi_{n,r}(k) g_{m,r}(k)\right| - \left|\sum_{k=0}^{M_r-1} \phi_{n,r}(k) g_{m,r}(k)\right|$$
$$> r \log(2) - 1 - C_5$$

for all $r \geq \max(m, n, 3)$. Therefore, we have proved (9).

Next, for $n, m \geq 1$, we will show that for all $(\hat{n}, \hat{m}) \neq (n, m)$ we have

$$\sup_{r \in \mathbb{N}} \left|\sum_{k=0}^{N_r(\hat{n},\hat{m})} \phi_n(k) g_m(k)\right| < \infty. \tag{18}$$

Let $r \in \mathbb{N}$, $r \geq \max\{m, n, 3\}$ be arbitrary. We have

$$\begin{aligned}\left|\sum_{k=0}^{N_r(\hat{n},\hat{m})} \phi_n(k) g_m(k)\right| &= \left|\sum_{k=0}^{N_r(\hat{n},\hat{m})} \phi_{n,r}(k) g_{m,r}(k)\right| \\ &\leq \left|\sum_{k=0}^{M_r-1} \phi_{n,r}(k) g_{m,r}(k)\right| + \left|\sum_{k=M_r}^{N_r(\hat{n},\hat{m})} \phi_{n,r}(k) g_{M,r}(k)\right|, \end{aligned} \tag{19}$$

because $\phi_n(k) = \phi_{n,r}(k)$ and $g_m(k) = g_{m,r}(k)$ for $0 \leq k \leq N_r(\hat{n}, \hat{m})$. For the first sum in (19) we have

$$\left|\sum_{k=0}^{M_r-1} \phi_{n,r}(k) g_{m,r}(k)\right| \leq C_5, \tag{20}$$

according to (14). Next, we treat the second sum in (19). There exists exactly one natural number $l(n, m, r)$, $1 \leq l(n, m, r) \leq l_r$, such that

$$w_n^{(r)}(l(n, m, r)) w_m^{(r)}(l(n, m, r)) > 0.$$

Hence, for k in the interval

$$M_r \leq k < M_{r+1},$$

the expression $\phi_{n,r}(k) g_{m,r}(k)$ can be non-zero only for those k satisfying

$$M_r + (l(n, m, r) - 1)(2N_r + 1) \leq k < M_r + l(n, m, r)(2N_r + 1). \tag{21}$$

We have $\phi_{n,r}(k)g_{m,r}(k) = 0$ for all $k \geq M_r$ that do not satisfy (21), in particular for all k in the interval

$$M_r + (l(\hat{n}, \hat{m}, r) - 1)(2N_r + 1) \leq k < M_r + l(\hat{n}, \hat{m}, r)(2N_r + 1).$$

If $N_r(\hat{n}, \hat{m}) < M_r + (l(n, m, r) - 1)(2N_r + 1)$, we consequently have

$$\sum_{k=M_r}^{N_r(\hat{n},\hat{m})} \phi_{n,r}(k)g_{M,r}(k) = 0.$$

If $N_r(\hat{n}, \hat{m}) > M_r + l(n, m, r)(2N_r + 1)$, we have

$$\begin{aligned}
&\left|\sum_{k=M_r}^{N_r(\hat{n},\hat{m})} \phi_{n,r}(k)g_{M,r}(k)\right| \\
&\quad= \left|\sum_{k=M_r+(l(n,m,r)-1)(2N_r+1)}^{M_r+l(n,m,r)(2N_r+1)-1} \phi_{n,r}(k)g_{M,r}(k)\right| \\
&\quad= \left|\sum_{k=M_r+(l(n,m,r)-1)(2N_r+1)}^{M_r+l(n,m,r)(2N_r+1)-1} p_{N_r}(k - (l(n, m, r) - 1)(2N_r + 1) - M_r)\times\right. \\
&\qquad\qquad\left.\times\, q_{N_r}(k - (l(n, m, r) - 1)(2N_r + 1) - M_r)\right| \\
&\quad= \left|\sum_{k=0}^{2N_r} p_{N_r}(k)q_{N_r}(k)\right| \\
&\quad\leq C_4,
\end{aligned} \tag{22}$$

where we used (7) in the last inequality. Hence from (19), (20), and (22) we see that

$$\left|\sum_{k=0}^{N_r(\hat{n},\hat{m})} \phi_n(k)g_m(k)\right| \leq C_5 + C_4,$$

where both constants are independent of r. This completes the proof. □

Now we are in the position to prove Theorem 1.

