Banach–Steinhaus Theory Revisited: Lineability and Spaceability

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Abstract

In this paper we study the divergence behavior of linear approximation processes in general Banach spaces. We are interested in the structure of the set of divergence creating functions. The Banach–Steinhaus theory gives some information about this set, however, it cannot be used to answer the question whether this set contains subspaces with linear structure. We give necessary and sufficient conditions for the lineability and the spaceability of the set of divergence creating functions.

Keywords: Approximation, Divergence, Banach spaces, Spaceability, Lineability, Banach–Steinhaus theorem

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1. Introduction and Notation

A central problem in approximation theory is the approximation of a bounded linear operator $T$ by a sequence of operators $\{T_N\}_{N \in \mathbb{N}}$. We consider the following setting. Let $B_1$ and $B_2$ be two Banach spaces and let $T: B_1 \to B_2$ be a bounded linear operator. Given a sequence of bounded linear operators $\{T_N\}_{N \in \mathbb{N}}$ mapping from $B_1$ into $B_2$, we are interested in whether, for all $f \in B_1$, $T_N f$ converges to $T f$ in the norm of $B_2$. According to the Banach–Steinhaus theorem, the answer to this question is “yes” if and only if there exists a con-
stant $C_1$ such that $\|T_N\|_{B_1 \to B_2} \leq C_1$ for all $N \in \mathbb{N}$ and we have $T_N f \to Tf$ as $N$ tends to infinity for all $f$ from a dense subspace of $B_1$.

On the other hand, the Banach–Steinhaus theorem also implies the principle of condensation of singularities: If there exists a function $f \in B_1$ for which we have divergence, i.e., $\limsup_{N \to \infty} \|T_N f\|_{B_2} = \infty$, then we have divergence for all functions from a residual and therefore dense subset of $B_1$.

Since the publication of Banach and Steinhaus [4, 3], the Banach–Steinhaus theory has been developed further and has today become an important part of functional analysis. There also have been efforts to extend the Banach–Steinhaus theory into different directions [15, 7, 8, 16].

Although the Banach–Steinhaus theorem gives some information about the size of the set of divergence creating functions, it gives no information about the structure of this set. In particular, it would be interesting to know if it possesses a linear structure. Such a linear structure is important in application, because it implies that any linear combination of functions, which is not the zero function, leads to divergence as well.

Note that it is significantly more difficult to show a linear structure in the set of functions with divergent approximation process compared to showing a linear structure in the set of functions with convergent approximation process. If we have two functions $f_1$ and $f_2$, for which $T_N f_1$ and $T_N f_2$ converge, it is clear that the sum of both functions, i.e., $f_1 + f_2$, is a function for which we have convergence as well. Hence, any finite linear combination of functions with convergent approximation process will be a function with convergent approximation process. However, for divergence this is not true. Given two functions $g_1$ and $g_2$ for which $T_N g_1$ and $T_N g_2$ diverge, we cannot conclude that the sum of both functions, i.e., $g_1 + g_2$, is a function for which the approximation process diverges. This can be easily seen by choosing $g_2 = f_1 - g_1$, where $f_1$ is any function with convergent approximation process and $g_1$ any function with divergent approximation process. Obviously, for the sum $g_1 + g_2 = f_1$ we do not have divergence.

The above example shows that in general we cannot expect that the set of
functions with divergent approximation process is a linear space. However, we can ask if this set contains an infinite dimensional subspace with linear structure. Lineability and Spaceability are two mathematical concepts that are suitable to study this question.

Lineability and spaceability were recently introduced and used for example in [10, 12, 2, 5, 1]. Both describe the structure of some given subset of an ambient normed space or, more generally, topological space. A set $S$ in a linear topological space $X$ is said to be spaceable if $S \cup \{0\}$ contains a closed infinite dimensional subspace of $X$. A closely related concept is lineability. A set $S$ in a linear topological space $X$ is said to be lineable if $S \cup \{0\}$ contains an infinite dimensional subspace.

Results about lineability and spaceability have been obtained for different problems. In [11] it was proved that the set of continuous nowhere differentiable functions on $\mathbb{R}$ is lineable. Later, it was shown that the set of continuous nowhere differentiable functions on $C[0, 1]$ is spaceable [10]. The divergence of Fourier series was analyzed in [5], where it was shown that the set of functions in $L^1(\partial \mathbb{D})$, whose Fourier series diverges everywhere on $\partial \mathbb{D}$ is spaceable.

2. General Setting

In this paper we present results about the lineability and spaceability of certain sets. In order to be able to discuss the problems, we introduce some notation, which will be used throughout the rest of this paper.

By $V$ we denote a general vector space. Further, $V_P$ is a subspace of $V$. In the applications that we will discuss, $V_P$ is the subspace of all elements in $V$ that have a certain property $P$. The property $P$ can be different for different applications. Note that we implicitly assume that the set of all elements that have the property $P$ forms a vector space, i.e., that $V_P$ has a linear structure. This assumption is, as we will see, justified for many applications. By $D_P = V \setminus V_P$ we denote the complement of the set $V_P$, i.e., the set of all elements in $V$ that do not have the property $P$. 

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We further will use the concept of Hamel basis. A subset $S$ of a vector space $V$ is called a Hamel basis for $V$, if $S$ is finitely linear independent and \( \text{span}(S) = \left\{ \sum_{n=1}^{N} \alpha_n s_n : N \in \mathbb{N}, s_n \in S, \alpha_n \in \mathbb{C} \right\} = V \). Every vector space has a Hamel basis. By $L_V$ we denote a Hamel basis of the vector space $V$. Further, it is easy to see that there exists a Hamel basis $L_{V_P}$ of the subspace $V_P \subset V$ such that $L_{V_P} \subset L_V$. Whenever we use $L_{V_P}$, we assume that it has the property $L_{V_P} \subset L_V$. We will also use the fact that every subspace $W$ of $V$ is a direct summand $V = W \oplus U$ for some subspace $U$ of $V$.

3. Lineability

In this section we will show that, under natural assumption, the set of functions with divergent approximation process is lineable. Before we state this main result in Section 3.2, we will derive general results about the lineability of certain sets in Section 3.1.

3.1. General Result

The first lemma shows that the complement $D_P$ of a subspace $V_P \subsetneq V$ always contains contains a non-trivial vector space when the zero element $\{0\}$ is added.

**Lemma 1.** If $V_P \subsetneq V$, then the set $D_P \cup \{0\} = (V \setminus V_P) \cup \{0\}$ contains a non-trivial vector space.

**Remark 1.** The vector space that is contained in $D_P \cup \{0\}$ does not need to be infinite dimensional even if $V$ is infinite dimensional.

**Proof.** Let $L_{V_P}$ be a Hamel basis for $V_P$, and let $L_V$ be a Hamel basis for $V$ with $L_{V_P} \subset L_V$. Then $L_V \setminus L_{V_P}$ is a Hamel basis for a vector space $U_P \subset V$, and we have $V = V_P \oplus U_P$. Let $f \in U_P$, $f \neq 0$. Since $f \notin V_P$, $f$ does not have the property $P$. \[\square\]

The next lemma gives us a decomposition result for the complement $D_P$ of a subspace $V_P \subsetneq V$. 

