Mathematics of Signal Design for Communication Systems

Holger Boche and Ezra Tampubolon

Abstract. Orthogonal transmission schemes constitute the foundations of both our present - , and future communication standards. One of the major drawbacks of orthogonal transmission schemes is their high dynamical behaviour, which can be measured by the so-called Peak-to-Average power value – the ratio between the peak value (i.e. $L^\infty$-norm) and the average power (i.e. $L^2$-norm) of a signal. This undesired behaviour of orthogonal schemes has remarkable negative impacts to the performance - , the energy-efficiency - , and the maintain cost of the transmission systems. In this work, we give some discussions concerning to the problem of reduction of the high dynamics of an orthogonal transmission scheme. We show that this problem is connected with some mathematical fields, such as functional analysis (Hahn-Banach Theorem and Baire Category), additive combinatorics (Szeméredi Theorem, Green-Tao Theorem on arithmetic progressions in the primes, sparse Szeméredi type Theorems, by Conlon and Gowers, and the famous Erdős problem on arithmetic progressions), and both trigonometric - and non-trigonometric harmonic analysis.

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1. Introduction

The rapid development of technologies and the astronomic growth in data usage over the past two decades are inter alia the driving force for the development of flexible and efficient transmission technologies. The latter can certainly not be imagined without the development of the orthogonal transmission scheme, which can roughly be described as the techniques with which several data can be transmitted instantaneously orthogonally (orthonormally) in one single shot within a given time frame. Specifically, given a duration $T_s > 0$ of a transmit signal, and given a finite transmit data $\{a_k\}_{k=1}^{N}$, which constitute simply a sequence in $\mathbb{C}$. The transmit signal of an orthogonal transmission scheme has the form:

$$s(t) = \sum_{k=1}^{N} a_k \phi_k(t), \quad t \in [0, T_s],$$

(1)
where \( \{ \phi_n \}_{n=1}^N \) constitutes an orthonormal system (ONS), in the space of square integrable functions on \([0,T_s]\). Each function in the collection \( \{ \phi_n \}_{n=1}^N \) is also referred as wave function, and the expression (1) as waveform. In the literature on wireless communications, each \( \{ \phi_n \}_{n=1}^N \) is also, according to its purpose - to carry the information-bearing coefficients \( \{a_n\}_{n=1}^N \), referred as carrier. In the context of communications engineering, the way to process the information-bearing data by means of functions (in the space of square integrable functions on \([0,T_s]\)) generally, viz. the functions do not necessarily form an ONS, in the manner (1) for transmission purposes is also called information modulation. A quite popular choices of the wave functions in communications engineering are \( \phi_n(\cdot) = e^{i2\pi(n-1)\cdot} \), and the Walsh functions (see Def. 6.1). The former case is referred by Fourier/OFDM case, and the latter by Walsh/CDMA case. The origin of the terms OFDM and CDMA shall be introduced in the next section. To give a better understanding, the block diagram of the information modulation unit of the orthogonal transmission scheme for the OFDM case is provided in Fig. 1. The coefficients \( \{a_n\}_{n=1}^N \) in (1)

\[
\sum_{n=1}^{N} a_n \phi_n(\cdot)
\]

might for instance be information sequences, which are available a fore serially in time, or for each \( k \in \{1, \ldots, N\} \), \( a_k \) corresponds to one of the information symbol of a user. For convenience, we mostly consider the time duration \( T_s = 1 \), since any other cases can be derived by simple rescaling.

In this work, we are mainly interested in the dynamical behaviour of the waveforms formed by orthonormal systems, which is measured by the ratio between the peak-value - and the average power, called peak-to-average-power ratio (PAPR). It is well-known, both theoretically and practically, that the waveforms of orthogonal transmission schemes possess high dynamical behaviour. A more detailed discussions on this aspect shall be given informally in the next section and formally in Subsec. 4.1. The so-called tone reservation method \([31, 32, 33]\) is without doubt one of the canonical ways to reduce the PAPR value of a waveform. There, the (indexes of) available wave functions (\([N]\), \( N \in \mathbb{N} \)) are separated into two (fixed) subsets, one which is reserved for those, which carry the information data (call \( I \)), called information set, and another which is reserved for those, which
should, by means of choices of the coefficients, reduce/compensate the peak value of the resulted waveforms, s.t. it is uniformly below a certain threshold constant $C_{Ex} > 0$, called *extension constant*. A more formal introduction of this method shall be given in Subsec. 4.3. We refer the applicability of tone reservation method with extension constant $C_{Ex}$ simply by the *solvability of the PAPR problem with extension constant $C_{Ex}$* (see Def. 4.3).

Further, in this work, we mostly keep the option open, that the information data and the wave functions, used for the compensation of the peak value, are of infinite number. We shall give in Subsec. 4.4 a discussion on the optimal extension constant, with which the PAPR reduction problem is solvable. Furthermore, we shall see in Subsec. 4.5 that the solvability of the PAPR problem is connected to an embedding problem of a certain closed subspace of $L^1([0,1])$ (Def. 4.8) into $L^2([0,1])$ (Thm. 4.10 and Prop. 4.13). In turn, that fact shall give a relation between the non-solvability of the PAPR problem for a given extension constant, and the existence of certain combinatorial objects in the information set. We will observe, that in the OFDM case, the corresponding object is the so-called arithmetic progression (Def. 5.1), which leads us to involve the famous Szemerédi Thm. on arithmetic progressions (see Thm. 5.3 and Thm. 5.6), and some asymptotic -, and infinite tightening due to Green and Tao, and Conlon and Gowers (see Thm. 5.4 and Thm. 5.7). Further, it shall be obvious, the corresponding combinatorial object in the CDMA case is the subset, which indexed a so-called perfect Walsh sum (Def. 6.7). We will give a condition on the size of the information set, s.t. such a combinatorial object exists therein (Thm. 6.10). By means of the former, we are able to derive an asymptotic statement concerning to the existence of that combinatorial object (Thm. 6.12). Those results give in turn some statements concerning to the non-solvability of the PAPR problem, both in the finite - (Thm. 6.2), asymptotic - (Thm. 6.14, Thm. 6.15)), and infinite case (Thm. 6.17).

### 2. Motivation of the PAPR Reduction Problems

In this work, we consider two transmission schemes, which use basically the previous mentioned idea, namely the *orthogonal frequency division multiplexing (OFDM)* and the *direct sequence code division multiple access (DS-CDMA)*.

The OFDM constitutes a transmission scheme dominating, both the present, and future communication systems. It has become an important part of various, both wireless and wire line, current -, and future standards, such as DSL, IEEE 802.11, DVB-T, LTE, and LTE-advanced/4G, and 5G. The wave functions of OFDM are basically complex sines of the form $\phi_n(\cdot) = \exp(2\pi i (n-1)(\cdot))$, $n \in \mathbb{N}$. One of the reasons for the attractiveness of OFDM for wireless applications is its robustness against multipath fading, which constitute major characteristic of wireless transmission channel. Roughly, the multipath fading of the signal transmitted through a wireless channel is caused by the fact, that the signal will reach the receiver not only via the direct path, but also as a result of reflections from non-moving objects such as buildings, hills, ground, water, and moving objects such
as automobiles and people, that are adjacent to the main direct path. Furthermore, the implementation of OFDM waveforms, and both the equalization and the decoding of the information contained in a received OFDM waveform, are fairly easy, since they all require only fast Fourier transform (FFT), inverse fast Fourier transform (IFFT), and simple multiplications. The latter is justified by the fact that the wave functions of OFDM are the eigenfunctions of a linear-time-invariant channel, which give a mathematical model of a (wireless) transmission channel. Notice that there are some minor technicalities, such as cyclic prefix insertion, and synchronization, to be considered. Also, making use of those techniques, OFDM communication scheme is proven to be robust against intersymbol interference (ISI), i.e. interference between consecutive waveforms/symbols. The occurrence of ISI in a communication system might impact the reliability of that system negatively. For details on previous mentioned aspects, we refer to standard textbooks on wireless communications. A rough sketch of OFDM transmission scheme, and the corresponding signal processing at the receiver is given by the block diagram depicted in Fig. 2.

![Block diagram of OFDM transmission scheme](image)

Figure 2. The block diagram of orthogonal transmission scheme and the corresponding signal processing at the receiver for multipath propagation channel. The sequence \((\hat{a}_1, \ldots, \hat{a}_K)\) denotes the estimates of the transmitted symbols \((a_1, \ldots, a_K)\).

DS-CDMA is certainly also a transmission scheme, which plays an important role in numerous present communications. This scheme has become an indispensable part of several communication standards, such as 3G, UMTS, GPS, and Galileo. DS-CDMA is used mainly for uplink communications, i.e. from users to a base station. There, a user \(n\) obtains an individual signature pulse \(\phi_n\), which basically a train of rectangular pulses. All of those pulses have to satisfy the orthonormality condition, which enables the base station receiver to separate each of the users separately. The principle mentioned previously works also for the down-
link communication between base station and several users. The combined signal (1) is noise alike, so that a potential jamming is aggravated to intercept a certain user. To overcome the effect of multipath propagation, one may use the so-called RAKE-receiver, with which the multipath components can even be detected. A detailed treatment about those aspects is without scope of this work. Thus, we refer to standard textbooks on wireless communications.

In this work we are rather interested in the so-called Peak-to-Average-Power-Ratio (PAPR) behavior of orthogonal transmission schemes, than in the technical implementations of that scheme. The PAPR of a waveform (or more generally: a signal) gives a proportion between the peak value of a waveform, which is measured by the essential supremum, and its energy, which is measured by the $L^2$-norm of the waveform. In this case, the energy can also be interpreted as the average (quadratic) value of the waveform. Thus, the PAPR of a waveform gives an insight into the behaviour of its peaks. In particular, a high PAPR value indicates the existence of extreme peaks in the considered waveform. Thus in some literature, PAPR is denoted more vividly by crest factor. High PAPR value might have negative impacts to the reliability of a transmission scheme: Commonly, a transmission signal is amplified, before sent through a channel (see Fig. 3). However, every (non-ideal) amplifier in practice has a certain magnitude threshold $M \in \mathbb{R}^+$, beyond which the input signal is not linearly amplified, but distorted or clipped. By clipping, we mean the operation, which leaves the part of the signal undisturbed, if its magnitude is below $M$, and which sets the magnitude of another part of the signal to the value $M$, if its magnitude is over $M$, while the phase is remained unchanged. For a depiction of the occurrence of clipping caused by a non-ideal amplifier, see Fig. 4. Thus in case that an orthogonal transmission scheme having waveforms with high PAPR value, and an amplifier with low threshold value is used, then clipping might occur, which results in the alteration of the waveforms, and in particular in the destruction of the desired structure of the waveforms – the orthonormality of the wave functions. Another negative effects of clipping is for instance the occurrence of out-of-band radiation. For a more detailed discussions on those aspects, we refer to standard textbooks on wireless communications.

A naive solution to that problem is to use another amplifier with a higher threshold value. However, this is practically not the best solution, since an ampli-
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Figure 4. Impact of a realistic non-ideal amplifier to a waveform to be transmitted (clipping)

A high linear range is expensive, not only to purchase, but also to maintain. Such amplifiers would in general have inefficiently high power consumption, and accordingly, require batteries with high capacity and long lifetime. Energy efficiency of a communication system, which is partially reflected in the low power consumption, is without doubt an important issue both for the present and future. The importance of this issue can be seen in the reports [3, 22] of consulting firms, which estimates, that 2% of global CO₂ emissions are attributable to the use of information and communication technology, which is comparable to the CO₂ emissions due to avionic activities. Energy efficiency of communications systems is not only of environmental interests, but also of financial interests: Nowadays, energy cost of network operation can even make up to 50% of the total operational cost.

An orthogonal transmission scheme tends to possess waveforms with high PAPR value. This disadvantage might be caused by the fact that such waveforms are generated by a superposition of large numbers of wave functions. For instance, there are up to 2048 wave functions for the downlink communication in the LTE standard [1], which uses OFDM as a basic transmission scheme. There are several methods proposed for reducing the PAPR [17, 14]. However, the so-called tone reservation method [31, 32, 33] is without doubt a PAPR reduction method, which might have the potential to be popular, since it is canonical and robust, in the way that the only information required on the receiver’s side is the indexes of the information-bearing coefficients. To understand this, let us first describe the tone reservation method. There, the (indexes of) available carriers ([N], N ∈ ℕ) are separated into (fixed) two subsets, one which is reserved for those, which carry the information data, and one which is reserved for those, which should, by means of choices of the coefficients, reduce/compensate the peak value of the resulted signal. A more formal description of the task shall be given in Def. 4.3. The auxiliary coefficients may simply be ignored by the receiver. Thus there is no need for additional overhead in the transmission symbols. In the practical scenario, some further requirements for the compensation set have to be considered. Therefore, the compensation set can also be used for channel estimation purposes [34].
At last, for some further discussions on PAPR, we recommend the recent comprehensive overview article [36], which gives for instance a discussion on alternative metrics beyond the PAPR, which are relevant to the behaviour of the energy consumption of a transmission system, and new mathematical concepts aiming to overcome the high PAPR behavior of an orthogonal transmission scheme.