Proof of Theorem 1. Let $\{\phi_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ be the two sequences of functions from Lemma 1 with the properties 1–3, stated in Lemma 1. For $n \in \mathbb{N}$ and $t \in \mathbb{R}$ let

$$\xi_n^{(1)}(t) = \frac{1}{2^n 2C_1}\phi_n(t),$$

$$h_n^{(1)}(t) = \frac{1}{2^n 2 C_2} g_n(t),$$

and

$$e_n(t) = \frac{\sin(\pi(t-2^n))}{\pi(t-2^n)}.$$

According to Paley's theorem [9, p. 104], $\{e_n\}_{n\in\mathbb{N}}$ is a basic sequence in $\mathcal{PW}_\pi^1$. Further, $\{\hat{e}_n\}_{n\in\mathbb{N}}$ is a basic sequence in $L^\infty[-\pi,\pi]$ [14, p. 247]. Now we consider

$$\xi_n(t) = \xi_n^{(1)}(t) + e_n(t)$$

and

$$h_n(t) = h_n^{(1)}(t) + e_n(t).$$

We have $\|e_n^*\|_{\mathcal{PW}_\pi^\infty} = 1$ and $\|\hat{e}_n^*\|_{L^1[-\pi,\pi]} = 1$. Thus, it follows that

$$\sum_{n=1}^{\infty} \|e_n^*\|_{\mathcal{PW}_\pi^\infty} \|\xi_n - e_n\|_{\mathcal{PW}_\pi^1} = \frac{1}{2} < 1$$

and

$$\sum_{n=1}^{\infty} \|\hat{e}_n^*\|_{L^1[-\pi,\pi]} \|\hat{h}_n - \hat{e}_n\|_{C[-\pi,\pi]} = \frac{1}{2} < 1.$$

Hence, $\{\xi_n\}_{n\in\mathbb{N}}$ is a basic sequence for $\mathcal{PW}_\pi^1$ that is equivalent to $\{e_n\}_{n\in\mathbb{N}}$, and $\{\hat{h}_n\}_{n\in\mathbb{N}}$ is a basic sequence for $L^\infty[-\pi,\pi]$ that is equivalent to $\{\hat{e}_n\}_{n\in\mathbb{N}}$ [8, p. 46]. Further, there exists a constant $C_6 > 0$ such that

$$C_6 \left(\sum_{n=1}^{\infty} |a_n|^2 \right)^{\frac{1}{2}} \leq \left\| \sum_{n=1}^{\infty} a_n e_n \right\|_{\mathcal{PW}_\pi^1} \leq \left(\sum_{n=1}^{\infty} |a_n|^2 \right)^{\frac{1}{2}}.$$

Let D_{sig} denote the closure in the $\mathcal{PW}_\pi^1$-norm of the set

$$\left\{ \sum_{n=1}^{M} a_n \xi_n \colon a_n \in \mathbb{R}, M \in \mathbb{N} \right\}.$$

We have $f \in D_{\text{sig}}$ if and only if $\sum_{n=1}^{\infty} |a_n|^2 < \infty$. For every $f \in D_{\text{sig}}$ there exists a unique l^2-sequence $\{a_n\}_{n\in\mathbb{N}}$ such that

$$f = \sum_{n=1}^{\infty} a_n \xi_n.$$

Further let $\hat{D}_{\text{sys1}}$ denote the closure in the $C[-\pi,\pi]$-norm of the set

$$\left\{\sum_{n=1}^{M} b_n \hat{h}_n \colon b_n \in \mathbb{R}, M \in \mathbb{N}\right\}.$$

We have $\hat{h} \in \hat{D}_{\text{sys1}}$ if and only if $\hat{h}$ has a coefficient sequence $\{b_n\}_{n\in\mathbb{N}}$ with $\sum_{n=1}^{\infty}|b_n| < \infty$. The coefficient sequence defines $\hat{h}$ uniquely. Clearly, every $\hat{h}$ uniquely defines a stable LTI system $T = Q^{-1}\hat{h}$. We denote the corresponding space of LTI systems by $D_{\text{sys1}} = Q^{-1}\hat{D}_{\text{sys1}}$.