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Lemma 2. Let $V_P \subsetneq V$, $L_V$ be a Hamel basis for $V$, $L_{V_P}$ a Hamel basis for $V_P$ with $L_{V_P} \subset L_V$, and $U_P = \text{span}(L_V \setminus L_{V_P})$. Then $D_P = V \setminus V_P$ has the representation

$$D_P = \{g + d: g \in V_P, d \in U_P, d \neq 0\}.$$  

Proof. Since $V = V_P \oplus U_P$, it is clear that every element in $D_P \subset V$ can be decomposed as stated in the lemma. It remains to show that for all $g \in V_P$ and all $d \in U_P$, $d \neq 0$, we always have $g + d \not\in V_P$. Assume that we have $\phi = g + d \in V_P$. Since, $V_P$ is a vector space, it follows that $\phi - g \in V_P$, and consequently that $d \in V_P$. However, we have that $d \in U_P$ and $d \neq 0$. This is a contradiction because $V_P \cap U_P = \{0\}$. \hfill \Box

In Theorem 1 we have a necessary and sufficient condition for the lineability of the complement $D_P$ of a subspace $V_P \subset V$.

Theorem 1. Let $L_V$ be a Hamel basis for $V$, $L_{V_P}$ be a Hamel basis for $V_P$ with $L_{V_P} \subset L_V$, and $U_P = \text{span}(L_V \setminus L_{V_P})$. Then the set $D_P = V \setminus V_P$ is lineable if and only if $U_P$ is infinite dimensional.

Proof. “⇒”: We have $U_P \subset (V \setminus V_P) \cup \{0\}$, because $U_P$ has the Hamel basis $L_V \setminus L_{V_P}$. Since $U_P$ is an infinite dimensional vector space, $D_P$ is lineable.

“⇐”: Suppose that $D_P$ is lineable and $U_P$ is a finite dimensional vector space. Let $A_P$ denote the dimension of $U_P$, and let $\{\varphi_1, \ldots, \varphi_{A_P}\}$ be a basis of $U_P$. Since $D_P$ is lineable, there exists an infinite dimensional vector space $U_P^* \subset D_P \cup \{0\}$. Further, let $I^{(0)} = \{f_k^{(0)}\}_{k \in \mathbb{N}} \subset U_P^*$ be a countably infinite and linearly independent set. Based on $I^{(0)}$ we construct a new set $I^{(1)}$ according to the following procedure. For $r \in \mathbb{N}$ we consider the pairs $(f_{2r-1}^{(0)}, f_{2r}^{(0)})$. Since $U_P^* \subset V$ and $V = V_P \oplus U_P$, we have

$$f_{2r-1}^{(0)} = g_{2r-1}^{(0)} + d_{2r-1}^{(0)}$$

with suitable elements $g_{2r-1}^{(0)} \in V_P$ and $d_{2r-1}^{(0)} \in U_P$, $l = 0, 1$. Let

$$d_{2r-1}^{(0)} = \sum_{n=1}^{A_P} a_{2r-1,n}^{(0)} \varphi_n$$

and
be the basis expansion of \( d_{2l-1}^{(0)} \), \( l = 0, 1 \). We have to distinguish three cases:

1. If the elements \( d_{2l-1}^{(0)} \) and \( d_{2l}^{(0)} \) are independent of the element \( \varphi_1 \), i.e., if 
   \[ a_{2l-1,1}^{(0)} = 0 \] and \( a_{2l,1}^{(0)} = 0 \), then we include \( f_{2l-1}^{(0)} \) and \( f_{2l}^{(0)} \) into the set \( I^{(1)} \).

2. If only one of the elements \( d_{2l-1}^{(0)} \) and \( d_{2l}^{(0)} \) depends on the element \( \varphi_1 \), i.e., 
   if either \( a_{2l-1,1}^{(0)} \neq 0 \) or \( a_{2l,1}^{(0)} \neq 0 \), then we include the element \( f_{2l-1}^{(0)} \) into the set \( I^{(1)} \) if \( a_{2l-1,1}^{(0)} = 0 \), or the element \( f_{2l}^{(0)} \) into the set \( I^{(1)} \) if \( a_{2l,1}^{(0)} = 0 \).

3. If both of the elements \( d_{2l-1}^{(0)} \) and \( d_{2l}^{(0)} \) depend on the element \( \varphi_1 \), i.e., if 
   both \( a_{2l-1,1}^{(0)} \neq 0 \) and \( a_{2l,1}^{(0)} \neq 0 \), then we include
   \[ h_r^{(0)} = a_{2l-1,1}^{(0)} f_{2l-1}^{(0)} - a_{2l,1}^{(0)} f_{2l-1}^{(0)} \]
   into the set \( I^{(1)} \). Clearly, \( h_r \neq 0 \) because \( \{f_k^{(0)}\}_{k \in \mathbb{N}} \) is linearly independent.
   Moreover, since \( h_r^{(0)} \in U_p \subset V \) and \( V = V_p \oplus U_p \), we have
   \[ h_r^{(0)} = g_r^{(0)} + d_r^{(0)} \]
   with suitable elements \( g_r^{(0)} \in V_p \) and \( d_r^{(0)} \in U_p \). Due to the construction of \( h_r^{(0)} \), \( d_r^{(0)} \) is independent of \( \varphi_1 \).

We have \( I^{(1)} \subset U_p \) and \( I^{(1)} \) is linearly independent. For each pair from the set \( I^{(0)} \), we added at least one element to the set \( I^{(1)} \). Due to the linear independence all those elements are different. Hence \( I^{(1)} \) is a countably infinite set. According to the same procedure we can construct a countably infinite, linearly independent set \( I^{(2)} = \{f_k^{(2)}\}_{k \in \mathbb{N}} \subset U_p^* \), such that the elements \( d_k^{(2)} \) in the decomposition
   \[ f_k^{(2)} = g_k^{(2)} + d_k^{(2)} \]
   \( g_k^{(2)} \in V_p, d_k^{(2)} \in U_p \), are independent of \( \varphi_1 \) and \( \varphi_2 \). We repeat this process \( A_P \) times, and obtain a countably infinite, linearly independent set \( I^{(A_P)} = \{f_k^{(A_P)}\}_{k \in \mathbb{N}} \subset U_p^* \), such that the elements \( d_k^{(A_P)} \) in the decomposition
   \[ f_k^{(A_P)} = g_k^{(A_P)} + d_k^{(A_P)} \]
   \( g_k^{(A_P)} \in V_p, d_k^{(A_P)} \in U_p \), are independent of \( \varphi_1, \ldots, \varphi_{A_P} \), which implies that 
   \( d_k^{(A_P)} = 0 \). Thus, we have \( f_k^{(A_P)} \in V_p \) and \( f_k^{(A_P)} \neq 0 \) for all \( k \in \mathbb{N} \), which is a contradiction, because \( f_k^{(A_P)} \in U_p \subset D_p \cup \{0\} \). \( \square \)
3.2. Lineability of the Set of Functions with Divergent Approximation Processes

Now we come to our main result, which shows that the set of functions with divergent approximation process is lineable under very general assumptions.

**Theorem 2.** Let $B_1$ and $B_2$ be two Banach spaces and $T: B_1 \to B_2$ a bounded linear operator. Further, let $\{T_N\}_{N \in \mathbb{N}}$ be a sequence of bounded linear operators, mapping from $B_1$ into $B_2$, with:

(i) $\limsup_{N \to \infty} \|T_N\|_{B_1 \to B_2} = \infty$, and

(ii) there exists a dense subset $K$ of $B_1$ such that $\lim_{N \to \infty} \|Tf - T_Nf\|_{B_2} = 0$ for all $f \in K$.

Then the set

$$D_{UB} = \left\{ f \in B_1 : \limsup_{N \to \infty} \|T_N f\|_{B_2} = \infty \right\}$$

is lineable.