3. Basic Notions and - Notations

For $N \in \mathbb{N}$, we denote the set $\{1, 2, \ldots, N\}$ simply by $[N]$. Let $\mathcal{I} \subset \mathbb{N}$ be an index set, we denote the space of square-summable sequences in $\mathbb{C}$ indexed by $\mathcal{I}$ by $l^2(\mathcal{I})$. For ease of notations, we denote sequences $\{c_n\}_{n \in \mathcal{I}}$ in $\mathbb{C}$, simply by bold letters $\mathbf{c}$, and vice versa.

Operators between Banach Spaces. We call a mapping between vector spaces as an Operator. For an operator $\Phi : X \rightarrow Y$ between Banach spaces, we define the norm by:

$$\|\Phi\|_{X \rightarrow Y} := \sup_{\|x\|_X \leq 1} \|\Phi x\|_Y.$$ 

Clearly, in case $\Phi$ is linear, we can write:

$$\|\Phi\|_{X \rightarrow Y} := \sup_{x \in X, x \neq 0} \frac{\|\Phi x\|_Y}{\|x\|_X} = \sup_{\|x\|_X = 1} \|\Phi x\|_Y.$$ 

Lebesgue Spaces. For $T > 0$, we denote the Lebesgue space (Banach space) of $p$-integrable functions on $[0, T]$ by $L^p([0, T])$. As usual, the Lebesgue space has to be understand as equivalence classes of functions, which differ in a set of (Lebesgue-)measure zero, i.e. almost everywhere (a.e.). $L^p([0, T])$ is as usual equipped with the norm $\|f\|_{L^p([0, T])} := [(1/T)\int_0^T |f(t)|^p]^{1/p}$, in case $p \in [1, \infty)$, and in case $p = \infty$, the norm is defined by $\|f\|_{L^\infty([0, 1])} = \text{ess sup}_{t \in [0, T]} |f(t)|$. As already mentioned in the introduction, it is sufficient only to consider $T = 1$, since any other cases can be treated by means of simple rescaling.

Given a sequence $\{\phi_n\}_{n \in \mathcal{I}}$, where $\mathcal{I} \subset \mathbb{N}$ is an index set, of functions in $L^2([0, 1])$. $\{\phi_n\}_{n \in \mathcal{I}}$ is said to be an orthonormal system (ONS) in $L^2([0, 1])$, if $\int_0^1 \phi_k(t)\phi_l(t)dt = 0$, if $k \neq l$, and 1, if $k = l$, where $k, l \in \mathcal{I}$. A collection of functions $\{\phi_n\}_{n \in \mathcal{I}}$ in $L^2([0, 1])$ is said to be a complete orthonormal system (CONS) for $L^2([0, 1])$, if $\{\phi_n\}_{n \in \mathcal{I}}$ is an ONS in $L^2([0, 1])$, and if every function $f \in L^2([0, 1])$ can be represented as the sum $f = \sum_{k \in \mathcal{I}} c_k \phi_k$, where $c \in l^2(\mathcal{I})$ is given by $c_n = \int_0^1 f(t)\overline{\phi_n(t)}dt$, $\forall n \in \mathcal{I}$. The convergence of the sum has to be interpreted in the sense of $L^2([0, 1])$, i.e. w.r.t. $\|\cdot\|_{L^2([0, 1])}$.

4. PAPR Problem for Orthogonal Transmission Schemes
4.1. Basic Bounds for PAPR of Orthogonal Transmission Schemes. Given a signal \( f \in L^2([0,1]) \). The Peak-to-Power-Average-Ratio of \( f \) is defined generally as the ratio between peak value and the energy of the signal \( f \):

\[
PAPR(f) := \frac{\| f \|_{L^\infty([0,1])}}{\| f \|_{L^2([0,1])}}.
\]

Since we only consider ourselves with signals, which are basically the linear combination of orthonormal functions, we emphasize the following definition of PAPR for the corresponding subclass of \( L^2([0,1]) \). Further, we allows in the definition that the considered signal is generated by infinite numbers of orthonormal functions.

**Definition 4.1.** Let \( \{ \phi_k \}_{k \in \mathbb{K}} \) be a set of orthonormal functions, where \( \mathbb{K} \subset \mathbb{N} \). We define the Peak-to-Average Power Ratio (PAPR) of a set of coefficients \( a \in \mathbb{C}^\mathbb{K} \), \( a \neq 0 \) (w.r.t. \( \{ \phi_k \}_{k \in \mathbb{K}} \)) as the quantity:

\[
PAPR(\{ \phi_k \}_{k \in \mathbb{K}}, a) = \text{ess sup} \left\{ \frac{\sum_{k \in \mathbb{K}} a_k \phi_k(t)}{\|a\|_{l^2(\mathbb{N})}} \right\}_{t \in [0,1]}
\]

By the orthonormality of \( \{ \phi_k \} \), it is obvious that the PAPR of a sequence \( a \in l^2(\mathbb{K}) \) is equal to the PAPR of the signal \( s \in L^2([0,1]) \), given by \( s = \sum_{k \in \mathbb{K}} a_k \phi_k \).

The behaviour of the peak value of orthonormal systems might, as already discussed informally in the introduction, be worse. Indeed, one can show \([4, 7]\), that for finite number of orthonormal functions, the following worst-case behaviour holds:

\[
\sqrt{N} \leq \sup_{\|a\|_{l^2(\mathbb{N})} \leq 1} \text{PAPR}(\{ \phi_k \}_{k \in [N]} ; a).
\] (2)

Furthermore, a corresponding sequence \( a \), with \( \|a\|_{l^2([N])} = 1 \), for which the above inequality holds, can easily be constructed.

Such a behavior is of course not tolerable, since the waveforms of an orthogonal transmission scheme typically consists a large number of wave functions. One of the canonical and robust way to reduce the PAPR of a signal is the so-called tone reservation method. We will formalize the method and discuss it soon. But, let us first give some remarks concerning to the PAPR behaviour of an ONS in the following subsection.

Furthermore, it is also interesting to ask whether such a bad behaviour can occur for orthonormal single-carrier systems, for instance the systems with which the information-bearing signals are carried by shifted kernels. For single-carrier systems generated by \( N \) number of mutually distinct sinc-kernels, it was recently shown in \([5]\) (Thm. 2.1), that in case that the information coefficients are chosen i.i.d. by Gaussian normal distribution with zero expectation, and a given variance, then the expected value of the PAPR of resulted single-carrier signal is comparable with \( \sqrt{\log N} \). By some additional requirements on the information coefficients, one can even have the result, that expectation the PAPR of the resulted signal is comparably with \( \log \log N \). Those statements might asserts, that the PAPR behaviour of single-carrier systems (compared to the multi-carrier orthogonal transmission systems) is fairly good.
4.2. General Remarks on Coefficients of ONS. For ONS in $L^2([0,1])$, we have already mentioned in the previous subsection, that PAPR of some non-zero coefficients in $l^2(N)$ are bigger than $\sqrt{N}$, which is surely not tolerable. This asserts that some efforts have to be done to prevent such an undesired behaviour. From the mathematical point of view, the problem does not look so helpless, since by the Nazarov’s solution of the coefficients problem [19], we have the following statement:

**Theorem 4.2** (Nazarov [19]). Let $\{\phi_n\}_{n \in \mathbb{N}}$ be an ONB, for which $\|\phi_n\|_{L^1([0,1])} \geq C_1$, $\forall n \in \mathbb{N}$, for some constants $C_1 > 0$. Then there exists a constant $C > 0$, such that for every coefficients $a \in l^2(N)$, there exists a function $f_* \in L^\infty([0,1])$, with:

1. $\|f_*\|_{L^\infty([0,1])} \leq C \|a\|_{l^2(N)}$.
2. $\left\| \int_0^1 f_*(t) \overline{\phi_n(t)} dt \right\| \geq |a_n|$, $n \in \mathbb{N}$.

Thus, in case that the considered ONS is in addition complete, one can construct for arbitrary $a \in l^2(N)$, another sequence $b \in l^2(N)$, with $|b_n| \geq |a_n|$, $\forall n \in \mathbb{N}$, by means of suitable enlargement of the absolute value - and phase change of each of its members, such that another waveform $f_*$ is obtained, whose peak value can be controlled by a certain factor $C > 0$, depending only of the choice of ONS $\{\phi_n\}_{n \in \mathbb{N}}$:

$$f_* := \sum_{k=1}^{\infty} b_n \phi_n \quad \text{and} \quad \|f_*\|_{L^\infty([0,1])} \leq C \|a\|_{l^2([0,1])}.$$

However, this result is not applicable for communication systems, since the inverse transformation of the information-bearing coefficients $a$ from the signal formed by $b$ after a transmission for instance through a noisy channel is generally not possible.

For other applications in electrical engineering, Nazarov’s solution might be very interesting: In some cases, it is crucial to construct waveforms (1), formed by means of a given ONB $\{\phi_n\}_{n \in \mathbb{N}}$, s.t. the modulus of its coefficients is lower bounded, and its $L^\infty$-norm can be controlled. For instance, to identify parameters of an unknown (wireless) channel, which can be seen as a linear-time invariant system, one may send a suitable known waveforms, called pilot signals. In particular, the identification of channel parameters constitute an important step for a reliable communications. Based on the knowledge of the those - and the resulting received signals, one hopes to estimate those unknowns (e.g. see [34], which also gives some novel ideas). To fulfill that task, the pilot signals have of course to possess some desired properties, i.e.: their peak values have to lie within an admissible range, which is in particular determined by the hardware of the transmitter, i.e. amplifier, filter, and antennas. In section 2, we have already discussed about the effect of

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1 Actually, Nazarov’s Thm. ensures the existence of a function $f_* \in L^\infty([0,1])$. But since $\{\phi_n\}_{n \in \mathbb{N}}$ is an ONB for $L^2([0,1])$, and $L^\infty([0,1]) \subset L^2([0,1])$, $f_*$ can be represented as the series $f = \sum_{n=1}^{\infty} b_n \phi_n$ converges w.r.t. $\|\cdot\|_{L^2([0,1])}$, with $b_n := \int_0^1 f_*(t) \overline{\phi_n(t)} dt$. 
the distortion of the amplified signal, in case that the peak value of it does not lie within the linear range of the amplifier. The occurrence of such effect in the process of channel measurement is clearly undesired. Furthermore, it is in some cases desired, that the energy, i.e. the $L^2$-norm, is spread over all the coefficients of the pilot signal, e.g. see the notion of block-type pilot model in the textbooks on wireless engineering. A necessary condition for the fulfillment of the latter condition is certainly, that the modulus of the coefficients of pilot signal lies uniformly over a certain threshold. For instance, if the pilot signals are desired to have the energy of 1, one may require, that the modulus of their coefficients are lower bounded by $1/\sqrt{N}$ (in case that $N$ wave functions are available). Nazarov's result shows, that this is basically possible. The occurrence of such effect in the process of channel measurement is clearly undesired. Furthermore, it is in some cases desired, that the energy, i.e. the $L^2$-norm, is spread over all the coefficients of the pilot signal, e.g. see the notion of block-type pilot model in the textbooks on wireless engineering. A necessary condition for the fulfillment of the latter condition is certainly, that the modulus of the coefficients of pilot signal lies uniformly over a certain threshold. For instance, if the pilot signals are desired to have the energy of 1, one may require, that the modulus of their coefficients are lower bounded by $1/\sqrt{N}$ (in case that $N$ wave functions are available). Nazarov's result shows, that this is basically possible.

The Nazarov's solution might also be interesting for designing test signals: After preparation of a system (e.g. circuits, chips, amplifier, etc.), one aims to examine, whether the implemented system possesses the desired property. This task is done by injecting input signals, which are admissible for later applications of the system. For instance, to check whether an implemented broadband amplifier fulfills the given requirements, one may inject a broadband signal, which is in our context a signal of the form (1), whose coefficients are non-zero for a large index set. Furthermore, it is desired that the peak value of the test broadband signal lies within the linear range of the designed broadband amplifier. A similar approach is also suitable for the testing of the small-signal behaviour of certain circuits.

Furthermore, in measurement-technological applications, it is often desired to construct waveforms $f$ of the form (1), for a specific ONS $\{\phi_n\}_{n=1}^N$, and a certain conditions on the coefficients (for instance they should be of modulus 1), which possess flatness property, in the sense that the modulus of $f$ is nearly constant. For the ONS $\{e^{i2\pi(n-1)\cdot}\}_{n=1}^N$, this problem is connected to the Littlewood’s flatness problem [18], which can be stated as follows:

Given a polynomial $s_N$ of the form (1), with $\phi_n(\cdot) = e^{i2\pi(n-1)\cdot}$, $n \in [N]$, whose coefficients are either complex and fulfill $|a_n| = 1$, $\forall n \in [N]$ (such a polynomial is called complex unimodular polynomial), or $a_n \in \{-1, 1\}, \forall n \in [N]$ (such a polynomial is also called real unimodular polynomial). How close can $s_N$ come to satisfying $|s_N(t)| = \sqrt{N}$, $\forall t \in [0,1]$? Specifically, for a given possibly small $\epsilon > 0$, we seek for an unimodular polynomial $s_N$, which is $\epsilon$-flat, i.e. $s_N$ fulfills the condition:

$$(1 - \epsilon)\sqrt{N} \leq |s_N(t)| \leq (1 + \epsilon)\sqrt{N}.$$  

Equivalently, one may also seek for a sequence of polynomials $\{s_{N_k}\}_{k \in \mathbb{N}}$, whose
members are each unimodular, and of degree $N_k \in \mathbb{N}, \forall k \in \mathbb{N}$, and which fulfills:

$$(1 - \epsilon_k) \sqrt{N_k} \leq |s_{N_k}(t)| \leq (1 + \epsilon_k) \sqrt{N_k}, \quad \forall k \in \mathbb{N},$$

for a sequence $\{\epsilon_{N_k}\}_{k \in \mathbb{N}}$ tending to 0. Such a sequence of unimodular polynomials is also called $\{\epsilon_{N_k}\}_{k \in \mathbb{N}}$ - ultraflat. Erdős conjectured in 1957 [9], that such a task is impossible, in the sense that every unimodular polynomials $s_N$ of degree $N$ have the property:

$$\max_{t \in [0,1]} |s_N(t)| \geq (1 + \epsilon) \sqrt{N},$$

where $\epsilon > 0$ is a constant, which is independent on $N$. In the case that the considered unimodular polynomials are complex, Kahane [15] showed the existence of a sequence of complex unimodular polynomials $\{s_N\}_{N \in \mathbb{N}}$, which is $\{\epsilon_N\}_{N \in \mathbb{N}}$ - ultraflat, where $\epsilon_N$ tends basically not faster to zero than $N^{-1/47} \sqrt{\log(N)}$, as $N$ goes to infinity. This statement disproves the Erdős conjecture, and sheds light to the solvability of the Littlewood’s flatness problem, in the case, where the considered unimodular polynomials are complex. Further discussions on Kahane’s result is given in [2]. For real unimodular polynomials, Erdős conjecture is still unsolved.