Let $f \in D_{\text{sig}}$, $f \not\equiv 0$, and $\hat{h} \in \hat{D}_{\text{sys1}}$, $\hat{h} \not\equiv 0$, both be arbitrary but fixed. Then we have the expansions

$$f(t) = \sum_{n=1}^{\infty} a_n(f)\xi_n(t), \quad t \in \mathbb{R},$$

and

$$\hat{h}(\omega) = \sum_{n=1}^{\infty} b_n(h)\hat{h}_n(\omega), \quad \omega \in [-\pi,\pi].$$

Let n_0 denote the smallest natural number n such that $|a_n(f)| > 0$, and m_0 denote the smallest natural number m such that $|b_m(h)| > 0$. Clearly, we have

$$f(t) = \sum_{n=n_0}^{\infty} a_n(f)\xi_n(t) = \underbrace{\sum_{n=n_0}^{\infty} a_n(f)e_n(t)}_{=A(t)} + \underbrace{\sum_{n=n_0}^{\infty} a_n(f)\xi_n^{(1)}(t)}_{=F_1(t)}$$

and

$$h(t) = \sum_{m=m_0}^{\infty} b_m(h)h_m(t) = \underbrace{\sum_{m=m_0}^{\infty} b_m(h)e_m(t)}_{=B(t)} + \underbrace{\sum_{m=m_0}^{\infty} b_m(h)h_m^{(1)}(t)}_{=G_1(t)}.$$

For $N \in \mathbb{N}$ we consider

$$\begin{aligned}\sum_{k=0}^{N} f(k)h(k) = \sum_{k=0}^{N} A(k)B(k) + \sum_{k=0}^{N} A(k)G_1(k)&\\ + \sum_{k=0}^{N} B(k)F_1(k) + \sum_{k=0}^{N} G_1(k)F_1(k).&\end{aligned}$$

Since

$$\sum_{n=n_0}^{\infty} |a_n(f)|^2 < \infty,$$

we have $A \in \mathcal{PW}_{\pi}^2$, and since

$$\sum_{m=m_0}^{\infty} |b_m(h)| < \infty,$$

we also have $B \in \mathcal{PW}_{\pi}^2$. It follows that

$$\left|\sum_{k=0}^{N} A(k)B(k)\right| \leq \left(\sum_{k=0}^{N} |A(k)|^2\right)^{\frac{1}{2}} \left(\sum_{k=0}^{N} |B(k)|^2\right)^{\frac{1}{2}} = C_7,$$

where the constant C_7 is independent of N. The same argumentation is valid for

$$\sum_{k=0}^{N} A(k)G_1(k),$$

i.e., we have

$$\left|\sum_{k=0}^{N} A(k)G_1(k)\right| \leq C_8.$$

Further, we have

$$\sum_{k=0}^{N} B(k)F_1(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{F}_1(\omega) \left(\sum_{k=0}^{N} B(k)\,\mathrm{e}^{ik\omega}\right) \mathrm{d}\omega. \tag{23}$$