**Remark 2.** According to Lemma 2, we have $f \in D_{UB}$ if and only if $f$ can be written as $f = g + d$ with $g \in V_{UB}$ and $d \in U_{UB}$, $d \neq 0$. Hence, a Hamel basis of $U_{UB}$ creates all the elements that are responsible for divergence.

Based on Theorem 1 and Theorem 2, we obtain the following corollary.

**Corollary 1.** Let $B_1$ and $B_2$ be two Banach spaces and $T: B_1 \to B_2$ a bounded linear operator. Further, let $\{T_N\}_{N \in \mathbb{N}}$ be a sequence of bounded linear operators, mapping from $B_1$ into $B_2$, with:

(i) $\limsup_{N \to \infty} \|T_N\|_{B_1 \to B_2} = \infty$, and

(ii) there exists a dense subset $K$ of $B_1$ such that $\lim_{N \to \infty} \|Tf - T_Nf\|_{B_2} = 0$ for all $f \in K$.

Let $V_B$ be the vector space of all elements $f \in B_1$ with $\limsup_{N \to \infty} \|T_N f\|_{B_2} < \infty$. Then every vector space $U_B \subset B_1$ with $B_1 = V_B \oplus U_B$ is lineable.

**Proof.** We have $D_{UB} = B_1 \setminus V_B$, and from Theorem 2 we know that $D_{UB}$ is lineable. Hence, Theorem 1 implies that every vector space $U_B \subset B_1$ with $B_1 = V_B \oplus U_B$ is infinite dimensional and hence lineable.

\[\square\]
In the proof of Theorem 2 we will explicitly construct an infinite dimensional subspace of $D_{UB}$. For the proof of Theorem 2 we need the following lemma.

**Lemma 3.** Let $B_1$ and $B_2$ be two Banach spaces and $T: B_1 \to B_2$ a bounded linear operator. Further, let $\{T_N\}_{N \in \mathbb{N}}$ be a sequence of bounded linear operators, mapping from $B_1$ into $B_2$, with:

(i) $\limsup_{N \to \infty} \|T_N\|_{B_1 \to B_2} = \infty$, and

(ii) there exists a dense subset $\mathcal{K}$ of $B_1$ such that $\lim_{N \to \infty} \|Tf - T_Nf\|_{B_2} = 0$ for all $f \in \mathcal{K}$.

Then there exist a sequence of finitely linearly independent functions $\{\varphi_n\}_{n \in \mathbb{N}} \subset B_1$ with

(iii) $\|\varphi_n\|_{B_1} = 1$ for all $n \in \mathbb{N}$,

and a constant $C_2 > 0$, such that for all $n \in \mathbb{N}$ there exists a sequence of natural numbers $\{N_k(n)\}_{k \in \mathbb{N}}$ with:

(iv) $\limsup_{k \to \infty} \|T_{N_k(n)}\varphi_n\|_{B_2} = \infty$, and

(v) $\sup_{k \in \mathbb{N}} \|T_{N_k(m)}\varphi_n\|_{B_2} \leq C_2$ for all $m \neq n$.

**Proof of Lemma 3.** We construct the functions $\{\phi_n\}_{n \in \mathbb{N}}$ iteratively. We start with $r = 1$. Let $C_1(1) = 1$ and let $N_1(1)$ be the smallest natural number such that $\|T_{N_1(1)}\| > 1^3$. Let $g_{1,1} \in \mathcal{K}$, $\|g_{1,1}\|_{B_1} = 1$, be such that $\|T_{N_1(1)}g_{1,1}\|_{B_2} > 1^3$. We set

$\phi_{1,1} = g_{1,1}$.

Let $\overline{N}_1(1)$ be the smallest natural number such that $\overline{N}_1(1) \geq N_1(1)$ and $\|T_N\phi_{1,1}\|_{B_2} < 1 + \|T\|$ for all $N \geq \overline{N}_1(1)$. Since

$$\|T_N\phi_{1,1}\|_{B_2} = \|T_N g_{1,1}\|_{B_2}$$

$$\leq \|T_N g_{1,1} - T g_{1,1}\|_{B_2} + \|T g_{1,1}\|_{B_2}$$

$$\leq \|T_N g_{1,1} - T g_{1,1}\|_{B_2} + \|T\|,$$

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and we have $g_{1,1} \in \mathcal{K}$, this is possible because of condition (ii).

Now we come to the second iteration step, i.e., $r = 2$. Let $C_2(1) = \max_{1 \leq N \leq \mathcal{N}_1(1)} \|T_N\|$. Since $\max_{1 \leq N \leq \mathcal{N}_1(1)} \|T_N\| \geq \|T_{N_1(1)}\| > 1$, it follows that $C_2(1) > 1$. Let $N_2(1)$ be the smallest natural number such that $N_2(1) \geq \mathcal{N}_1(1)$ and $\|T_{N_2(1)}\| > 2^3C_2(1)$. Let $g_{1,2} \in \mathcal{K}$, $\|g_{1,2}\|_{B_1} = 1$, be such that $\|T_{N_2(1)}g_{1,2}\|_{B_2} > 2^3C_2(1)$. We set

$$\phi_{1,2} = g_{1,1} + \frac{1}{2^2C_2(1)}g_{1,2}.$$ 

Let $\mathcal{N}_2(1)$ be the smallest natural number such that $\mathcal{N}_2(1) \geq N_2(1)$ and $\|T_N\phi_{1,2}\|_{B_2} < 1 + \|T\|C_3$ for all $N \geq \mathcal{N}_2(1)$, where $C_3 = \sum_{i=1}^{\infty} \frac{1}{2^i}$. Since

$$\|T_N\phi_{1,2}\|_{B_2} \leq \|T_N\phi_{1,2} - T\phi_{1,2}\|_{B_2} + \|T\| \phi_{1,2} \|_{B_2}$$

$$\leq \|T_N\phi_{1,2} - T\phi_{1,2}\|_{B_2} + \|T\| \left( \|g_{1,1}\|_{B_1} + \frac{1}{2^2C_2(1)}\|g_{1,2}\|_{B_1} \right)$$

$$\phi_{1,2} \in \mathcal{K},$$

and

$$\|g_{1,1}\|_{B_1} + \frac{1}{2^2C_2(1)}\|g_{1,2}\|_{B_1} \leq 1 + \frac{1}{2^2} < C_3$$

because $C_2(1) \geq 1$, this is possible.

Let $C_2(2) = \max_{1 \leq N \leq \mathcal{N}_2(1)} \|T_N\|$. Let $N_2(2)$ be the smallest natural number such that $N_2(2) \geq \mathcal{N}_2(1)$ and $\|T_{N_2(2)}\| > 2^3C_2(2)$. Let $g_{2,2} \in \mathcal{K}$, $\|g_{2,2}\|_{B_1} = 1$, be such that $\|T_{N_2(2)}g_{2,2}\|_{B_2} > 2^3C_2(2)$. We set

$$\phi_{2,2} = \frac{1}{2^2C_2(2)}g_{2,2}.$$ 

Let $\mathcal{N}_2(2)$ be the smallest natural number such that $\mathcal{N}_2(2) \geq N_2(2)$ and $\|T_N\phi_{2,2}\|_{B_2} < 1 + \|T\|C_3$ for all $N \geq \mathcal{N}_2(2)$. This is possible by the same arguments as before.