Kahane’s result asserts both the possibility to reduce the peak value of an OFDM waveform, and to obtain another waveform having nearly constant envelope, only by changing the phase of its coefficients. To see this, consider for instance the normalized OFDM waveforms $f$, viz. $\|f\|_{L^2([0,1])} = 1$. Furthermore, we consider such signals, whose energy are spread equally among those coefficients, i.e. the coefficients $\{a_n\}$ in the representation (1) yields, $|a_n| = 1/\sqrt{N}, \forall n \in [N]$. By the triangle inequality, and by the fact that complex exponentials are of modulus 1, one can show that the peak value of such signals is upper bounded by $\sqrt{N}$. If we set $a_n = 1/\sqrt{N}$, for each $n \in [N]$, we obtain a waveform, which achieves such a worst case behaviour. As asserted in the previous paragraph, for each $k \in \mathbb{N}$, we can find $\{\beta^{(k)}_n\}_{n \in [N_k]}$, such that the $L^\infty$-norm of the following sequence of waveforms:

$$f_{N_k}(t) = \frac{1}{\sqrt{N_k}} \sum_{n=1}^{N_k} e^{i\beta^{(k)}_n} e^{i2\pi(n-1)t}, \quad k \in \mathbb{N},$$

tends to 1, i.e. $\lim_{k \to \infty} \|f_{N_k}\|_{L^\infty([0,1])} = 1$. Furthermore, as it has already been discussed in previous paragraph, the existence of such a sequence, whose minimum value tends to 1, as the index increases, i.e. $\lim_{k \to \infty} \min_{t \in [0,1]} |f_{N_k}(t)| = 1$. Summarily, we obtain OFDM waveforms $\{f_{N_k}\}_{k \in \mathbb{N}}$, each possessing envelopes approximately constant, by only changing the phase of their coefficients. An interesting application of this result is certainly the design of pilot signals in OFDM system, which we have already discussed in the beginning of this section. The obtained OFDM waveforms might serve as a suitable pilot signals. Furthermore, a broadband amplifier for sending the pilot signal can be designed relatively easy, since the pilot signal has nearly constant envelope.
4.3. PAPR Reduction Problem and Tone Reservation Method. As already mentioned in the introduction, the strategy considered in this paper is to reserve one subset of orthonormal functions for carrying the information-bearing coefficients, and to determine the coefficients for the remaining orthonormal functions, s.t. the resulted sum of functions has a peak-value smaller than a given threshold:

Definition 4.3 (PAPR Reduction Problem). Given $\mathcal{K} \subset \mathbb{N}$. Let $\{\phi_n\}_{n \in \mathcal{K}}$ be an orthonormal system, and $\mathcal{I} \subset \mathcal{K}$. We say the PAPR reduction problem is solvable for the pair $(\{\phi_n\}_{n \in \mathcal{K}}, \mathcal{I})$ with constant $C_{\text{Ex}} > 0$, if for every $a \in l^2(\mathcal{I})$, there exists $b \in l^2(\mathcal{I}^c)$ (the complementation is of course w.r.t. $\mathcal{K}$), satisfying $\|b\|_{l^2(\mathcal{I}^c)} \leq C_{\text{Ex}} \|a\|_{l^2(\mathcal{I})}$, for which the following holds:

$$\esssup_{t \in [0,1]} \left| \sum_{k \in \mathcal{I}} a_k \phi_k(t) + \sum_{k \in \mathcal{I}^c} b_k \phi_k(t) \right| \leq C_{\text{Ex}} \|a\|_{l^2(\mathcal{I})} \tag{3}$$

We further refer $\mathcal{I}$ as information set, $\mathcal{I}^c$ as compensation set, $\{\phi_n\}_{n \in \mathcal{I}}$ as information tones, and respectively $\{\phi_n\}_{n \in \mathcal{I}^c}$ as compensation tones. We refer the quantity $|\mathcal{I}| / |\mathcal{K}|$ as the density of information set (in $\mathcal{K}$). Unless otherwise stated, $\{\phi_n\}_{n \in \mathbb{N}}$ is a ONS for $L^2([0,1])$, $\mathcal{K}$ is a subset of $\mathbb{N}$, $\mathcal{I} \subset \mathcal{K}$, and $\mathcal{I}^c$ is complemented w.r.t. $\mathcal{K}$. We shall always ignore the uninteresting case, where $a = 0$. A necessary condition for the solvability of the PAPR reduction problem is surely, that $\{\phi_n\}_{n \in \mathcal{I}}$ is uniformly bounded, in the sense that $\phi_n \in L^\infty([-\pi, \pi])$, for all $n \in \mathcal{I}$. Otherwise, one can easily construct a sequence $a \in l^2(\mathcal{I})$, for which the peak-value of the signal $\sum_{k \in \mathcal{I}} a_k \phi_k$ is unbounded. To avoid any further undesirable behaviour, we assume not only that the restricted orthonormal system $\{\phi_n\}_{n \in \mathcal{I}}$ is bounded, but that all of the considered orthonormal systems are bounded. In practice, the compensation tones might also be used for another purposes, such as the estimation of the transmission channel (e.g. [34]).

Sometimes, we refer the PAPR reduction problem simply as PAPR problem. Notice that if (3) is fulfilled, then the PAPR of the combined signal is below the given threshold value $C_{\text{Ex}}$. To see this, notice that the $L^2$-norm of the combined signal is simply $\sqrt{\|a\|_{l^2(\mathcal{I})}^2 + \|b\|_{l^2(\mathcal{I}^c)}^2}$. Correspondingly, we have:

$$\esssup_{t \in [0,1]} \left| \sum_{k \in \mathcal{I}} a_k \phi_k(t) + \sum_{k \in \mathcal{I}^c} b_k \phi_k(t) \right| \leq \esssup_{t \in [0,1]} \left| \sum_{k \in \mathcal{I}} a_k \phi_k(t) \right| + \esssup_{t \in [0,1]} \left| \sum_{k \in \mathcal{I}^c} b_k \phi_k(t) \right| \leq C_{\text{Ex}},$$

by assumption, that the PAPR reduction problem is solvable with constant $C_{\text{Ex}}$. The choice of the extension constant is dependent on the design of the communication scheme, which is in particular constrained by some factors, such as the admissible maximum energy consumption, and the linear range of the amplifier. In case that the PAPR reduction problem is solvable with a certain extension constant, the corresponding compensation coefficients carried by the compensation tones might be computed by means of linear program [31, 32, 33].
The condition $\|b\|_{l^2(I^c)} \leq C_{Ex} \|a\|_{l^2(I)}$ might seem at first sight to come out of nothing. However, it can be shown [6], that if (3) holds for some coefficients $b$, then $\|b\|_{l^2(I^c)}$ has to be less or equal than $C_{Ex} \|a\|_{l^2(I)}$. Thus, the requirement $\|b\|_{l^2(I^c)} \leq C_{Ex} \|a\|_{l^2(I)}$ serves in some sense as a restriction of the possible solutions of the PAPR reduction problem. Notice also, that we allow infinitely many carriers for the compensation of the PAPR value. This is not only of mathematical-, but also of practical interests, since the solvability of the PAPR reduction problem in this setting is a necessary condition for the solvability of the PAPR reduction problem in the setting, where the available compensation tones are of finite number.

In some cases, it is advantageous to consider the following restricted form of the PAPR reduction problem:

**Definition 4.4 (Restricted PAPR (RPAPR) Reduction Problem).** Given $\mathcal{K} \subset \mathbb{N}$. Let $\{\phi_n\}_{n \in \mathcal{K}}$ be an orthonormal system, and $\mathcal{I} \subset \mathcal{K}$. We say the restricted PAPR reduction problem is solvable for the pair $\{\{\phi_n\}_{n \in \mathcal{K}}, \mathcal{I}\}$ with constant $C_{Ex} > 0$, if for every $a \in l^2(\mathcal{I})$, with $\|a\|_{l^2(\mathcal{I})} \leq 1$, there exists $b \in l^2(\mathcal{I}^c)$, satisfying $\|b\|_{l^2(\mathcal{I}^c)} \leq C_{Ex}$, for which it holds:

$$\text{ess sup}_{t \in [0,1]} \left| \sum_{k \in \mathcal{I}} a_k \phi_k(t) + \sum_{k \in \mathcal{I}^c} b_k \phi_k(t) \right| \leq C_{Ex} \|a\|_{l^2(\mathcal{I})}. \quad (4)$$

Clearly, if the PAPR reduction problem is solvable, then the restricted PAPR reduction problem is also solvable. To give a clear distinction between the restricted - and the PAPR reduction problem, we sometimes refer the latter as the general PAPR reduction problem. By reason of energy efficiency, practitioners might be interested rather in the restricted PAPR reduction problem, than in the general PAPR reduction problem, since they would ensure that the energy of the transmission waveforms, resulted from the designed scheme, is below a certain threshold. Thus, it makes sense, only to consider information-bearing waveform under a certain energy threshold w.l.o.g. 1, i.e. $a \in l^2(\mathcal{I})$, with $\|a\|_{l^2(\mathcal{I})} \leq 1$.

The restricted PAPR reduction problem is indeed related with the general PAPR reduction problem in the following manner:

**Remark 4.5.** The solvability of the restricted PAPR problem implies also the solvability of the general PAPR problem. To see this, suppose that the restricted problem is solvable for $\{\{\phi_n\}_{n \in \mathcal{K}}, \mathcal{I}\}$, with an extension constant $C_{Ex} > 0$. Let $a \in l^2(\mathcal{I})$, $\|a\|_{l^2(\mathcal{I})} \neq 0$, be arbitrary. Now consider the sequence $a_1 := a/\|a\| \in l^2(\mathcal{I})$. By the solvability of the restricted PAPR reduction problem, we can find a sequence $b \in l^2(\mathcal{I}^c)$, for which (4) holds, with $a_1$ instead of $a$. Thus we have:

$$\text{ess sup}_{t \in [0,1]} \left| \sum_{k \in \mathcal{I}} a_k \phi_k(t) + \sum_{k \in \mathcal{I}^c} b_k \|a\|_{l^2(\mathcal{I})} \phi_k(t) \right| \leq C_{Ex} \|a\|_{l^2(\mathcal{I})}.$$

Since $a$ is arbitrary, previous observation implies that the general PAPR problem is solvable for $\{\{\phi_n\}_{n \in \mathcal{K}}, \mathcal{I}\}$, with extension constant $C_{Ex}$. The fact that the solvability of the general PAPR reduction problem implies the solvability of the restricted PAPR reduction problem with the same extension constant is trivial.
4.4. On the Optimal Extension Constant. In this subsection we aim to give some discussion about the extension constant. An elementary observation is that the extension constant with which the PAPR reduction problem is solvable is always bigger than 1. To see this, suppose that the PAPR reduction problem is solvable for \( \{ \phi_n \}_{n \in K}, I \) with \( C_{\text{Ex}} > 0 \). Then for an information-bearing coefficients \( a \in l^2(I) \), \( a \neq 0 \), we can obtain \( b \in l^2(I^c) \), s.t. the combined signal has the peak value less than \( C_{\text{Ex}} \| a \|_{l^2(I)} \). Thus, we have:

\[
\frac{\sum_{k \in I} |a_k|^2 + \sum_{k \in I^c} |b_k|^2}{\|a\|^2_{l^2(I)}} \leq \text{ess sup}_{t \in [0,1]} \left| \sum_{k \in I} a_k \phi_k(t) + \sum_{k \in I^c} b_k \phi_k(t) \right|^2 \leq C_{\text{Ex}}^2,
\]

where the first inequality follows from the usual estimation of the integral by means of essential supremum, and the orthonormality of \( \{ \phi_n \}_{n \in \mathbb{N}} \). The above inequality shows that \( C_{\text{Ex}} \geq 1 \), as desired.