Let m_N the largest natural number satisfying $2^{m_N} \leq N$. If $m_N < m_0$ then we have

$$\sum_{k=0}^{N} B(k)\,\mathrm{e}^{ik\omega} = 0$$

for all $\omega \in [-\pi, \pi]$, i.e., (23) is equal to zero. For $m_N > m_0$ we have

$$\left|\sum_{k=0}^{N} B(k)\,\mathrm{e}^{ik\omega}\right| = \left|\sum_{l=l_0}^{m_N} B(2^l)\,\mathrm{e}^{i2^l\omega}\right| \leq \sum_{l=l_0}^{m_N} |B(2^l)| \leq \sum_{m=m_0}^{m_N} |b_m(h)| = C_9,$$

where l_0 denotes the smallest natural number such that $2^{l_0} \geq m_0$. It follows that

$$\begin{aligned}
\left|\sum_{k=0}^{N} B(k)F_1(k)\right| &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{F}_1(\omega)| \left|\sum_{k=0}^{N} B(k)\,\mathrm{e}^{ik\omega}\right| \mathrm{d}\omega \\
&\leq C_9 \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{F}_1(\omega)|\,\mathrm{d}\omega \\
&= C_{10}.
\end{aligned}$$

Consequently, we have

$$\left|\sum_{k=0}^{N} f(k)h(k) - \sum_{k=0}^{N} F_1(k)G_1(k)\right| \leq C_7 + C_8 + C_{10} = C_{11}, \tag{24}$$

independently of N.

For $N \in \mathbb{N}$ we have

$$\begin{aligned}
\sum_{k=0}^{N} F_1(k)G_1(k) &= \sum_{k=0}^{N} \left(\sum_{n=n_0}^{\infty} a_n(f)\xi_n^{(1)}(k)\right) \left(\sum_{m=m_0}^{\infty} b_m(h)h_m^{(1)}(k)\right) \\
&= a_{n_0}(f)b_{m_0}(h) \sum_{k=0}^{N} \xi_{n_0}^{(1)}(k)h_{m_0}^{(1)}(k) \\
&\quad + \sum_{k=0}^{N} a_{n_0}(f)\xi_{n_0}^{(1)}(k) \left(\sum_{m=m_0+1}^{\infty} b_m(h)h_m^{(1)}(k)\right) \\
&\quad + \sum_{k=0}^{N} b_{m_0}(h)h_{m_0}^{(1)}(k) \left(\sum_{n=n_0+1}^{\infty} a_n(f)\xi_n^{(1)}(k)\right) \\
&\quad + \sum_{k=0}^{N} \left(\sum_{n=n_0+1}^{\infty} a_n(f)\xi_n^{(1)}(k)\right) \left(\sum_{m=m_0+1}^{\infty} b_m(h)h_m^{(1)}(k)\right).
\end{aligned} \tag{25}$$

For the second term we obtain

$$\begin{aligned}
&\left|\sum_{k=0}^{N_r(n_0,m_0)} a_{n_0}(f)\xi_{n_0}^{(1)}(k) \left(\sum_{m=m_0+1}^{\infty} b_m(h)h_m^{(1)}(k)\right)\right| \\
&\quad \leq \frac{|a_{n_0}(f)|}{C_1 2^{n_0+1}} \sum_{m=m_0+1}^{\infty} \frac{|b_m(h)|}{C_2 2^{m+1}} \left|\sum_{k=0}^{N_r(n_0,m_0)} \phi_{n_0}^{(1)}(k)g_m^{(1)}(k)\right| \\
&\quad \leq \frac{C_3|a_{n_0}(f)|}{C_1 C_2 2^{n_0+1}} \sum_{m=m_0+1}^{\infty} \frac{|b_m(h)|}{2^{m+1}} \\
&\quad = C_{12}.
\end{aligned} \tag{26}$$

For the third term we obtain

$$\begin{aligned}
&\left|\sum_{k=0}^{N_r(n_0,m_0)} b_{m_0}(f) h_{m_0}^{(1)}(k)\left(\sum_{n=n_0+1}^{\infty} a_n(f)\xi_n^{(1)}(k)\right)\right| \\
&\quad \leq \frac{|b_{m_0}(h)|}{C_2 2^{m_0+1}} \sum_{n=n_0+1}^{\infty} \frac{|a_n(f)|}{C_1 2^{n+1}} \left|\sum_{k=0}^{N_r(n_0,m_0)} \phi_n^{(1)}(k) g_{m_0}^{(1)}(k)\right| \\
&\quad \leq \frac{C_3|b_{m_0}(h)|}{C_1 C_2 2^{m_0+1}} \sum_{n=n_0+1}^{\infty} \frac{|a_n(f)|}{2^{n+1}} \\
&\quad \leq \frac{C_3|b_{m_0}(h)|}{C_1 C_2 2^{m_0+1}} \left(\sum_{n=n_0+1}^{\infty} |a_n(f)|^2\right)^{\frac{1}{2}} \left(\sum_{n=n_0+1}^{\infty} \frac{1}{2^{2n+2}}\right)^{\frac{1}{2}} \\
&\quad = C_{13}.
\end{aligned} \tag{27}$$