Now assume that for some $r \in \mathbb{N}$, $r \geq 2$, we have constructed the following objects: $g_{1,r}, g_{2,r}, \ldots, g_{r,r}$; $N_r(1), \ldots, N_r(r)$; $\mathcal{N}_r(1), \ldots, \mathcal{N}_r(r)$; $C_r(1), \ldots, C_r(r)$; $\phi_{1,r} = \phi_{1,r-1} + \frac{1}{r^2C_{r-1}(1)}g_{1,r}$, $\phi_{r-1,r} = \phi_{r-1,r-1} + \frac{1}{r^2C_{r-1}(r-1)}g_{r-1,r}$, $\phi_{r,r} = \frac{1}{r^2C_{r}(r)}g_{r,r}$. Let $C_{r+1}(1) = \max_{1 \leq N \leq \mathcal{N}_r(1)} \|T_N\|$. Let $N_{r+1}(1)$ be the smallest natural number such that $N_{r+1}(1) \geq \mathcal{N}_r(r)$ and $\|T_{N_{r+1}(1)}\| > (r + 1)^3C_{r+1}(1)$.
Let \( g_{1,r+1} \in \mathcal{K} \), \( \|g_{1,r+1}\|_{B_1} = 1 \), be such that \( \|T_{N_{r+1}(1)}g_{1,r+1}\|_{B_2} > (r + 1)^3C_{r+1}(1) \). We set
\[
\phi_{1,r+1} = \phi_{1,r} + \frac{1}{(r + 1)^2C_{r+1}(1)}g_{1,r+1}.
\]
Let \( \overline{N}_{r+1}(1) \) be the smallest natural number such that \( \overline{N}_{r+1}(1) \geq N_{r+1}(1) \) and \( \|T_N\phi_{1,r+1}\|_{B_2} < 1 + \|T\|C_3 \) for all \( N \geq \overline{N}_{r+1}(1) \).

Let \( C_{r+1}(2) = \max_{1 \leq N \leq \overline{N}_{r+1}(1)}\|T_N\| \). Let \( N_{r+1}(2) \) be the smallest natural number such that \( N_{r+1}(2) \geq \overline{N}_{r+1}(1) \) and \( \|T_{N_{r+1}(2)}\| > (r + 1)^3C_{r+1}(2) \). Let \( g_{2,r+1} \in \mathcal{K} \), \( \|g_{2,r+1}\|_{B_1} = 1 \), be such that \( \|T_{N_{r+1}(2)}g_{2,r+1}\|_{B_2} > (r + 1)^3C_{r+1}(2) \). We set
\[
\phi_{2,r+1} = \phi_{2,r} + \frac{1}{(r + 1)^2C_{r+1}(2)}g_{2,r+1}.
\]
Let \( \overline{N}_{r+1}(2) \) be the smallest natural number such that \( \overline{N}_{r+1}(2) \geq N_{r+1}(2) \) and \( \|T_N\phi_{2,r+1}\|_{B_2} < 1 + C_3\|T\| \) for all \( N \geq \overline{N}_{r+1}(2) \). This procedure is continued.

Now assume, for some \( l \leq r \), we have already constructed the functions \( g_{l,r+1}, \phi_{l,r+1} \) and the numbers \( N_{r+1}(l), \overline{N}_{r+1}(l), \) and \( C_{r+1}(l) \). The functions and numbers for \( l+1 \) are defined as follows. Let \( C_{r+1}(l+1) = \max_{1 \leq N \leq \overline{N}_{r+1}(l)}\|T_N\| \). Further, let \( N_{r+1}(l+1) \) be the smallest natural number such that \( N_{r+1}(l+1) \geq \overline{N}_{r+1}(l) \) and \( \|T_{N_{r+1}(l+1)}\| > (r + 1)^3C_{r+1}(l + 1) \). Let \( g_{l+1,r+1} \in \mathcal{K} \), \( \|g_{l+1,r+1}\|_{B_1} = 1 \), be such that \( \|T_{N_{r+1}(l+1)}g_{l+1,r+1}\|_{B_2} > (r + 1)^3C_{r+1}(l + 1) \). If \( l < r \) we set
\[
\phi_{l+1,r+1} = \phi_{l+1,r} + \frac{1}{(r + 1)^2C_{r+1}(l + 1)}g_{l+1,r+1}
\]
and if \( l = r \) we set
\[
\phi_{r+1,r+1} = \frac{1}{(r + 1)^2C_{r+1}(r + 1)}g_{r+1,r+1}.
\]
Let \( \overline{N}_{r+1}(l+1) \) be the smallest natural number such that \( \overline{N}_{r+1}(l+1) > N_{r+1}(l+1) \) and \( \|T_N\phi_{l+1,r+1}\|_{B_2} < 1 + C_3\|T\| \) for all \( N \geq \overline{N}_{r+1}(l + 1) \).

Thus, following the above construction, for every number \( r \in \mathbb{N} \), we have created the corresponding mathematical objects \( g_{l,r}, N_r(l), C_r(l), \) and \( \phi_{l,r} \), \( 1 \leq l \leq r \). For notational convenience, for all \( r \in \mathbb{N} \), we set \( N_r(l) = N_r(r) \) if
For $1 \leq n \leq r$, we have
\[
\phi_{n,r} = \sum_{k=n}^{r} \frac{1}{k^2 C_k(n)} g_{n,k},
\]
and it follows that there exist functions $\{\phi_n\}_{n \in \mathbb{N}}$ such that
\[
\lim_{r \to \infty} \| \phi_n - \phi_{n,r} \|_{B_1} = 0
\]
for all $n \in \mathbb{N}$. For arbitrary $n \in \mathbb{N}$ and $k > n$ we have
\[
T_{N_k(n)} \phi_n = T_{N_k(n)} \left( \sum_{r=n}^{k-1} \frac{1}{r^2 C_r(n)} T_{N_k(n)} g_{n,r} + \frac{1}{k^2 C_k(n)} T_{N_k(n)} g_{n,k} \right) + \sum_{r=k+1}^{\infty} \frac{1}{r^2 C_r(n)} T_{N_k(n)} g_{n,r}
\]
\[
= T_{N_k(n)} \phi_{n,k-1} + \frac{1}{k^2 C_k(n)} T_{N_k(n)} g_{n,k} + \sum_{r=k+1}^{\infty} \frac{1}{r^2 C_r(n)} T_{N_k(n)} g_{n,r},
\]
and consequently that
\[
\| T_{N_k(n)} \phi_{n,k-1} \|_{B_2} < 1 + C_3 \| T \|
\]
because $N_k(n) \geq N_{k-1}(n)$. Further, we have
\[
\left\| \sum_{r=k+1}^{\infty} \frac{1}{r^2 C_r(n)} T_{N_k(n)} g_{n,r} \right\|_{B_2} \leq \sum_{r=k+1}^{\infty} \frac{1}{r^2 C_r(n)} \| T_{N_k(n)} \| \| g_{n,r} \|_{B_1}
\]
\[
\leq \sum_{r=k+1}^{\infty} \frac{1}{r^2} < \frac{1}{k} < 1,
\]
because $\| g_{n,r} \|_{B_1} = 1$ for all $r \geq n$, and $\| T_{N_k(n)} \| \leq C_r(n)$ for all $r > k$, which follows from $N_k(n) \leq N_r(n-1)$. It follows that
\[
\| T_{N_k(n)} \phi_n \|_{B_2} > \frac{1}{k^2 C_k(n)} \| T_{N_k(n)} g_{n,k} \|_{B_2} - 2 - C_3 \| T \|
\]
\[
> k - 2 - C_3 \| T \|
\]
because \( \|T_{N_k(n)}g_{n,k}\|_{B_2} > k^3C_k(n) \), and consequently that
\[
\limsup_{k \to \infty} \|T_{N_k(n)}\phi_n\|_{B_2} = \infty.
\]
Similarly, we show that there exists a constant \( C_4 > 0 \) such that, for all \( m,n,k \in \mathbb{N}, m \neq n \), we have
\[
\|T_{N_k(m)}\phi_n\|_{B_2} \leq C_4.
\]
We have to distinguish three cases: \( k < n \), \( k > n \), and \( k = n \). For \( k < n \) we have
\[
\|T_{N_k}(\phi_n)\|_{B_2} = \left\| \sum_{r=n}^{\infty} \frac{1}{r^2C_r(n)} g_{n,r} \right\|_{B_2}
\]
\[
\leq \sum_{r=n}^{\infty} \frac{1}{r^2C_r(n)} \|T_{N_k}\| \|g_{n,r}\|_{B_2}
\]
\[
\leq \sum_{r=n}^{\infty} \frac{1}{r^2} \leq \frac{\pi^2}{6},
\]
because \( \|g_{n,r}\|_{B_2} = 1 \) for all \( r \geq n \), and \( \|T_{N_k}\| \leq C_r(n) \) for all \( r > k \), which follows from \( N_k(n) \leq N_r(n-1) \).