Suppose that the restricted PAPR reduction problem is solvable for the pair \( \{ \phi_n \}_{n \in K}, I \) with a given constant \( C_{\text{Ex}} \geq 1 \). It is natural to ask, whether there is an lowest possible extension constant \( C_{\text{Ex}}^* \), less than \( C_{\text{Ex}} \), with which the PAPR reduction problem for \( \{ \phi_n \}_{n \in K}, I \) is still solvable. It is also natural to ask the same question for the general PAPR reduction problem, where the optimal constant is denoted by \( C_{\text{Ex}}^* \). Furthermore, it is interesting to see whether those quantities differ, or are exactly the same. Before we answer those question, let us first give the following preliminaries.

We define the operator, \( E_I : l^2(I) \rightarrow L^\infty([0,1]) \), which assign each \( a \in l^2(I) \) a waveform:

\[
E_I a := \sum_{k \in I} a_k \phi_k + \sum_{k \in I^c} b_k \phi_k,
\]

where \( b \) is a suitable sequence in \( \mathbb{C} \) indexed by \( I^c \) (where the complement is w.r.t \( K \)), as an extension operator (EP). By means of that operator, we may say that the restricted PAPR reduction problem is solvable for \( \{ \phi_n \}_{n \in K}, I \), with extension norm \( C_{\text{Ex}} \geq 1 \), if there exists an extension operator \( E_I \), fulfilling:

\[
\|E_I a\|_{L^\infty([0,1])} \leq C_{\text{Ex}} \|a\|_{l^2(I)} , \quad \forall \|a\|_{l^2(I)} \leq 1.
\]

Notice that such an operator is not necessarily unique, and in general not linear. By means of Remark 4.5, we can give the following observation:

**Remark 4.6.** Furthermore, by Remark 4.5, an operator \( E_I \), for which (5) holds, induces another operator \( E'_I : l^2(I) \rightarrow L^\infty([0,1]) \) assigning each \( a \in l^2(I) \) to the waveform:

\[
E'_I a := \sum_{k \in I} a_k \phi_k + \sum_{k \in I^c} b_k \|a\|_{l^2(I)} \phi_k,
\]

which gives the solution of the general PAPR problem for \( \{ \phi_n \}_{n \in K}, I \) with \( C_{\text{Ex}} \).

By the following quantity, which corresponds to an extension operator \( E_I \), we may give another more compact formulation of the solvability of the restricted
PAPR problem:
\[ \|E_I\|_{L^\infty(0,1)} = \sup_{\|a\|_{I^2} \leq 1} \|E_Ia\|_{L^\infty(0,1)}. \] (6)
Thus by (5), we may say that the restricted PAPR reduction problem is solvable for \((\{\phi_n\}_{n \in K}, I)\), with extension norm \(C_{E_{Ex}} \geq 1\), if there exists an extension operator \(E_I\), fulfilling:
\[ \|E_I\|_{L^\infty(0,1)} \leq C_{E_{Ex}}. \]
For a reformulation of the general PAPR reduction problem, we define the following quantity:
\[ \|E_I\|_{L^\infty(0,1)} = \sup_{\|a\|_{I^2} \neq 0} \frac{\|E_Ia\|_{L^\infty(0,1)}}{\|a\|_{I^2}}. \] (7)
As similar as done in the previous paragraph, we may give the following helpful formulation of the general PAPR reduction problem: We say that the PAPR reduction problem is solvable for \((\{\phi_n\}_{n \in K}, I)\) with extension constant \(C_{E_{Ex}}\), if there exists an extension operator \(E_I\), for which
\[ \|E_I\|_{L^\infty(0,1)} \leq C_{E_{Ex}}. \]
Back to our actual aim, we can define the optimal constant \(C_{E_{Ex}}\), which was already mentioned in the beginning of this subsection in the context of the solvability of the restricted PAPR reduction problem as follows:
\[ C_{E_{Ex}} := \inf \{ C_{E_{Ex}} > 0 : \text{RPAPR prob. is solv. for } (\{\phi_n\}_{n \in K}, I) \text{ with } C_{E_{Ex}} \} . \] (8)
By means of the extension operator, and \(\|\cdot\|_{L^\infty(0,1)}\), we can write the above expression by:
\[ C_{E_{Ex}} := \inf \{ \|E_I\|_{L^\infty(0,1)} : E_I \text{ is an extension operator} \}. \] (9)
Notice that solvability of the restricted PAPR reduction problem with the optimum extension constant \(C_{E_{Ex}}\), and accordingly/equivalently the existence of an extension operator giving a solution is not yet ensured. Further, in the interest of the general PAPR reduction problem, we may consider the following quantity:
\[ C_{E_{Ex}}^* := \inf \{ C_{E_{Ex}} > 0 : \text{PAPR prob. is solv. for } (\{\phi_n\}_{n \in K}, I) \text{ with } C_{E_{Ex}} \}, \]
which can be written by means of extension operators and \(\|\|_{I^2} \rightarrow L^\infty(0,1)\) as follows:
\[ C_{E_{Ex}}^* := \inf \{ \|E_I\|_{L^\infty(0,1)} : E_I \text{ is an extension operator} \}. \] (10)
The case \(C_{E_{Ex}} = \infty\) (resp. \(C_{E_{Ex}}^* = \infty\)), means that the restricted (resp. general) PAPR reduction problem is not solvable for \((\{\phi_n\}_{n \in K}, I)\) (with any extension constant \(C_{E_{Ex}} > 0\)). Those cases can also clearly be formalized by means of the non-existence of an extension operator giving the solution. Without ensuring the solvability of the PAPR reduction problem in the optimum, we are able show that \(C_{E_{Ex}}\) and \(C_{E_{Ex}}^*\) are essentially the same:
Lemma 4.7. Let be \( \{\phi_n\}_{n \in \mathbb{N}}, \mathcal{K} \subset \mathbb{N}, \) and \( \mathcal{I} \subset \mathcal{K}. \) It holds for the quantities (9) and (10):

\[
C_{Ex}^* = C_{Ex}^*.
\]

Proof. In case that that the restricted (resp. general) PAPR reduction problem is not solvable for \( \{\phi_n\}_{n \in \mathcal{K}}, \mathcal{I} \), it follows immediately, that the general (resp. restricted) PAPR reduction problem \( \{\phi_n\}_{n \in \mathcal{K}}, \mathcal{I} \) is not solvable. In both cases, we have \( C_{Ex}^* = C_{Ex}^* = \infty. \) For the remaining, assume that the general/restricted (by Remark 4.5, both are equivalent) PAPR reduction problem is solvable for \( \{\phi_n\}_{n \in \mathcal{K}}, \mathcal{I} \) with an extension constant \( C_{Ex} \geq 1. \)

By straightforward computation involving the definitions of \( C_{Ex}^* \) and \( C_{Ex}^* \), it is clear that \( C_{Ex}^* \geq C_{Ex}^*. \) To show the reverse inequality, let \( \epsilon > 0 \) be arbitrary. By the property of the infimum, there exists an extension operator \( E_{Ex}^{(c)} \), with:

\[
C_{Ex}^* + \epsilon \geq \left\| P_{Ex}^{(c)} \right\|_{L^2(I) \rightarrow L^\infty([0,1])} \geq C_{Ex}^*.
\]

Now take an arbitrary \( a \in L^2(\mathcal{I}), \) with \( \|a\|_{L^2(\mathcal{I})} \neq 0. \) Subsequently, define the sequence \( a_1 := a/\|a\|. \) Clearly \( \|a_1\|_{L^2(\mathcal{I})} = 1. \) Thus we have:

\[
\left\| P_{Ex}^{(c)} a_1 \right\| \leq \left\| E_{Ex}^{(c)} \right\|_{L^2(I) \rightarrow L^\infty([0,1])} \leq C_{Ex}^* + \epsilon. \tag{11}
\]

We can represent \( E_{Ex}^{(c)} a_1 \) as follows:

\[
E_{Ex}^{(c)} a_1 = \sum_{k \in \mathcal{I}} \frac{a_k}{\|a\|_{L^2(\mathcal{I})}} \phi_k + \sum_{k \in \mathcal{I}^c} b^{(c)}(a) \phi_k, \tag{12}
\]

where \( b^{(c)} := \{b^{(c)}\}_{n \in \mathbb{N}} \in L^2(\mathcal{I}^c). \) Combining (11) and (12), we have:

\[
\left\| \sum_{k \in \mathcal{I}} a_k \phi_k + \sum_{k \in \mathcal{I}^c} b^{(c)}(a) \|a\|_{L^2(\mathcal{I})} \phi_k \right\|_{L^\infty([0,1])} \leq (C_{Ex}^* + \epsilon) \|a\|_{L^2(\mathcal{I})}.
\]

Of course, the above relation holds for all \( a \in L^2(\mathcal{I}), \) with \( \|a\|_{L^2(\mathcal{I})} \neq 0. \) Thus we have an operator \( E_{Ex}^{(c)} : L^2(\mathcal{I}) \rightarrow L^\infty([0,1]) \) given by:

\[
a \mapsto \sum_{k \in \mathcal{I}} a_k \phi_k + \sum_{k \in \mathcal{I}^c} b^{(c)}(a) \|a\|_{L^2(\mathcal{I})} \phi_k,
\]

for which it holds:

\[
\left\| E_{Ex}^{(c)} \right\|_{L^2(I) \rightarrow L^\infty([0,1])} \leq C_{Ex}^* + \epsilon.
\]

Taking infimum over the left hand side of the above inequality, it yields \( C_{Ex}^* \leq C_{Ex}^* + \epsilon, \) and since \( \epsilon > 0 \) is arbitrary, we have \( C_{Ex}^* \leq C_{Ex}^* \) as desired. \( \square \)

Thus, to analyze the behaviour of the PAPR reduction problem in the optimal case, it is unnecessary to give a distinction between the restricted version -, and the general version of the problem.
4.5. Necessary and Sufficient Condition for solvability of the PAPR Reduction Problem. We aim in this section to give a necessary condition for the solvability of the PAPR reduction problem. In case, that \( \{ \phi_n \}_{n \in \mathbb{N}} \) is an orthonormal basis for \( L^2([0,1]) \), the condition given later is even sufficient. For ease of notations, let us first define the following subspaces of \( L^1([0,1]) \):

**Definition 4.8.** For an \( \mathcal{I} \subset \mathbb{N} \), and an ONS \( \{ \phi_n \}_{n \in \mathbb{N}} \), we define the following subspaces of \( L^1([0,1]) \):

\[
\mathcal{F}^1(\mathcal{I}) := \left\{ f \in L^1([0,1]) : f = \sum_{k \in \mathcal{I}} a_k \phi_k, \text{ for a } \{a_k\}_{k \in \mathcal{I}} \text{ in } \mathbb{C} \right\}
\]

\[
\mathcal{F}_c^1(\mathcal{I}) := \left\{ f \in L^1([0,1]) : f = \sum_{k \in \mathcal{I}} a_k \phi_k, \text{ where } a_k \neq 0, \text{ for finitely many } k \in \mathcal{I} \right\}
\]

Since, we are possibly dealing with infinite index set \( \mathcal{I} \), the expression \( f = \sum_{k \in \mathcal{I}} a_k \phi_k \) in the definition of \( \mathcal{F}^1(\mathcal{I}) \) has to be understood w.r.t. the norm structure of \( \mathcal{F}^1(\mathcal{I}) \), viz. inherited from \( L^1([0,1]) \). Specifically, \( f = \sum_{k \in \mathcal{I}} a_k \phi_k \) means that the finite partial sum of \( \sum_{k \in \mathcal{I}} a_k \phi_k \) converges to \( f \), w.r.t. \( \| \cdot \|_{L^1([0,1])} \). Notice that subspaces \( \mathcal{F}^1(\mathcal{I}) \) and \( \mathcal{F}_c^1(\mathcal{I}) \) depend in particular on the choices of the ONS. We will not emphasize the choice of the ONS in the notation for both mentioned subspaces, since it will be clear from the context. The following characterization of those subspaces is elementary to show:

**Lemma 4.9.** For a \( \mathcal{I} \subset \mathbb{N} \), the following statements holds:

1. \( \mathcal{F}^1(\mathcal{I}) \) is a closed subspace of \( L^1(\mathcal{I}) \)

2. \( \mathcal{F}^1(\mathcal{I}) \) is the closure of \( \mathcal{F}_c^1(\mathcal{I}) \).

To show the first statement, one can simply take a sequence \( \{f_n\}_{n \in \mathbb{N}} \) in \( \mathcal{F}^1(\mathcal{I}) \), which converges to an \( f \in L^1([0,1]) \). By involving the fact that \( \phi_n \in L^\infty([0,1]) \), \( n \in \mathbb{N} \), one can show by simple application of the Hölder’s inequality, that for arbitrary \( k \in \mathcal{I} \), \( \int_0^1 f(t) \phi_k(t)dt = \lim_{n \to \infty} \int_0^1 f_n(t) \phi_k(t)dt \) (in \( \mathbb{C} \)). Finally, by noticing \( \int_0^1 f_n(t) \phi_k(t)dt = 0, \forall n \), the statement is shown. The second statement can be shown, simply by approximating each \( f \in \mathcal{F}^1(\mathcal{I}) \), which is an (infinite) linear combination of some members of \( \{ \phi_n \} \) by its finite sum.