For the fourth term we have

$$\begin{aligned}
&\sum_{k=0}^{N_r(n_0,m_0)} \left(\sum_{n=n_0+1}^{\infty} a_n(f)\xi_n^{(1)}(k)\right)\left(\sum_{m=m_0+1}^{\infty} b_m(h) h_m^{(1)}(k)\right) \\
&\quad = \sum_{n=n_0+1}^{\infty} a_n(f) \sum_{m=m_0+1}^{\infty} b_m(h) \sum_{k=0}^{N_r(n_0,m_0)} \xi_n^{(1)}(k) h_m^{(1)}(k) \\
&\quad = \sum_{n=n_0+1}^{\infty} \frac{a_n(f)}{2^{n+1}C_1} \sum_{m=m_0+1}^{\infty} \frac{b_m(h)}{2^{n+1}\ C_2} \sum_{k=0}^{N_r(n_0,m_0)} \phi_n^{(1)}(k) g_m^{(1)}(k).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\left|\sum_{k=0}^{N_r(n_0,m_0)} \left(\sum_{n=n_0+1}^{\infty} a_n(f)\xi_n^{(1)}(k)\right)\left(\sum_{m=m_0+1}^{\infty} b_m(h) h_m^{(1)}(k)\right)\right| \\
&\quad \leq \sum_{n=n_0+1}^{\infty} \frac{|a_n(f)|}{2^{n+1}C_1} \sum_{m=m_0+1}^{\infty} \frac{|b_m(h)|}{2^{m+1}C_2} \left|\sum_{k=0}^{N_r(n_0,m_0)} \phi_n^{(1)}(k) g_m^{(1)}(k)\right| \\
&\quad \leq C_3 \sum_{n=n_0+1}^{\infty} \frac{|a_n(f)|}{2^{n+1}C_1} \sum_{m=m_0+1}^{\infty} \frac{|b_m(h)|}{2^{m+1}C_2} \\
&\quad \leq \frac{C_3}{C_1 C_2} \left(\sum_{n=n_0+1}^{\infty} |a_n(f)|^2\right)^{\frac{1}{2}} \left(\sum_{n=n_0+1}^{\infty} \frac{1}{2^{2n+2}}\right)^{\frac{1}{2}} \\
&\qquad \left(\sum_{m=m_0+1}^{\infty} |b_m(h)|^2\right)^{\frac{1}{2}} \left(\sum_{m=m_0+1}^{\infty} \frac{1}{2^{2n+2}}\right)^{\frac{1}{2}} \\
&\leq C_{14},
\end{aligned} \tag{28}$$

where we used Lemma 1 in the second inequality. Combining (24) and (25)–(28), we see that

$$\left|\sum_{k=0}^{N_r(n_0,m_0)} f(k)h(k)\right| \geq \left|a_{n_0}(f)b_{m_0}(h) \sum_{k=0}^{N_r(n_0,m_0)} \xi_{n_0}^{(1)}(k)h_{m_0}^{(1)}(k)\right| - C_{11} - C_{12} - C_{13} - C_{14}.$$