Next, we treat the case \( k > n \).

\[
\|T_{N_k(m)}\phi_n\|_{B_2}
\]
\[
= \left\| \sum_{r=n}^{k-1} \frac{1}{r^2C_r(n)} g_{n,r} + \frac{1}{k^2C_k(n)} g_{n,k} + \sum_{r=k+1}^{\infty} \frac{1}{r^2C_r(n)} g_{n,r} \right\|_{B_2}
\]
\[
\leq \|T_{N_k(m)}\phi_{n,k-1}\|_{B_2} + \frac{1}{k^2C_k(n)} \|T_{N_k(m)}g_{n,k}\|_{B_2}
\]
\[
+ \sum_{r=k+1}^{\infty} \frac{1}{r^2C_r(n)} \|T_{N_k(m)}g_{n,r}\|_{B_2}.
\]
For the first term we have
\[
\|T_{N_k(m)}\phi_{n,k-1}\|_{B_2} \leq 1 + C_3\|T\|,
\]
because \( N_k(m) \geq N_{k-1}(n) \). For the second term we have to distinguish \( n > m \) and \( n < m \). For \( n > m \) we have
\[
\frac{1}{k^2C_k(n)} \|T_{N_k(m)}g_{n,k}\|_{B_2} \leq \frac{1}{k^2C_k(n)} \|T_{N_k(m)}\| \leq \frac{1}{k^2} < 1,
\]
while for \( n < m \) we have
because $C_k(n) \geq \|T_{N_k(m)}\|$ due to $N_k(m) \leq N_k(n-1)$. For $n < m$ we have
\[
\frac{1}{k^2C_k(n)}\|T_{N_k(m)}g_{n,k}\|_{B_2} \leq 1 + C_3\|T\|,
\]
because $N_k(m) \geq N_k(n)$. For the third term we have
\[
\sum_{r=k+1}^{\infty} \frac{1}{r^2C_r(n)}\|T_{N_k(m)}g_{n,r}\|_{B_2} \leq \sum_{r=k+1}^{\infty} \frac{1}{r^2C_r(n)}\|T_{N_k(n)}\|\|g_{n,r}\|_{B_1}
\]
\[
\leq \sum_{r=k+1}^{\infty} \frac{1}{r^2} < \frac{1}{k} < 1,
\]
because $\|g_{n,r}\|_{B_1} = 1$ for all $r \geq n$, and $\|T_{N_k(n)}\| \leq C_r(n)$ for all $r > k$, which follows from $N_k(n) \leq N_r(n-1)$. Combining all partial results, it follows that
\[
\|T_{N_k(m)}\phi_n\|_{B_2} \leq 2(1 + C_3\|T\|) + 1 = C_4
\]
for all $m, n, k \in \mathbb{N}$, $m \neq n$.

The case $k = n$ is treated like the case $k > n$, except that now the first sum in (1) is not present.

Finally, by a simple scaling we can normalize all functions $\{\phi_n\}_{n \in \mathbb{N}}$ to one, which gives the desired functions $\{\varphi_n\}_{n \in \mathbb{N}}$ that have the properties (iii)–(v) of the lemma.

Now we are in the position to prove Theorem 2.

**Proof of Theorem 2.** We use the functions $\{\varphi_n\}_{n \in \mathbb{N}}$ from Lemma 3. The set $\{\varphi_n\}_{n \in \mathbb{N}}$ is finitely linearly independent. Let $D = \text{span}(\{\varphi_n\}_{n \in \mathbb{N}})$. Then $D$ is an infinite dimensional vector space in $B_1$. Let $f \in D$, $f \neq 0$, be arbitrary. There exists exactly one finite set $I \subset \mathbb{N}$ and complex numbers $\{a_k\}_{k \in I}$ such that $f = \sum_{k \in I} a_k \varphi_k$. Without loss of generality, we can assume that $a_k \neq 0$ for all $k \in I$. Then, for all $k_1 \in I$, we have according to Lemma 3 that
\[
\|T_{N_m(k_1)}f - a_{k_1}T_{N_m(k_1)}\varphi_{k_1}\|_{B_2} = \left\| \sum_{k \in I \setminus \{k_1\}} a_k T_{N_m(k_1)}\varphi_k \right\|_{B_2}
\]
\[
\leq C_2 \sum_{k \in I \setminus \{k_1\}} |a_k|.
\]
It follows that
\[ \|T_{N_m}(k_1)f\|_{B_2} \geq |a_{k_1}| \|T_{N_m}(k_1)\varphi_{k_1}\|_{B_2} - C_2 \sum_{k \in I \setminus \{k_1\}} |a_k|, \]
and consequently, using Lemma 3, that
\[ \limsup_{m \to \infty} \|T_{N_m}(k_1)f\|_{B_2} = \infty. \]

Since \( f \in D, f \neq 0 \), was arbitrary, we have proved the lineability of \( D_{UB} \).

In the following examples, where we use different possibilities for choosing the property \( P \), we illustrate the preceding results.

**Example 1.** Let \( B_1 \) and \( B_2 \) be two Banach spaces, \( T: B_1 \to B_2 \) be a bounded linear operator, and \( \{T_N\}_{N \in \mathbb{N}} \) a sequence of bounded linear operators mapping from \( B_1 \) into \( B_2 \) with the properties (i) and (ii) from Theorem 2. We say \( f \in B_1 \) has the property \( P_1 \) if
\[ \lim_{N \to \infty} \|T_Nf - Tf\|_{B_2} = 0 \]
holds. Then \( V_{P_1} \) is a linear subspace of \( B_1 \), and \( D_{P_1} = B_1 \setminus V_{P_1} \) is the set of all functions \( f \in B_1 \) with
\[ \limsup_{N \to \infty} \|T_Nf - Tf\|_{B_2} > 0, \]
i.e., the set of all functions \( f \in B_1 \) for which we do not have convergence. From Theorem 2 it follows that \( D_{P_1} \) is lineable, because \( D_{P_1} \supset D_{UB} \).