Now, we are ready to give a necessary condition for the solvability of the PAPR reduction problem:

**Theorem 4.10** (Thm. 5 in [6]). Let \( \{ \phi_n \}_{n \in \mathbb{N}} \) be an ONS in \( L^2([0,1]) \). Given a subset \( \mathcal{I} \subset \mathbb{N} \) and a constant \( C_{E_0} > 0 \). Assume that the PAPR reduction problem is solvable for \( \{ \phi_n \}_{n \in \mathbb{N}}, \mathcal{I} \) with extension constant \( C_{E_0} \). Then:

\[
\| f \|_{L^1([0,1])} \leq C_{E_0} \| f \|_{L^1([0,1])}, \forall f \in \mathcal{F}^1(\mathcal{I})
\]

(13)
Proof. For \( f \in \mathfrak{F}^1(I) \), (13) was shown in [7]. It remains to show (13) holds generally for \( \mathfrak{F}^1(I) \). Let be \( f \in \mathfrak{F}^1(I) \) arbitrary. Lemma 4.9 asserts, that there exists a sequence \( \{f_n\}_{n \in \mathbb{N}} \) in \( \mathfrak{F}^1(I) \), which converges to \( f \) w.r.t. \( \| \cdot \|_{L^1([0,1])} \). Now we claim that \( \{f_n\}_{n \in \mathbb{N}} \) converges to \( f \) w.r.t. \( \| \cdot \|_{L^2([0,1])} \). To show this claim, notice that since (13) holds for functions in \( \mathfrak{F}^1_k(I) \), \( \{f_n\} \) is a Cauchy sequence in \( L^2([0,1]) \). Thus by completeness of \( L^2([0,1]) \), there exists \( g \in L^2([0,1]) \), for which \( \{f_n\}_{n \in \mathbb{N}} \) converges to \( g \), w.r.t. \( \| \cdot \|_{L^2([0,1])} \). It is well known that the convergence of sequence of functions in \( L^p \)-spaces implies the convergence of those almost everywhere (a.e.). Thus, there exists subsequences \( \{n_k\} \subset \mathbb{N} \), and \( \{\bar{n}_k\} \) of \( \mathbb{N} \), for which:
\[
\lim_{k \to \infty} f_{n_k}(t) = g(t) \quad \text{and} \quad \lim_{k \to \infty} f_{\bar{n}_k}(t) = g(t), \quad \text{a.e. } t \in [0,1],
\]
which gives \( f(t) = g(t) \), a.e. \( t \in [0,1] \), accordingly \( f = g \), which gives the claim.
So by previous claim, sequential continuity of norm, and the fact that (13) holds for functions in \( \mathfrak{F}^1_k(I) \), we have:
\[
\|f\|_{L^2([0,1])} = \lim_{n \to \infty} \|f_n\|_{L^2([0,1])} \leq C_{\text{ex}} \lim_{n \to \infty} \|f_n\|_{L^1([0,1])} = \|f\|_{L^1([0,1])}.
\]

Before we continue, let us first give the following remarks:

Remark 4.11. By Remark 4.5, we can infer that it is also adequate in the above Thm. only to require that the restricted -, instead of the general PAPR reduction problem is solvable.

Remark 4.12. Now, suppose that there exists a constant \( C_{\text{ex}} > 0 \), s.t. the following holds:
\[
\|f\|_{L^2([0,1])} \leq C_{\text{ex}} \|f\|_{L^1([0,1])}, \quad \forall f \in \mathfrak{F}^1(I), \quad \|f\|_{L^1([0,1])} \leq 1. \quad (14)
\]
It is obvious that (13) holds. Indeed, to see this, take an arbitrary \( f \in \mathfrak{F}^1(I) \), with \( \|f\|_{L^1([0,1])} \neq 0 \). By setting the function \( f/\|f\|_{L^1([0,1])} \) in (14), and subsequent elementary computations, the claim holds. Thus to show that the PAPR reduction problem is not solvable with an extension constant \( C_{\text{ex}} > 0 \), it is sufficient to show the "restricted" norm equivalence (14). Further, that (13) implies (14), is trivial. Summarily, we can infer that the condition (14) is equivalent with (13).

In case that \( \{\phi_n\}_{n \in \mathbb{N}} \) forms additionally an orthonormal basis, we have also the converse of Thm. 4.10:

Proposition 4.13 (Thm. 5 in [6]). Let \( \{\phi_n\}_{n \in \mathbb{N}} \) be an orthonormal basis for \( L^2([0,1]) \), and let be \( I \subset \mathbb{N} \), and \( C_{\text{ex}} > 0 \). If the following condition is fulfilled:
\[
\|f\|_{L^2([0,1])} \leq C_{\text{ex}} \|f\|_{L^1([0,1])}, \quad \forall f \in \mathfrak{F}^1(I),
\]
then the PAPR reduction problem is solvable for \( (\{\phi_n\}_{n \in \mathbb{N}}, I) \) with extension constant \( C_{\text{ex}} \).
Proof. Let be \( f \in \mathcal{F}^1(\mathcal{I}) \) arbitrary, having the representation \( f = \sum_{k \in \mathcal{I}} c_k \phi_k \) w.r.t. \( ||f||_{L^1([0,1])} \); i.e. \( \sum_{k \in \mathcal{I}} c_k \phi_k \) converges to \( f \) w.r.t. \( ||f||_{L^1([0,1])} \), for a sequence \( c \) in \( \mathbb{C} \). Since (15) holds by assumption, \( f = \sum_{k \in \mathcal{I}} c_k \phi_k \) holds also w.r.t. \( ||f||_{L^2([0,1])} \).

Now, take an arbitrary \( a \in \mathcal{F}(\mathcal{I}) \) and define the mapping \( \Psi_a : \mathcal{F}^1(\mathcal{I}) \to \mathbb{C} \) by \( \Psi_a f := \sum_{k \in \mathcal{I}} c_k \phi_k \). Linearity of \( \Psi_a \) is obvious. Further \( \Psi_a \) is bounded, since:

\[
||\Psi_a f|| \leq ||a||_{\mathcal{F}(\mathcal{I})} ||c||_{\mathcal{F}(\mathcal{I})} = ||a||_{\mathcal{F}(\mathcal{I})} \sum_{k \in \mathcal{I}} \phi_k \in \mathcal{F}(\mathcal{I}) \leq ||a||_{\mathcal{F}(\mathcal{I})} C_{Ex} ||f||_{L^1([0,1])} < \infty,
\]

where the equality follows from the fact that \( f = \sum_{k \in \mathcal{I}} c_k \phi_k \) w.r.t. \( ||f||_{L^2([0,1])} \), and the assumption that \( \{\phi_n\}_{n \in \mathbb{N}} \) is orthonormal, and the third inequality from (15). \( \mathcal{F}^1(\mathcal{I}) \) is clearly a subspace of \( L^1([0,1]) \), and as we have already seen, \( \Psi_a \) is linear and bounded. Thus the Hahn-Banach Thm. asserts the existence of a linear and bounded mapping \( \tilde{\Psi} : L^1([0,1]) \to \mathbb{C} \), for which:

\[
\tilde{\Psi} f = \Psi_a f, \quad \forall f \in \mathcal{F}^1(\mathcal{I}), \quad \text{and} \quad \left\| \tilde{\Psi} \right\|_{L^1([0,1]) \to \mathbb{C}} = ||\Psi_a||_{\mathcal{F}^1(\mathcal{I}) \to \mathbb{C}},
\]

holds.

Further, since the dual space of \( L^1([0,1]) \) is \( L^\infty([0,1]) \), we can find a unique \( g \in L^\infty([0,1]) \), for which the following holds:

\[
\tilde{\Psi} f = \int_0^1 f(t) g(t) dt, \quad \forall f \in L^1([0,1]), \quad \text{and} \quad ||g||_{L^\infty([0,1])} = \left\| \tilde{\Psi} \right\|_{L^\infty([0,1]) \to \mathbb{C}}.
\]

As \( L^\infty([0,1]) \subset L^2([0,1]) \), it follows that \( g \) can be represented by means of the series \( g = \sum_{k=1}^\infty d_k \phi_k \), for a sequence \( d \) in \( l^2(\mathbb{N}) \). By the first statement in (17), and the orthonormality of \( \{\phi_n\}_{n \in \mathbb{N}} \), one can imply that \( a_k = d_k \), for every \( k \in \mathcal{I} \).

Define a sequence \( b \in l^2(\mathcal{I}^c) \), by setting \( b_k = d_k \), \( \forall k \in \mathcal{I}^c \). Thus we have:

\[
\left\| \sum_{k \in \mathcal{I}} a_k \phi_k + \sum_{k \in \mathcal{I}^c} b_k \phi_k \right\|_{L^\infty([0,1])} = ||g||_{L^\infty([0,1])} = \left\| \tilde{\Psi} \right\|_{L^\infty([0,1]) \to \mathbb{C}} = ||\Psi_a||_{L^\infty([0,1]) \to \mathbb{C}} \leq C_{Ex} ||a||_{\mathcal{F}(\mathcal{I})},
\]

where the 2. equality follows from (16), the 3. from (17), and the inequality from (16), as desired.

\[\square\]

Remark 4.14. By Remark 4.12, the condition (15) in the above Thm. can clearly be softened by the condition (14).

4.6. Further Discussions on the Optimal Extension Constant and the Solvability of the PAPR reduction problem in the Optimum. Proposition 4.13 sheds light on the discussions made in Subsection 4.4, about the optimal constant \( C_{Ex} \), which gives the lower bound of the extension constant, for which both the restricted - and the general PAPR reduction problem is solvable: Assume
that \( \{\phi_n\}_{n \in \mathbb{N}} \) forms an ONB for \( L^2([0, 1]) \). Notice that \( C_{\text{Ex}} \) is basically the operator norm of the embedding \( \operatorname{Emb} : \mathcal{F}^1(I) \to L^2([0, 1]), \ f \mapsto f \). Formally, we have:

\[
C_{\text{Ex}} := \inf \left\{ c > 0 : \| \operatorname{Emb}f \|_{L^2([0, 1])} \leq c \| f \|_{L^1([0, 1])}, \ \forall f \in \mathcal{F}^1(I) \right\}.
\]

In particular, if \( I \) is finite, \( C_{\text{Ex}} \) is always finite. By Thm. 4.10, the former case also occurs, if \( I \) is infinite, and PAPR reduction problem is solvable for \( (\{\phi_n\}, I) \) for some constant \( C_{\text{Ex}} \).

In case that the considered ONS is complete, we can even ensure the solvability of the PAPR reduction problem in the optimal case:

**Proposition 4.15.** Let \( \{\phi_n\}_{n \in \mathbb{N}} \) be an ONB. Assume that the PAPR reduction problem is solvable for \( (\{\phi_n\}_{n \in \mathbb{N}}, I) \) with a certain extension constant \( C_{\text{Ex}} > 0 \), then it is also solvable with the optimal extension constant \( C_{\text{opt}} \).

**Proof.** Let \( \epsilon > 0 \) be arbitrary. Then by the solvability assumption, and the definition of the infimum, there exists an extension operator \( E^{(\epsilon)}_I \), for which:

\[
\| E^{(\epsilon)}_I \|_{\mathcal{L}(I \to L^\infty([0, 1]))} \leq C_{\text{opt}} + \epsilon.
\]

Consequently, by Thm. 4.10, it follows that:

\[
\| f \|_{L^2([0, 1])} \leq (C_{\text{opt}} + \epsilon) \| f \|_{L^1([0, 1])}, \ \forall f \in \mathcal{F}^1(I).
\]

The left hand side of the above inequality does not depend on \( \epsilon \). Thus it follows that \( \| f \|_{L^2([0, 1])} \leq C_{\text{opt}} \| f \|_{L^1([0, 1])} \). Finally, Prop. 4.13 asserts the solvability of PAPR reduction problem for \( (\{\phi_n\}_{n \in \mathbb{N}}, I) \) with \( C_{\text{opt}} \), as desired. \( \square \)

### 4.7. On a Weaker Formulation of the PAPR reduction problem.

An interesting and weaker formulation of the PAPR reduction problem can be given as follows:

**Problem 4.16.** Given \( I \subset \mathbb{N} \), and an ONS \( \{\phi_n\}_{n \in \mathbb{N}} \). Let \( a \in l^2(I) \) be fixed, but arbitrary. Does there exists an \( b \in l^2(I^c) \), for which:

\[
\left\| \sum_{k \in I} a_k \phi_k + \sum_{k \in I^c} b_k \phi_k \right\|_{L^\infty([0, 1])} < \infty,
\]

holds? (19)

Notice that in the above formulation, we merely require the solvability of the PAPR reduction problem only for some coefficients of interests, rather than the solvability of the PAPR reduction problem for all sequences in \( l^2(I) \), and the corresponding "uniform" control. With the weak formulation of the PAPR reduction problem, we identify the following optimal constant:

\[
C_{\text{opt}}(a) := \inf_{b \in l^2(I^c)} \left\| \sum_{k \in I} a_k \phi_k + \sum_{k \in I^c} b_k \phi_k \right\|_{L^\infty([0, 1])}.
\]
Notice that $C_{\text{opt}}$ can be seen as a functional $C_{\text{opt}} : l^2(\mathcal{I}) \to \mathbb{R}$, where $\mathbb{R} = \mathbb{R} \cup \{-\infty, \infty\}$ denotes the extended real line. It is obvious that $C_{\text{opt}}$ is non-negative. Thus $C_{\text{opt}}$ can not take the value $-\infty$. In case that $C_{\text{opt}}(\mathbf{a}) = \infty$, for an $\mathbf{a} \in l^2(\mathcal{I})$ means, that one can not find $\mathbf{b} \in l^2(\mathcal{I}')$, for which (19) holds. Further, $C_{\text{opt}}$ depends on the choice of information set $\mathcal{I}$, and the choice of orthonormal system $\{\phi_n\}_{n \in \mathbb{N}}$. The functional $C_{\text{opt}}$ possesses some nice properties, which can be shown straightforwardly:

**Proposition 4.17.** Let $\mathcal{I} \subset \mathbb{N}$, and $\{\phi_n\}_{n \in \mathbb{N}}$ be an ONS. The corresponding functional $C_{\text{opt}}$ is convex, in the sense that:

1. $C_{\text{opt}}(\mathbf{a}) \geq 0$
2. For $\lambda \in (-1, 1)$, $\lambda \neq 0$, and $\mathbf{a} \in l^2(\mathcal{I})$, it holds: $C_{\text{opt}}(\lambda \mathbf{a}) = |\lambda| C_{\text{opt}}(\mathbf{a})$.
3. For $\mathbf{a}^{(1)}, \mathbf{a}^{(2)} \in l^2(\mathcal{I})$. It holds: $C_{\text{opt}}(\mathbf{a}^{(1)} + \mathbf{a}^{(2)}) \leq C_{\text{opt}}(\mathbf{a}^{(1)}) + C_{\text{opt}}(\mathbf{a}^{(2)})$.