Now since

$$\left|a_{n_0}(f)b_{m_0}(h) \sum_{k=0}^{N_r(n_0,m_0)} \xi_{n_0}^{(1)}(k)h_{m_0}^{(1)}(k)\right| = \frac{|a_{n_0}(f)b_{m_0}(h)|}{C_1C_2 2^{n_0+m_0+2}} \left|\sum_{k=0}^{N_r(n_0,m_0)} \phi_{n_0}(k)g_{m_0}(k)\right|,$$

where $|a_{n_0}(f)b_{m_0}(h)| > 0$, and

$$\limsup_{r\to\infty} \left|\sum_{k=0}^{N_r(n_0,m_0)} \phi_{n_0}(k)g_{m_0}(k)\right| = \infty,$$

according to Lemma 1, it follows that

$$\limsup_{r\to\infty} \left|a_{n_0}(f)b_{m_0}(h) \sum_{k=0}^{N_r(n_0,m_0)} \xi_{n_0}^{(1)}(k)h_{m_0}^{(1)}(k)\right| = \infty,$$

and consequently that

$$\limsup_{r\to\infty} \left|\sum_{k=0}^{N_r(n_0,m_0)} f(k)h(k)\right| = \infty.$$

Since $h(k) = 0$ for $k < 0$, this implies that

$$\limsup_{N\to\infty} \left|\sum_{k=-N}^{N} f(k)h(k)\right| = \infty.$$

To complete the proof, we consider the space $D_{\text{sys}} = Q^{-1}RQD_{\text{sys1}}$, where $R\colon f \mapsto f(-\,\cdot\,)$ denotes the time-reversal operator. D_{sys} is an infinite dimensional closed subspace of $\mathcal{T}$ and we have

$$\limsup_{N\to\infty} \left|\sum_{k=-N}^{N} f(k)h_T(-k)\right| = \infty$$

for all $f \in D_{\text{sig}}$, $f \not\equiv 0$, and all $T \in D_{\text{sys}}$, $T \not\equiv 0$. $\square$

Proof of Corollary 1. From Theorem 1 we know that there exist an infinite dimensional closed subspace $D_{\mathrm{sig}} \subset \mathcal{PW}_\pi^1$ and an infinite dimensional closed subspace $D_{\mathrm{sys}} \subset \mathcal{T}$, such that for all $f \in D_{\mathrm{sig}}$, $f \not\equiv 0$, and all $T \in D_{\mathrm{sys}}$, $T \not\equiv 0$, we have

$$\limsup_{N\to\infty} \left| \sum_{k=-N}^{N} f(k) h_T(-k) \right| = \infty. \tag{29}$$

Let $t \in \mathbb{R}$ be arbitrary but fixed, and consider the operator $U\colon L^\infty[-\pi,\pi] \to L^\infty[-\pi,\pi], \hat{h}_T \mapsto \hat{h}_T \,\mathrm{e}^{-i\,\cdot\,t}$. U is a bounded, linear, and invertible operator with bounded inverse. Hence, $D_{\mathrm{sys2}} = Q^{-1}UQD_{\mathrm{sys}}$ is an infinite dimensional closed subspace of $\mathcal{T}$. Let $f \in D_{\mathrm{sig}}$ and $T_2 \in D_{\mathrm{sys2}}$ be arbitrary but fixed. Further, let $\hat{h}_T = U^{-1}\hat{h}_{T_2}$. For $N \in \mathbb{N}$ we have

$$\begin{aligned}
\sum_{k=-N}^{N} f(k) h_{T_2}(t-k) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{h}_{T_2}(\omega)\,\mathrm{e}^{i\omega t} \sum_{k=-N}^{N} f(k)\,\mathrm{e}^{-i\omega k}\ \mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} (U^{-1}\hat{h}_{T_2})(\omega) \sum_{k=-N}^{N} f(k)\,\mathrm{e}^{-i\omega k}\ \mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{h}_{T}(\omega) \sum_{k=-N}^{N} f(k)\,\mathrm{e}^{-i\omega k}\ \mathrm{d}\omega \\
&= \sum_{k=-N}^{N} f(k) h_T(-k).
\end{aligned}$$

Since $T \in D_{\mathrm{sys}}$, it follows from (29) that

$$\limsup_{N\to\infty} \left| \sum_{k=-N}^{N} f(k) h_{T_2}(t-k) \right| = \infty. \qquad \square$$

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