**Example 2.** Let the assumptions be the same as in Example 1. But now, we define a different property. We say \( f \in B_1 \) has the property \( P_2 \) if
\[ \limsup_{N \to \infty} \|T_Nf - Tf\|_{B_2} < \infty \]
holds. Then \( V_{P_2} \) has a linear structure, i.e., \( V_{P_2} \) is a subspace of \( B_1 \), and for all \( f \in D_{P_2} = B_1 \setminus V_{P_2} \) we have
\[ \limsup_{N \to \infty} \|T_Nf - Tf\|_{B_2} = \infty. \]
This is exactly the situation that was treated in Theorem 2, and therefore we know that \( D_{P_2} \) is lineable.
**Example 3.** Let $B_1$ and $B_2$ be two Banach spaces and $\{T_n\}_{n \in \mathbb{N}}$ a sequence of bounded linear operators mapping from $B_1$ into $B_2$. Further, let

$$\limsup_{N \to \infty} \|T_N\|_{B_1 \to B_2} = \infty.$$ 

We say $f \in B_1$ has the property $P_3$ if

$$\limsup_{N \to \infty} \|T_N f\|_{B_2} < \infty.$$ 

Then $V_{P_3}$ has a linear structure, i.e., $V_{P_3}$ is a subspace of $B_1$. According to the Banach–Steinhaus theorem, $D_{P_3} = B_1 \setminus V_{P_3}$ is a residual set. However, $D_{P_3}$ is not necessarily lineable. That is, we can have

$$D_{P_3} = \{ f \in V : f = g + d, g \in V_{P_3}, d \in U_{P_3}, d \neq 0 \}$$

with $U_{P_3}$ being a finite dimensional space. This situation is further illustrated in Example 4.

**Example 4.** Let $B_1 = B_2 = C$, where $C$ denotes the space of continuous $2\pi$-periodic functions, equipped with the maximum norm. Let

$$c_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} \, dt, \quad k \in \mathbb{Z},$$

denote the Fourier coefficients, and, for $N \in \mathbb{N}$, let

$$(T_N f)(t) = N c_0(f) + f(t) - c_0(f).$$

Then we have $\|T_N\|_{C \to C} \geq N$, which implies that there exists a residual set $D$ in $C$ such that

$$\limsup_{N \to \infty} \|T_N f\|_C = \infty$$

for all $f \in D$. We say $f \in C$ has the property $P_4$ if

$$\lim_{N \to \infty} \|T_N f - f\|_C = 0.$$ 

We have

$$V_{P_4} = \{ f \in C : \lim_{N \to \infty} \|T_N f - f\|_C = 0 \}$$

$$= \{ f \in C : c_0(f) = 0 \},$$

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and \( V_{P_4} \) is a closed subspace of \( C \). Further,

\[
U_{P_4} = \{ f \in C : f \equiv \lambda, \lambda \in \mathbb{C} \}
\]
is a one dimensional subspace of \( C \), and we have \( C = V_{P_4} \oplus U_{P_4} \). Hence, according to Theorem 1, \( D_{P_4} \) in this example is not lineable.

4. Spaceability

4.1. General Result

In the previous section we have seen examples where \( D_P \) is not lineable, and consequently also not spaceable. Further, we have given a complete characterization of lineability in Theorem 1. In the next theorem we will present a necessary and sufficient condition for the spaceability of the set \( D_P \).

**Theorem 3.** Let \( V_P \subset V \). The set \( D_P = V \setminus V_P \) is spaceable if and only if there exists a vector space \( U_P \subset V \) such that \( V = V_P \oplus U_P \) and \( U_P \) is spaceable.

**Remark 3.** Note that there is a difference between Theorem 1 and Theorem 3 concerning the subspace \( U_P \). We have seen in Theorem 1 that the space \( U_P \) plays a significant role for lineability. In general, for given \( V \) and \( V_P \), the subspace \( U_P \) is not uniquely defined by the requirement \( V = V_P \oplus U_P \). In Theorem 1 we are in the situation that if one choice of \( U_P \) is infinite dimensional then all other possible choices of \( U_P \) are infinite dimensional as well. However, the proof of Theorem 3 does not give the corresponding statement for the spaceability of the subspace \( U_P \). Thus, it is not clear if the spaceability of one choice of \( U_P \) implies the spaceability of all other choices of \( U_P \).

**Proof of Theorem 3.** “\( \Leftarrow \)” Let \( U_P \subset V \) a spaceable vector space such that \( V = V_P \oplus U_P \). We have \( U_P \subset D_P \cup \{0\} \). Since \( U_P \) is spaceable, \( D_P \) is spaceable.

“\( \Rightarrow \)” Since \( D_P \) is spaceable, there exists an infinite dimensional closed subspace \( \overline{U} \) of \( D_P \). We have \( \overline{U} \cap V_P = \{0\} \). Let \( L_{\overline{U}} \) be a Hamel basis of \( \overline{U} \), and let \( L_{V_P} \) be a Hamel basis of \( V_P \). Then \( L_{\overline{U}} \cup L_{V_P} \) is a Hamel basis for the vector space \( V_2 = V_P \oplus \overline{U} \). If \( V_2 = V \) then we can choose \( U_P = \overline{U} \), i.e., \( U_P \) is spaceable. Next,
we study the case $V_2 \neq V$, i.e., $V_2 \subset V$. Let $L_V$ be a Hamel basis of $V$, $L_{V_P}$ a Hamel basis of $V_P$ with $L_{V_P} \subset L_V$, and $L_U$ a Hamel basis of $U$ with $L_U \subset L_V$. Then we have $L_V \supseteq (L_{V_P} \cup L_U)$. Let $L_{U_2} = L_V \setminus (L_{V_P} \cup L_U)$. Then $L_{U_2}$ is a Hamel basis for a vector space $U_2$, and we have $V = V_2 \oplus U_2 = V_P \oplus U \oplus U_2$, where $L_U \cap L_{U_2} = \emptyset$ and $L_U \cup L_{U_2} = L_V \setminus L_{V_P}$. It follows that we can choose $U_P = V \setminus V_P = U \oplus U_2$. The proof is complete, because $U_P$ contains the infinite dimensional closed subspace $U$, i.e., $U_P$ is spaceable.

4.2. Spaceability of the Set of Functions with Divergent Approximation Processes

While Theorem 3 gives a necessary and sufficient condition for the spaceability of the complement $D_P$ of a subspace $V_P \subset V$, the next theorem gives a sufficient condition for the spaceability of the set of functions with divergent approximation process.

**Theorem 4.** Let $B_1$ and $B_2$ be two Banach spaces and $T: B_1 \to B_2$ a bounded linear operator. Further, let $\{T_N\}_{N \in \mathbb{N}}$ be a sequence of bounded linear operators, mapping from $B_1$ into $B_2$, with:

(i) $\limsup_{N \to \infty} \|T_N\|_{B_1 \to B_2} = \infty$,

(ii) there exists a dense subset $\mathcal{K}$ of $B_1$ such that $\lim_{N \to \infty} \|Tf - T_Nf\|_{B_2} = 0$ for all $f \in \mathcal{K}$, and

(iii) there exists an infinite dimensional closed subspace $B_1$ of $B_1$ such that $\sup_{N \in \mathbb{N}} \|Tf\|_{B_2} \leq C_5 \|f\|_{B_1}$ for all $f \in B_1$.

Then, the set

$$D_{UB} = \left\{ f \in B_1 : \limsup_{N \to \infty} \|T_Nf\|_{B_2} = \infty \right\}$$

is spaceable.