An additional property for the functional $C_{\text{opt}}$, which we desire to have is that it is lower semi-continuous, in the sense that:

**Definition 4.18** (Lower Semi-continuity). Given a normed space $\mathcal{X}$, and a functional $p : \mathcal{X} \to \mathbb{R}$. Then $p$ is said to be lower semi-continuous at the point $x \in \mathcal{X}$, if $p(x_0) = -\infty$, or for each $h \in \mathbb{R}$, with $p(x_0) > h$, there exists $\delta > 0$, such that:

$$p(x) > h, \quad \forall x \in B_\delta(x_0),$$

where $B_\delta(x_0)$ denotes the open ball around $x_0$, with radius $\delta$, formally $B_\delta(x_0) := \{y \in \mathcal{X} : \|x_0 - y\|_{\mathcal{X}} < \delta\}$. In case that $p$ is lower semi-continuous at every point $x \in \mathcal{X}$, then we say $p$ is lower semi-continuous on $\mathcal{X}$.

Before we continue, let us first introduce the following notions: Let $\mathcal{B}$ be a Banach space, a set $\mathcal{M} \subseteq \mathcal{B}$ is said to be nowhere dense if $\text{int} \mathcal{M} = \emptyset$, i.e. if the inner of the closure of $\mathcal{M}$ is empty. A set $\mathcal{M} \subseteq \mathcal{B}$ is said to be of 1. category, if it can be represented as a countable union of nowhere dense sets. In case that a set is of 1. category, then it is said to be of 2. category. The complement of a set of 1. category is defined as a residual set. Topologically, sets of 1. category can be seen as a small set, in the sense that they are negligible if compared to the whole space, sets of 2. category as sets, which are not small, and residual sets, each as a complementary set of a set of 1. category, can be seen as a large set. The Baire category Thm. ensures that this categorization of sets of a Banach spaces is non-trivial, by showing that the whole Banach space $\mathcal{B}$ is not "small" in this sense, or can even not be "approximated" by such sets, i.e. it can not be written as the union of sets of 1. category, and that the residual sets are dense in $\mathcal{B}$, and closed under countable intersection. A property that holds for a residual subset of $\mathcal{B}$ is called a generic property. A generic property might not holds for all elements of $\mathcal{B}$, but for "typical" elements of $\mathcal{B}$. For more detailed treatment of the Baire category Thm., we refer to standard textbooks such as [21, 24, 25].

As an application of Gelfand’s Theorem (see e.g. Thm. 4 (1.VII) in [16]), we have the following characterization of the functional $C_{\text{opt}}$: 
Lemma 4.19. Let \( \{ \phi_n \}_{n \in \mathbb{N}} \) be an ONS, and \( \mathcal{I} \subset \mathbb{N} \). Further assume that the following holds:

1. \( C_{\text{opt}} \) is lower semi-continuous on \( l^2(\mathcal{I}) \).
2. There exists a set of 2. category \( \mathcal{M} \subset l^2(\mathcal{I}) \), s.t. \( C_{\text{opt}}(\mathbf{a}) < \infty \).

Then there exists a constant, for which the following holds:

\[
C_{\text{opt}}(\mathbf{a}) \leq C \| \mathbf{a} \|_{l^2(\mathcal{I})}, \quad \forall \mathbf{a} \in l^2(\mathcal{I}).
\]  

(20)

Clearly, the above lemma implies immediately the following statement, which gives a connection between \( C_{\text{opt}} \) and the optimal extension constant:

Proposition 4.20. Let be \( \mathcal{I} \subset \mathbb{N} \), and \( \{ \phi_n \}_{n \in \mathbb{N}} \) an ONS in \( L^2([0,1]) \). If \( C_{\text{opt}} \) is lower-semi continuous and is finite on a set of 2. category in \( l^2(\mathcal{I}) \), then the infimum of the constant \( C \), for which (20) hold is exactly \( C_{\text{Ex}} \), formally:

\[
\inf_{C > 0} \left\{ C : C_{\text{opt}}(\mathbf{a}) \leq C \| \mathbf{a} \|_{l^2(\mathcal{I})}, \quad \forall \mathbf{a} \in l^2(\mathcal{I}) \right\} = C_{\text{Ex}}.
\]

Proof. Since \( C_{\text{opt}} \) is lower semi-continuous, and finite on a set of second category, by Lemma 4.19, we can find a constant \( C > 0 \), for which it holds:

\[
\inf_{\mathbf{b} \in l^2(\mathcal{I}^c)} \left\| \sum_{k \in \mathcal{I}} a_k \phi_k + \sum_{k \in \mathcal{I}^c} b_k \phi_k \right\|_{L^\infty([0,1])} = C_{\text{opt}}(\mathbf{a}) \leq C \| \mathbf{a} \|_{l^2(\mathcal{I})}, \quad \forall \mathbf{a} \in l^2(\mathcal{I}),
\]

which shows the finiteness of \( C_{\text{opt}}^* \). For an arbitrary \( \epsilon > 0 \), we find an \( \mathbf{b}(\epsilon) \in l^2(\mathcal{I}) \), for which it holds:

\[
\left\| \sum_{k \in \mathcal{I}} a_k \phi_k + \sum_{k \in \mathcal{I}^c} b_k^{(\epsilon)} \phi_k \right\|_{L^\infty([0,1])} \leq C + \epsilon, \quad \forall \mathbf{a} \in l^2(\mathcal{I}), \quad \| \mathbf{a} \|_{l^2(\mathcal{I})} \neq 0,
\]

which gives the observation, that there exists an extension operator \( E_\mathcal{I}^{(\epsilon)} \), for which

\[
\| E_\mathcal{I}^{(\epsilon)} \|_{l^2(\mathcal{I}) \to L^\infty([0,1])} \leq C + \epsilon,
\]

and the correspondingly the solvability of the PAPR reduction problem for \( \{ \phi_n \}_{n \in \mathbb{N}}, \mathcal{I} \). Taking the infimum of \( \| \cdot \|_{l^2(\mathcal{I}) \to L^\infty([0,1])} \) over all extension operators, and subsequently noticing that \( \epsilon > 0 \) can be chosen arbitrarily (the infimum on the L.H.S. does not depend on \( \epsilon \!)) \), we have \( C_{\text{Ex}} \leq C \).

By taking the corresponding infimum over all constant \( C \), we have as desired \( C_{\text{Ex}} \leq C_{\text{opt}}^* \), as desired.

To show the reverse inequality, notice that by argumentations made in the beginning of the proof, we have that \( C_{\text{Ex}} \) is finite. Thus for each \( \epsilon > 0 \), we find an extension operator \( E_\mathcal{I}^{(\epsilon)} \), such that

\[
\| E_\mathcal{I}^{(\epsilon)} \|_{l^2(\mathcal{I}) \to L^\infty([0,1])} \leq C_{\text{Ex}} + \epsilon.
\]

The former
asserts, that for a fixed $a \in l^2(I)$, $\|a\|_{l^2(I)} \neq 0$, we can find $b^{(c)} \in l^2(I^c)$, for which:

$$\left\| \sum_{k \in I} a_k \phi_k + \sum_{k \in I^c} b^{(c)}_k \phi_k \right\|_{L^\infty([0,1])} \leq C_{Ex} + \epsilon.$$ 

Thus taking the infimum on the left hand side over all $b \in l^2(I)$, and since $\epsilon > 0$ was arbitrarily chosen, and now the left hand side does not depend on $\epsilon$, we have:

$$C_{opt}(a) \leq C_{Ex} \|a\|_{l^2(I)},$$

which shows that $C_{Ex} \geq C_{opt}^*$ (since $a \in l^2(I)$ was arbitrarily chosen).

In case that the functional $C_{opt}$ is lower semi-continuous, the finiteness of $C_{opt}$ in a subset of $l^2(I)$, which is not too small (in particular, it is sufficient to have finiteness of $C_{opt}$ on a ball in $l^2(I)$ with arbitrary small radius), implies already the solvability of Problem 4.16 for every sequences in $l^2(I)$.

**Theorem 4.21.** Let be $I \subset \mathbb{N}$, and $\{\phi_n\}_{n \in \mathbb{N}}$ an ONS in $L^2([0,1])$. Assume that $C_{opt}$ is lower-semi continuous. If there exists a set of 2. category $M$ in $l^2(I)$, for which $C_{opt}(a) < \infty$, $\forall a \in l^2(I)$, then the problem 4.16 is solvable for all $a \in l^2(I)$.

**Proof.** Since $C_{opt}$ is lower semi-continuous, and finite on a set of second category, Proposition 4.20 asserts that $C_{Ex}$ is finite. Thus, Prop. 4.15 asserts that the PAPR reduction problem is solvable for $\{(\phi_n)_{n \in \mathbb{N}}, I\}$, with $C_{Ex}$, and a fortiori Problem 4.16 for every $a \in l^2(I)$.

As an argumentum e contrario of the above Thm., and by noticing that sets, which are not of 2. category, is of 1. category, we have the following statement:

**Corollary 4.22.** Let be $I \subset \mathbb{N}$, and $\{\phi_n\}_{n \in \mathbb{N}}$ an ONS in $L^2([0,1])$. Assume that $C_{opt}$ is lower-semi continuous. If the problem 4.16 is not solvable for an $a \in l^2(I)$, then the set $M$, for which $C_{opt}(a) < \infty$, $\forall a \in M$, is of at most 2. category in $l^2(I)$.

The above Corollary gives in some sense a strong statement: The inability of compensating the peak value of only a single waveform formed by a sequence $a \in l^2(I)$ implies immediately the inability of compensating of ”typical” waveforms formed by sequences in $l^2(I)$. The latter is an implication of the fact that in this case $C_{opt}$ is finite only for sets of 1. category, thence it is infinite for residual sets.

5. Necessary Condition for Solvability of PAPR Reduction Problem for OFDM

Now we aim to analyze the PAPR reduction problem for OFDM systems. As already mentioned in the introduction, an OFDM transmission consists of superposition of sines weighted by information coefficients. The sines have the form:

$$\epsilon_n := e^{2\pi i (n-1)c}, \quad n \in \mathbb{N}.$$
Clearly, \( \{ e_n \}_{n \in \mathbb{N}} \) forms an ONS in \( L^2[0,1] \). Furthermore, it is well-known, that \( \{ e_n \}_{n \in \mathbb{N}} \) even forms an orthonormal basis for \( L^2([0,1]) \). Surprisingly, the PAPR reduction problem is connected to a deep result in mathematics, the so called Szemerédi Theorem [29], concerning to a certain subset of integers with additive structure, namely an arithmetic progression. The following subsection is devoted to that issue.

5.1. Deterministic approach to the PAPR problem. A problem in mathematics which has been raised interests in the last decades is the problem of finding or determining a so-called arithmetic progressions of a certain length in a given subset \( A \) of natural numbers. Let us first discuss about that object in the following:

**Szemerédi Theorem on Arithmetic Progressions.**

**Definition 5.1 (Arithmetic Progression).** Let be \( m \in \mathbb{N} \). An arithmetic progression of length \( m \) is defined as a subset of \( \mathbb{Z} \), which has the form:

\[
\{ a, a + d, a + 2d, \ldots, a + (m - 1)d \},
\]

for some integer \( a \) and some positive integer \( d \).

For sum sets, i.e. sets with specific structures such as \( A + A \), \( A + A + A \), or \( 2A - 2A \), for an \( A \subset \mathbb{N} \), there are some results concerning to the existence of arithmetic progressions within those sets. However, they require some insights into the structure of the subset \( A \). For some detailed discussions concerning to this aspect, we refer to the excellent textbook [30].

We are mostly interested in the following subset:

**Definition 5.2 ((\( \delta, m \))-Szemerédi Set).** Let \( I \) be a set of integers, \( \delta \in (0,1) \), and \( m \in \mathbb{N} \). The set \( I \) is said to be \((\delta, m)\)-Szemerédi, if every subset of \( I \) of cardinality at least \( \delta |I| \) contains an arithmetic progression of length \( m \).