**Proof of Theorem 4.** For the proof we use the basic fact that every infinite dimensional Banach space contains an infinite dimensional closed subspace with a Schauder basis [13, p. 4]. We apply this result to $B_1$. It follows that there exists
an infinite dimensional closed subspace $\mathcal{B}_1$ of $\mathcal{B}_1$, and functions $\{\phi_n\}_{n \in \mathbb{N}} \subset \mathcal{B}_1^*$, as well as continuous linear functionals $\{\Phi_n\}_{n \in \mathbb{N}} \subset \mathcal{B}_1^*$, such that

$$f = \sum_{n=1}^{\infty} \Phi_n(f)\phi_n$$

for all $f \in \mathcal{B}_1$, where the series converges in the $\mathcal{B}_1$-norm. We have

$$\sup_{N \in \mathbb{N}} \sup_{\|f\|_{\mathcal{B}_1} \leq 1} \left\| \sum_{n=1}^{N} \Phi_n(f)\phi_n \right\|_{\mathcal{B}_1} = C_6 < \infty$$

and

$$\|\Phi_n\|_{\mathcal{B}_1^*} \|\phi_n\|_{\mathcal{B}_1^*} \leq 2C_6$$

for all $n \in \mathbb{N}$. According to the Hahn–Banach theorem, the coefficient functionals $\{\Phi_n\}_{n \in \mathbb{N}}$ can be extended to continuous linear functionals $\{\Phi_n^{ex}\}_{n \in \mathbb{N}}$ defined on $\mathcal{B}_1$ which satisfy $\|\Phi_n\|_{\mathcal{B}_1^*} = \|\Phi_n^{ex}\|_{\mathcal{B}_1^*}, n \in \mathbb{N}$.

Let $q_n = \max\{1, \|\Phi_n^{ex}\|_{\mathcal{B}_1^*}\}$, and consider the functions

$$h_n = \phi_n + \frac{1}{2n+1}\varphi_n, \quad n \in \mathbb{N},$$

where $\{\varphi_n\}_{n \in \mathbb{N}}$ are the functions from Lemma 3. We have

$$\sum_{n=1}^{\infty} \|\Phi_n^{ex}\|_{\mathcal{B}_1^*} \|\phi_n - h_n\|_{\mathcal{B}_1} = \sum_{n=1}^{\infty} \|\Phi_n^{ex}\|_{\mathcal{B}_1^*} \left\| \frac{1}{2n+1}q_n \varphi_n \right\|_{\mathcal{B}_1}$$

$$= \sum_{n=1}^{\infty} \|\Phi_n^{ex}\|_{\mathcal{B}_1^*} \frac{1}{2n+1}q_n$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{2n+1} = \frac{1}{2} < 1.$$ 

It follows that $\{h_n\}_{n \in \mathbb{N}}$ is a basic sequence in $\mathcal{B}_1$ that is equivalent to $\{\phi_n\}_{n \in \mathbb{N}}$ [9, p. 46]. Thus, for a sequence $\{\alpha_n\}_{n \in \mathbb{N}}$ of complex numbers, $\sum_{n=1}^{\infty} \alpha_n h_n$ converges if and only if $\sum_{n=1}^{\infty} \alpha_n \phi_n$ converges. Let

$$D_1 = \left\{ f \in \mathcal{B}_1 : \exists \{\alpha_n\}_{n \in \mathbb{N}} \subset \mathbb{C} \text{ with } \lim_{N \to \infty} \left\| f - \sum_{n=1}^{N} \alpha_n h_n \right\|_{\mathcal{B}_1} = 0 \right\}.$$
Every \( f \in D_1 \) is uniquely determined by the corresponding sequence \( \{ \alpha_n \}_{n \in \mathbb{N}} \), and \( D_1 \) is a closed subspace of \( B_1 \). For \( f \in D_1 \) we have

\[
f = \sum_{n=1}^{\infty} \alpha_n h_n = \sum_{n=1}^{\infty} \alpha_n \phi_n + \sum_{n=1}^{\infty} \frac{1}{2^n + 1} q_n \varphi_n.
\]

(2)

The convergence of

\[
\sum_{n=1}^{\infty} \alpha_n \frac{1}{2^n + 1} q_n \varphi_n
\]

in the \( B_1 \)-norm follows from

\[
\sum_{n=1}^{N} \alpha_n h_n - \sum_{n=1}^{N} \alpha_n \phi_n = \sum_{n=1}^{N} \alpha_n \frac{1}{2^n + 1} q_n \varphi_n
\]

for all \( N \in \mathbb{N} \), and the convergence of both sums on the left hand side of the equation as \( N \) tends to infinity. Note that \( g \in B_1 \). Using (2) we obtain for \( N \in \mathbb{N} \) that

\[
T_N f = T_N g + T_N v
\]

and

\[
\|T_N f - T_N v\|_{B_2} = \|T_N g\|_{B_2} \leq C_5 \|g\|_{B_1},
\]

(3)

where the last inequality follows from assumption (iii) and \( B_1 \subseteq B_1 \). Let \( n_0 \) be the smallest index such that \( \alpha_{n_0} \neq 0 \). Then we have

\[
T_N v = \frac{\alpha_{n_0}}{2^{n_0 + 1} q_{n_0}} T_N \varphi_{n_0} + \sum_{n=n_0+1}^{\infty} \frac{\alpha_n}{2^{n+1} q_n} T_N \varphi_n.
\]

(4)

Since \( g = \sum_{n=1}^{\infty} \alpha_n \phi_n \) and \( \{ \phi_n \}_{n \in \mathbb{N}} \) is a basis of \( B_1 \), we have

\[
|\alpha_n| = |\Phi_n(g)| \leq \|\Phi_n\|_{B_1^*} \|g\|_{B_1} \leq q_n \|g\|_{B_1}.
\]
Hence, using Lemma 3, it follows that

\[
\left\| \sum_{n=n_0+1}^{\infty} \frac{\alpha_n}{2^{n+1}q_n} T_{N_k^{n_0}} \varphi_n \right\|_{B_2} \leq \sum_{n=n_0+1}^{\infty} \frac{|\alpha_n|}{2^{n+1}q_n} \left\| T_{N_k^{n_0}} \varphi_n \right\|_{B_2}
\]

\[
\leq \sum_{n=n_0+1}^{\infty} \frac{|g|_B}{2^{n+1}} \left\| T_{N_k^{n_0}} \varphi_n \right\|_{B_2}
\]

\[
\leq \sum_{n=n_0+1}^{\infty} \frac{|g|_B}{2^{n+1}} C_2 \left\| \varphi_n \right\|_{B_1}
\]

\[
= \frac{g|_B}{2} C_2 \sum_{n=n_0+1}^{\infty} \frac{1}{2^{n+1}}
\]

\[
\leq \frac{C_2}{2} \left\| g \right\|_{B_1}
\]

for all \( k \in \mathbb{N} \). From (4) and (5) we see that

\[
\left\| T_{N_k^{n_0}} v - \frac{\alpha_{n_0}}{2^{n_0+1}q_{n_0}} T_{N_k^{n_0}} \varphi_{n_0} \right\|_{B_2} \leq \frac{C_2}{2} \left\| g \right\|_{B_1}
\]

for all \( k \in \mathbb{N} \). Combining (3) and (6), it follows that

\[
\left\| T_{N_k^{n_0}} f \right\|_{B_2} \geq \frac{|\alpha_{n_0}|}{2^{n_0+1}q_{n_0}} \left\| T_{N_k^{n_0}} \varphi_{n_0} \right\|_{B_2} - C_5 \left\| g \right\|_{B_1} - \frac{C_2}{2} \left\| g \right\|_{B_1}
\]

for all \( k \in \mathbb{N} \), which implies that

\[
\limsup_{k \to \infty} \left\| T_{N_k^{n_0}} f \right\|_{B_2} = \infty.
\]

Since \( f \in D_1, f \neq 0 \), was chosen arbitrarily, we have proved spaceability, because \( D_1 \) is a infinite dimensional closed subspace of \( B_1 \).

5. Further Applications

In this section we will give further applications for the preceding theory.

Let \( \mathcal{D} \) denote the open unit disk, \( \overline{\mathcal{D}} \) the closure of \( \mathcal{D} \), and \( \partial \mathcal{D} = \overline{\mathcal{D}} \setminus \mathcal{D} \) the boundary of \( \mathcal{D} \). Further, let \( A(\mathcal{D}) = \{ f : f \text{ is analytic in } \mathcal{D} \text{ and continuous on } \overline{\mathcal{D}} \} \) denote the disk algebra, equipped with the norm \( \| f \|_{A(\mathcal{D})} = \max_{|z| \leq 1} |f(z)| \).
Corollary 2. Let \( \{ A_N \}_{N \in \mathbb{N}} \) be a sequence of bounded linear operators mapping from \( A(D) \) into \( A(D) \) with the properties:

(i) for every \( N \in \mathbb{N} \) there exist \( M \) pairwise distinct points

\[
I_N = \{ e^{i\Theta_1,N}, \ldots, e^{i\Theta_M,N} \}
\]

in \( \partial D \) such that if \( f_1(e^{i\Theta_k,N}) = f_2(e^{i\Theta_k,N}) \) for all \( 1 \leq k \leq M \) then we have \( A_N f_1 = A_N f_2 \), and

(ii) there exists a dense subset \( K \) of \( A(D) \) such that \( \lim_{N \to \infty} \| f - A_N f \|_{A(D)} = 0 \) for all \( f \in K \).

Then the set

\[
D(\{ A_N \}) = \left\{ f \in A(D) : \limsup_{N \to \infty} \| A_N f \|_{A(D)} = \infty \right\}
\]

is lineable.

Proof. Corollary 2 is a direct consequence of Theorem 2. From property (i) and (ii) it follows that \( \limsup_{N \to \infty} \| A_N \|_{A(D) \to A(D)} = \infty \) [14, 6]. Hence, we can apply Theorem 2. \( \square \)

Next we discuss the Lagrange interpolation, which can be seen as a special case of the setting in Corollary 2. For \( N \in \mathbb{N} \), let \( I_N = \{ e^{i\Theta_1,N}, \ldots, e^{i\Theta_N,N} \} \) be \( N \) pairwise distinct points in \( \partial D \). Further, let \( \mathcal{I} = \{ I_N \}_{N \in \mathbb{N}} \) denote the system of interpolation nodes, and let

\[
(L_N^T f)(z) = \sum_{n=1}^{N} f(e^{i\Theta_n,N})w_{n,N}(z), \quad z \in \mathbb{C},
\]

be the corresponding Lagrange interpolation, where

\[
w_{n,N}(z) = \frac{\phi_N(z)}{\phi_N'(e^{i\Theta_n,N})(z - e^{i\Theta_n,N})}
\]

with

\[
\phi_N(z) = \prod_{n=1}^{N} (z - e^{i\Theta_n,N}).
\]
Corollary 3. Let \( q_k(z) = z^{2^k}, \) \( z \in C, \ k \in \mathbb{N}. \) If the set \( \mathcal{I} \) of interpolation nodes satisfies
\[
\sup_{N \in \mathbb{N}} \sup_{k \in \mathbb{N}} \| L_N^T q_k \|_\infty < \infty
\]
then the set
\[
D(\{ L_N^T \}_{N \in \mathbb{N}}) = \left\{ f \in A(\mathcal{D}) : \limsup_{N \to \infty} \| L_N^T f \|_{A(\mathcal{D})} = \infty \right\}
\]
is spaceable.

Proof. Since the approximation process \( \{ L_N^T \}_{N \in \mathbb{N}} \) is a special instance of the approximation process \( A_N \) in Corollary 2, we have \( \limsup_{N \to \infty} \| L_N^T \|_{A(\mathcal{D}) \to A(\mathcal{D})} = \infty. \) Clearly, the set of polynomials is a dense subset of \( A(\mathcal{D}) \) for which we have \( \lim_{N \to \infty} \| f - L_N^T f \|_{A(\mathcal{D})} = 0. \) If we can show that
\[
\sup_{N \in \mathbb{N}} \| L_N^T f \|_{A(\mathcal{D})} \leq C_7 \| f \|_{A(\mathcal{D})}
\]
for all \( f \) in some infinite dimensional closed subspace \( B_1 \subset A(\mathcal{D}), \) then the assertion follows from Theorem 4.

We prove (8) next. Let
\[
c_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\Theta}) e^{-ik\Theta} \, d\Theta, \quad k \in \mathbb{Z}.
\]
Further, let \( B_1 \) denote the set of all \( f \in A(\mathcal{D}), \) satisfying \( c_k(f) = 0 \) for all \( k \in \{ n \in \mathbb{N} : n \neq 2^l \ \text{for some} \ l \in \mathbb{N} \}. \) Then \( B_1 \) is an infinite dimensional closed subspace of \( A(\mathcal{D}). \) For \( f \in B_1 \) we have
\[
C_8 \sum_{k=0}^{\infty} |c_{2^k}(f)| \leq \| f \|_{A(\mathcal{D})} \leq \sum_{k=0}^{\infty} |c_{2^k}(f)|,
\]
where \( C_8 \) is some positive constant [17]. Let \( \epsilon > 0 \) be arbitrary. Then there exists a natural number \( n_0 = n_0(\epsilon) \) such that
\[
\sum_{k=n_0+1}^{\infty} |c_{2^k}(f)| < \epsilon
\]
and, such that for
\[
f_\epsilon(e^{it}) = \sum_{k=0}^{n_0} c_{2^k}(f) e^{2^k t}
\]
we have
\[ \| f - f_\epsilon \|_{A(D)} < \epsilon. \]

Thus, for \( N > 2^{n_0} \), we have
\[ \| f - L_N^T f \|_{A(D)} = \| f - f_\epsilon + f_\epsilon - L_N^T f \|_{A(D)} \]
\[ = \| f - f_\epsilon + L_N^T f_\epsilon - L_N^T f \|_{A(D)} \]
\[ \leq \| f - f_\epsilon \|_{A(D)} + \| L_N^T (f_\epsilon - f) \|_{A(D)}. \]

Further, we have
\[ \| L_N^T (f - f_\epsilon) \|_{A(D)} = \left\| \sum_{k=0}^{\infty} c_{2^k} (f) L_N^T q_k \right\|_{A(D)} \]
\[ \leq \sum_{k=n_0+1}^{\infty} |c_{2^k} (f)| C(I) \]
\[ \leq \epsilon C(I), \]

where we used (9), and where \( C(I) = \sup_{N \in \mathbb{N}} \sup_{k \in \mathbb{N}} \| L_N^T q_k \|_\infty < \infty \), because of (7). It follows that
\[ \| f - L_N^T f \|_{A(D)} \leq \epsilon (1 + C(I)) \]
for all \( N > 2^{n_0} \). Since \( \epsilon > 0 \) was arbitrary, we have
\[ \lim_{N \to \infty} \| f - L_N^T f \|_{A(D)} = 0 \]
for all \( f \in \mathcal{B}_1 \). It follows from the Banach–Steinhaus theorem that
\[ \sup_{N \in \mathbb{N}} \sup_{\| f \|_{A(D)} \leq 1} \| L_N^T f \|_{A(D)} < \infty, \]

which completes the proof. \( \square \)
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