The celebrated Szemerédi Thm. [29] gives a connection on the size of the set consecutive numbers \([N]\), s.t. every subset \( I \) of \([N]\), viz. \(|I|/N\), contains an arithmetic progression of a given length:

**Theorem 5.3 (Szemerédi Theorem [29]).** For any \( m \in \mathbb{N} \), and any \( \delta \in (0,1) \), there exists \( N_{Sz} \in \mathbb{N} \), which depends on \( m \) and \( \delta \), s.t. for all \( N \geq N_{Sz} \), \([N]\) is \((\delta, m)\)-Szemerédi.

The cases \( m = 1,2 \) are merely trivial. The case \( m = 3 \) was already proven earlier by Roth [26], for which he was awarded the Fields Medal in 1958. Szemerédi proved the result firstly for \( m = 4 \) in 1969, and recent result [28] finally in 1975. Finding the correct constant \( N_{Sz} \) is quiet challenging. Gowers showed that \( N_{Sz}(\delta, m) \leq 2^{c_m/m} \), where \( c_m = 2^{2m+9} \). A lower bound of \( N_{Sz} \) is due to Rankin [23]. He has proven, that it holds \( N_{Sz}(\delta, m) \geq \exp(C(\log(1/\delta))^{1+\lfloor \log_2(m-1) \rfloor}) \), for some constant \( C > 0 \). A better lower bound might be derived from [20].
For the asymptotic case, Szemerédi Thm. is somehow unsatisfactory. It merely ensures the existence of arithmetic progressions of arbitrary length for subsets of \( \mathbb{N} \) with positive upper density. Specifically, an equivalent statement of the Szemerédi Thm. can be given as follows: Given a subset \( A \subseteq \mathbb{N} \), whose upper density is positive, i.e. \( \limsup_{N \to \infty} (|A \cap [N]|/N) > 0 \). Then, there exists an arithmetic progression of length \( k \), where \( k \) is an arbitrary natural number. A tightening of this statement is due to Green and Tao [11]. They showed, that the set of prime numbers \( \mathbb{P} \) contains arithmetic progressions of arbitrary length. It is well known that the density of prime numbers in \( [N] \), i.e. the quantity \( |\mathbb{P} \cap [N]|/N \), \( N \in \mathbb{N} \), is asymptotically \( 1/\log(N) \). Thus the density of the prime numbers in \( \mathbb{N} \) is 0. A more general statement than the previous one was already conjectured by Erdös (see Conjecture 6.16), which still remains unsettled. We shall later give a discussion in Subsection 6.3, and show that this conjecture holds true for Walsh case.

Recently, it was shown by Conlon and Gowers [8], that for arbitrary \( \delta \in (0,1) \) and \( m \in \mathbb{N} \), one can asymptotically give a "sparse" \((\delta, m)\)-Szemerédi, by choosing randomly the elements from \([N]\) by some arbitrary small probability \( p \) (Call the corresponding set \([N]_p\)):

**Theorem 5.4 (Conlon, Gowers [8]).** Given \( \delta > 0 \), and a natural number \( m \in \mathbb{N} \). There exists a constant \( C > 0 \), s.t.:

\[
\lim_{N \to \infty} \mathbb{P}([N]_p \text{ is } (\delta, m)\text{-Szemerédi}) = 1, \quad \text{if } p > C N^{-\frac{1}{m-1}}.
\]

Notice that the above Thm. ensures the existence of a sequence \( \{p_N\} \) in \((0,1)\), tending to zero, for which:

\[
\lim_{N \to \infty} \mathbb{P}([N]_{p_N} \text{ is } (\delta, m)\text{-Szemerédi}) = 1,
\]

which justifies in particular the notion, that such a \((\delta, m)\)-Szemerédi is asymptotically "sparse" in \( \mathbb{N} \), or specifically: has a density 0 a.s. in \( \mathbb{N} \), i.e. \( |[N]_p|/N = 0 \), as \( N \to \infty \).

**A Necessary Condition for Solvability of the PAPR Reduction Problem and Arithmetic Progressions.** The existence of an arithmetic progression in a subset \( \mathcal{I} \subseteq \mathbb{N} \) allows us to give a more specific necessary condition for the solvability of the PAPR reduction problem for OFDM systems than that given in Thm. 4.10:

**Lemma 5.5.** Let be \( \mathcal{I} \subseteq \mathbb{N} \). Assume that there exists an arithmetic progression of length \( m \) in \( \mathcal{I} \). Then, if the PAPR reduction problem is solvable for \( \{(e_n)_{n \in \mathbb{N}} : \mathcal{I}\} \) with a given \( C_{Ex} > 0 \), it follows:

\[
C_{Ex} > \frac{1}{\pi^2} \frac{\sqrt{m}}{\log \left( \frac{m}{2} \right) + C}, \quad (21)
\]

for a fixed constant \( C > 0 \).
Proof. Consider the signal \( f = \sum_{k=0}^{m-1} \frac{1}{\sqrt{m}} e_n + dk \). It is obvious, that \( f \in \mathcal{F}^1(I) \). Further, we have the following observation:

\[
\| f \|_{L^2([0,1])} = 1, \quad \| f \|_{L^1([0,1])} < \frac{\pi}{2m} + C \sqrt{m},
\]

for some absolute constant \( C > 0 \). The equality above follows from the orthonormality of \( \{ e_n \}_{n \in \mathbb{N}} \), and the inequality follows from usual upper bound for Dirichlet kernel, respectively. Finally, by the assumption that PAPR reduction problem is solvable for \( (\{ e_n \}_{n \in \mathbb{N}}, I) \) with constant \( C_{Ex} \), Thm. 4.10, and the fact \( f \in \mathcal{F}^1(I) \), we have:

\[
1 = \| f \|_{L^2([0,1])} \leq C_{Ex} \| f \|_{L^1([0,1])} < C_{Ex} \frac{\pi}{2m} + C',
\]

as desired. \( \square \)

In particular, the above lemma gives an insight into the structure of PAPR reduction problem: It asserts, that a necessary condition for the solvability of the PAPR reduction problem for \( (\{ e_n \}_{n \in \mathbb{N}}, I) \), with a certain constant \( C_{Ex} \), is that \( I \) does not contain an arithmetic progression of arbitrary large length \( m \), otherwise, the right hand side of the inequality (21) would dominate \( C_{Ex} \). Further, to ensure that the above statement makes sense, we need to ensure the existence of an arithmetic progression of a given length. Szemerédi Thm. gives the remaining arguments.

**Theorem 5.6.** Given \( \delta \in (0,1) \) and \( m \in \mathbb{N} \), then there exists an \( N_{Sz} \in \mathbb{N} \), depending on \( \delta \) and \( m \), s.t. for all \( N \geq N_{Sz} \), the following holds:

If the PAPR reduction problem is solvable for \( (\{ e_n \}_{n \in \mathbb{N}}, I) \) with \( C_{Ex} > 0 \), where \( I \subset [N] \), with \( |I| \geq \delta N \), then:

\[
C_{Ex} > \frac{\sqrt{m}}{4\pi} \log \left( \frac{m}{\delta} \right) + C'
\]

for some \( C > 0 \).

**Proof.** By Thm. 5.3, we can fix the constant \( N_{Sz} \) depend on the choices of \( m \) and \( \delta \). Thus, in a subset \( I \subset [N] \), where \( N \geq N_{Sz} \), for which \( |I| \geq \delta N \), we can find an arithmetic progression of length \( m \). Correspondingly, Lem. 5.5 gives the remaining of the argument. \( \square \)

Given a desired \( C_{Ex} > 0 \). One may conclude from the above Theorem, that there is a restriction to the size of the information set such that the PAPR reduction problem is solvable. Notice that the statement given in the Theorem is somehow stronger: it gives a necessary condition for solvability of the PAPR reduction problem not only for a certain information set, but for all information set having density bigger than \( \delta \in (0,1) \) in \([N]\).
5.2. Probabilistic and Asymptotic Approach to the PAPR Problem of the OFDM. Recently, there are some interests aroused in the tightened and probabilistic version of Thm. 5.6. We have already mentioned in the previous subsection that Szemerédi Thm. is unsatisfactory in the asymptotic case, since it ensures the existence of an arithmetic progressions of arbitrary length only in sets of positive upper density in \( \mathbb{N} \). Furthermore, Green and Tao [11] showed the existence of a set with density zero in \( \mathbb{N} \), viz. the prime numbers, in which there are arithmetic progressions of arbitrary length.

As asserted by Conlon and Gowers [8], a set of density zero having arithmetic progressions of arbitrary length, can be constructed in a probabilistic way. By means of that result, we are able to give a negative statement about the solvability of the PAPR reduction problem with arbitrary extension constant in the asymptotic setting:

**Theorem 5.7.** Let be \( m \in \mathbb{N} \), and \( \delta \in (0,1) \). Given a constant \( C_{Ex} > 0 \). Then, there is a constant \( C \), s.t.: 

\[
\lim_{N \to \infty} \mathbb{P}(A_{N,m,p}) = 1, \quad \text{if } p > \frac{C}{N^{\frac{1}{m-1}}},
\]

where \( A_{N,m,p} \) denotes the event: "The PAPR problem is not solvable for \( \{e_n\}_{n \in \mathbb{N}}, \mathcal{I} \) with

\[
C_{Ex} \leq \frac{\sqrt{m}}{\pi^2 \log \left( \frac{\pi}{2} \right)} + C,
\]

where \( C > 0 \) is an absolute constant, for every subset \( \mathcal{I} \subset \mathbb{N} \) of size \( |\mathcal{I}| \geq \delta N \)."

**Proof.** Choose \( m \) sufficiently large, s.t. (22) does not hold. Further, choose \( p \in (0,1) \), s.t. \( p > CN^{-\frac{1}{m-1}} \), with a suitable constant \( C > 0 \). Thm. 5.4 asserts that the set \( \mathcal{I} \), resulted by choosing elements of \( [N] \) independently by probability \( p \), is a \( (\delta,m) \)-Szemerédi with probability tends to 1 as \( N \) tends to infinity. By the definition of \( (\delta,m) \)-Szemerédi, the choice of \( m \), and Lemma 5.5, the result follows immediately. \( \square \)

The point of the above Thm. is that a set, for which the PAPR problem is not solvable for every subset having a relative density bigger than a given number, might be a large set, but still a sparse set in \( \mathbb{N} \), since the probability \( p \) can be decreased toward 0 as \( N \) increases.

5.3. Further Discussions and Outlooks. We have already seen that that the PAPR reduction problem for \( \{e_n\}_{n \in \mathbb{N}}, \mathcal{I} \), for a fixed information set \( \mathcal{I} \subset \mathbb{N} \), is not solvable with arbitrary (small) extension constant \( C_{Ex} \), although infinite compensation set is allowed. This gives of course a restriction to the solvability of the PAPR reduction problem for fixed information set, finite compensation set, and with a given extension constant.

Also an interesting question is, how does the PAPR reduction problem for OFDM systems behaves with fixed threshold constant \( C_{Ex} \), if the information set
and the compensation set is finite. An answer was given in [6]. To discuss this, let us define the following quantity, which depends on \( N \in \mathbb{N} \) and \( C_{\text{Ex}} > 0 \):

\[
\mathcal{E}_N(C_{\text{Ex}}) := \max \left\{ |I| : I \subset [N], \text{ PAPR prob. is solv. for } \left\{ e_n \right\}_{n=1}^N, I \text{ with } C_{\text{Ex}} \right\}.
\]

Notice that in the above definition, we allow only finite compensation set. One result (Thm. 2) given in [6] is that the following limit holds:

\[
\lim_{N \to \infty} \frac{\mathcal{E}_N(C_{\text{Ex}})}{N} = 0,
\]

which says that if a given PAPR bound \( C_{\text{Ex}} \) is always satisfied, and if we allow only finite compensation set, then the proportion between the possible information set, for which the PAPR reduction problem is solvable, and the number of available tones, goes toward zero, as the latter goes toward infinity. Thus the size of that possible information set does not scale with \( N \). For practical insight, (24) should be not strictly as an asymptotic statement. Rather, (24) has to be understand as restrictions on the existence of any arbitrarily large OFDM system of number \( N \), for which solvability occurs for a certain information set \( I_N \subset [N] \) with density \( |I_N|/N \) in \( N \), and for a given \( C_{\text{Ex}} > 0 \).

In case that the information set \( I \) is infinite, and the compensation set is the remaining \( \mathbb{N} \setminus I \), and the extension constant \( C_{\text{Ex}} \) is fixed, one can give in some sense a quantitative statement: If \( \limsup_{N \to \infty} (|S(N)|/N) > 0 \), where \( S(N) := I \cap [N] \), then the PAPR Problem is not solvable for \( \left\{ e_n \right\}_{n \in \mathbb{N}}, I \). A corresponding proof can be found in [6] (Thm. 6).

6. Solvability of PAPR problem for CDMA

We have already seen in the previous section (Lemma 5.5), that the solvability of the PAPR reduction problem in the OFDM case for an information set \( I \) is connected to the existence of a certain combinatorial object, viz. arithmetic progression of a certain length in \( I \). For DS-CDMA systems, whose carriers are the so-called Walsh functions, we shall see, that the derivation of an easy-to-handle necessary condition for the solvability of the PAPR reduction problem, based on a slightly different technique. In particular, it does not depend on the existence of an arithmetic progression in the considered information set \( I \), rather on the so called existence of the optimal Walsh sum of a given length in \( \mathfrak{F}(I) \) (or with abuse of notations: the existence of the optimal Walsh sum of a given length in the information set \( I \)). Before we go into detail, let us first define the Walsh functions, which serve as the carriers for CDMA systems.

Definition 6.1 (Rademacher -, Walsh Functions). The Rademacher functions \( r_n \), \( n \in \mathbb{N} \), on \([0,1]\) are defined as the functions:

\[
r_n(\cdot) := \text{sign}[\sin(\pi 2^n \cdot)] \],
\]
where $\text{sign}$ denotes simply the signum function, with the convention $\text{sign}(0) = -1$. By means of the Rademacher functions, we can define the so called \textit{Walsh Functions} $w_n, n \in \mathbb{N}$, on $[0,1]$ iteratively by:

$$w_{2^k+m} = r_k w_m, \quad k \in \mathbb{N}_0 \text{ and } m \in [2^k],$$

where $w_1$ is given by $w_1(t) = 1, \quad t \in [0,1]$.

Notice that the indexing of the Walsh functions used in this work differs slightly with the usual indexing, since it begins by the index 1 instead of 0. The Walsh functions form a multiplicative group with the identity $w_1$. Furthermore, the Walsh functions are each self-inverse, i.e. $w_kw_k = w_1$, for every $k \in \mathbb{N}$, and form an orthonormal basis for $L^2([0,1])$. The orthonormality of Walsh functions asserts that for every $n \in \mathbb{N} \setminus \{1\}$, $w_n$ (which can be written as $w_n w_1$), it holds:

$$\int_0^1 w_n(t)dt = 0. \quad (25)$$

A more detailed treatment concerning to those issues, and further properties of the Walsh functions can be read in [10, 27, 12]

\textbf{6.1. Deterministic approach.} A result concerning to the behavior of tone reservation scheme for Walsh systems is given in the following Theorem:

\textbf{Theorem 6.2.} Given $\delta \in (0,1)$, and assume that $N := 2^n, n \in \mathbb{N}$ fulfills:

$$N \geq 3 \left(\frac{2}{\delta}\right)^{2^m} \text{ for some } m \in \mathbb{N}.$$ 

If the PAPR problem is solvable for $(\{w_n\}_{n \in \mathbb{N}}, \mathcal{I})$ with constant $C_{Ex}$, for a subset $\mathcal{I} \subset [N]$ having the density $|\mathcal{I}|/N \geq \delta$, then it holds:

$$C_{Ex} \geq 2^{\frac{\delta}{2}}.$$ 

We still have a long way to proof above the statement. But first, let us gives an easy implications of the above Thm. concerning to the solvability of PAPR reduction problem.

\textbf{Corollary 6.3.} Let be $N := 2^n, n \in \mathbb{N}$. Assume that:

$$N \geq 3 \left(\frac{2}{\delta}\right)^{2^m} \text{ for some } m \in \mathbb{N}, \text{ and } \delta \in (0,1).$$

Given a desired $C_{Ex} > 0$. If $C_{Ex} < 2^{\frac{\delta}{2}}$, then the PAPR problem for $(\{w_n\}_{n \in \mathbb{N}}, \mathcal{I})$, where $|\mathcal{I}| \geq \delta N$ is not solvable with constant $C_{Ex}$.

\textbf{Remark 6.4.} Notice that, for a suitable $N$, which represent the number of the available carriers in a considered CDMA system, and a given maximum peak value $C_{Ex}$, the above Corollary gives a restriction on the number $m$ of the information bearing coefficients such that the PAPR reduction problem is solvable.
The first step to give a proof of the above results is to give the following definitions:

**Definition 6.5.** Let be $I \subset \mathbb{N}$ finite and $r \in \mathbb{N}$. The correlation between $w_r$ and $I$ is defined as the quantity:

$$
\text{Corr}(w_r, I) = \int_0^1 w_r(t) \left| \sum_{k \in I} w_k(t) \right|^2 dt.
$$

Furthermore, for $w_r$, $r \neq 1$, and $I$, we define the following sets:

- $\mathcal{M}(w_r, I) := \left\{ k \in I : w_k w_{\tilde{k}} = w_r, \text{ for a } \tilde{k} \in I \right\}$
- $\overline{\mathcal{M}}(w_r, I) := \left\{ k \in I : w_k w_{\tilde{k}} = w_r, \text{ for a } \tilde{k} \in I, \text{ with } \tilde{k} > k \right\}$
- $\mathcal{M}(w_r, I) := \mathcal{M}(w_r, I) \setminus \overline{\mathcal{M}}(w_r, I)$

Notice that for each $k \in \mathcal{M}(w_r, I)$, there exists exactly one $\tilde{k} \in \mathcal{M}(w_r, I)$, for which the requirement given in the definition of $\mathcal{M}(w_r, I)$ holds. This observation gives immediately the facts, that in case $\mathcal{M}(w_r, I) \neq \emptyset$, $\mathcal{M}(w_r, I)$ is always of even cardinality, and $\mathcal{M}(w_r, I)$ and $\overline{\mathcal{M}}(w_r, I)$ are non-empty. Some other nice properties regarding to the correlation function, can easily be given ([7]):

**Lemma 6.6.** Let be $N = 2^n$, $n \in \mathbb{N}$, $I \subset [N]$, and $r \in [N]$. The following holds:

1. $|\mathcal{M}(w_r, I)| = 2 |\overline{\mathcal{M}}(w_r, I)| = 2 |\mathcal{M}(w_r, I)|$
2. $\text{Corr}(w_r, I) = |\mathcal{M}(w_r, I)|$
3. $\sum_{r=1}^N \text{Corr}(w_r, I) = |I|^2$
4. $\arg \max_{r \in [N] \setminus \{1\}} \text{Corr}(w_r, I) \geq (|I|^2 - |I|)/N$

We have already show the first item in the above Lemma. The proof of the 2. and 3. item can be found in [7] (Lemma 4.4 and Lemma 4.6). To show the last item, notice first that by the orthonormality of Walsh functions, $\text{Corr}(w_1, I) = |I|$ holds. Further, we have the following computations, which gives the desired statement:

$$
|I| + N \arg \max_{r \in [N] \setminus \{1\}} \text{Corr}(w_r, I) = \text{Corr}(w_1, I) + N \arg \max_{r \in [N] \setminus \{1\}} \text{Corr}(w_r, I)
\geq \sum_{r=1}^N \text{Corr}(w_r, I) = |I|^2
$$

Otherwise, suppose that there exists $\tilde{k}_1, \tilde{k}_2 \in \mathcal{M}(w_r, I)$, $\tilde{k}_1 \neq \tilde{k}_2$, for which it holds:

$$
w_{\tilde{k}_1} w_{\tilde{k}_2} = w_r = w_{\tilde{k}_2} w_{\tilde{k}_1}.
$$

Multiplying the above equation by $w_k$, and involving the fact that Walsh functions are self inverse, we obtain the contradiction, $w_{\tilde{k}_1} = w_{\tilde{k}_2}$ as desired.
where the first equality follows from previous observation, and the second equality from item 3 in the above lemma.

Now, we define the object, which plays an important role (as important as the arithmetic progression for the proof of Lemma 5.5 and Thm. 5.6), for the proof of Thm. 6.2, is defined in the following:

**Definition 6.7.** Let be $I \subset \mathbb{N}$, and $m \in \mathbb{N}$. We say that $F^1(I)$ contains an perfect Walsh sum (PWS) of the size $2^m$, if there exists $f \in F^1(I)$, which can be written in the forms:

$$f = w_{l_0} \prod_{n=1}^{m} (1 + w_{l_n}) = w_{l_0} (1 + \sum_{n=1}^{2^m - 1} w_{l_n})$$

(26)

for a $l_0 \in \mathbb{N}, l_1, \ldots, l_{2^m - 1} \in \mathbb{N} \setminus \{1\}$ mutually distinct, and $k_n \in \mathbb{N}$, for $n \in [m]$.

We call also the function (26) a perfect Walsh sum of the size $2^m$. With abuse of notation, we also say $I$ is a PWS of size $2^m$. Given a set $\tilde{I} \subset [N]$. We say $\tilde{I}$ contains a PWS of size $2^m$, if $\tilde{I}$ has a subset, which is also a PWS of size $2^m$. The adjective "perfect" is due to the factorability of the PWS. Further, the norms of PWS can be computed explicitly:

**Lemma 6.8.** Let $m \in \mathbb{N}$. For an optimal Walsh sum $f$ of the size $2^m$, it holds:

$$\|f\|_{L^1([0,1])} = 1 \quad and \quad \|f\|_{L^2([0,1])} = 2^{\frac{m}{2}}$$

*Proof.* Let $f$ be an PWS of size $2^m$, i.e. it has the representation (26). Then by computation, we have:

$$\|f\|_{L^1([0,1])} = \frac{1}{0} \int |f(t)| \, dt = \frac{1}{0} \int |w_{l_0}(t)| \left| \prod_{n=1}^{m} (1 + w_{l_n}(t)) \right| \, dt = \frac{1}{0} \prod_{n=1}^{m} (1 + w_{l_n}(t)) \, dt$$

$$= 1 + \sum_{k=1}^{2^m - 1} \int_0^1 w_{l_k}(t) \, dt = 1.$$

The 3. equality follows from the fact, that Walsh functions are always of modulus 1 and non-negative if added by 1, since they take values between $\{-1, +1\}$. The 4. inequality follows from (25). The second statement is also not hard to established. Indeed, by setting $l_0 = 1$, since the Walsh functions are of modulus 1, and the orthonormality of Walsh functions, we have:

$$\|f\|^2_{L^2([0,1])} = \frac{1}{0} \int |w_{l_0}(t)|^2 \left| \sum_{n=0}^{2^m - 1} w_{l_n} \right|^2 \, dt = \sum_{k,l=0}^{2^m - 1} \int_0^1 w_{l_k}(t)w_{l_k}(t) \, dt = 2^m.$$

$\square$
Remark 6.9. As the above Lemma asserts, the existence of PWS in an information set, allows one to give the (lowest) extension constant $C_{\text{ex}}$ for which the following embedding inequality:
\[
\|f\|_{L^2([0,1])} \leq C_{\text{ex}} \|f\|_{L^1([0,1])}, \quad \forall f \in \mathcal{F}^1(I),
\] (27) explicitly, in the following sense: Assume $I \subset \mathbb{N}$ is a PWS of size $2^m$, $m \in \mathbb{N}$. It can be shown, that $C_{\text{ex}} \leq \sqrt{|I|}$ (This inequality holds also, if the assumption that $I$ is a PWS is dropped, and if another ONS is considered, instead of Walsh functions), and correspondingly $C_{\text{ex}} \leq 2^{\frac{m}{2}}$. Further, by setting the corresponding PWS $f \in \mathcal{F}^1(I)$ into (27), and by involving lemma 6.8, one obtains $C_{\text{ex}} \geq 2^{\frac{m}{2}}$.

Thus $C_{\text{ex}} = 2^{\frac{m}{2}}$. In the subsequent work, we shall a slightly stronger statement: For a $I \subset [2^n]$, $n \in \mathbb{N}$, it holds $I$ is a PWS if and only if it holds for the minimum constant $C_{\text{ex}}$ in (27), $C_{\text{ex}} = \sqrt{|I|} = 2^{\frac{m}{2}}$, for a $m \in \mathbb{N}$.

The most important step for proving Thm. 6.2 is given in the following statement:

Theorem 6.10. Let be $N = 2^n$, $n \in \mathbb{N}$, and $\delta \in (0,1)$. Then, for every subset $I \subset [N]$ fulfilling:
\[
|I| \geq \delta N \quad \text{and} \quad |I| \geq 3 \left( \frac{2}{\delta} \right)^{2^m-1},
\]
for an $m \in \mathbb{N}$, $\mathcal{F}^1(I)$ contains a perfect Walsh sum of size $2^m$, or more explicitly, $I$ contains a PWS.

The proof of the above Thm. shall be given in the subsequent work. In particular, it is based on Lemma 6.6 and basic properties of Walsh sums. A construction of such an object, provided that the assumption in the above Thm. is fulfilled, can explicitly be given. Roughly, above Thm. says that if an information set $I$ is large enough, then it has a subset, which is a PWS. Now we are to give the desired result:

Proof of Thm. 6.2. Let be $\delta$, $N$, $m$ as required in the Theorem. Take an $I \subset [N]$ having density in $[N]$ bigger than $\delta$. Assume that the PAPR problem is solvable for $(\{w_n\}_{n \in \mathbb{N}}, I)$, with a desired extension constant $C_{\text{ex}} > 0$. By Thm. 4.10, we know that:
\[
||f||_{L^2([0,1])} \leq C_{\text{ex}} \|f\|_{L^1([0,1])}, \quad \forall f \in \mathcal{F}^1(I).
\] (28)
Further by the assumption that $|I| \geq \delta N$, and $N \geq (3/2)(2/\delta)^{2^m}$, which assert that $|I| \geq 3(2/\delta)^{2^m-1}$, Thm. 6.10 asserts that $I$ contains a PWS of size $2^m$. Let denote the corresponding sum $f \in \mathcal{F}^1(I)$, by $f$. Noticing that $f$ have the norms as given in Lemma 6.8 and setting those to (28), $2^{\frac{m}{2}} \leq C_{\text{ex}}$ has to hold.

6.2. Probabilistic approach. In spirit of Thm. 5.7, we aim to give some asymptotic statements regarding to the solvability of the PAPR reduction problem. The first step to establish such a statement, is to give the following remark:
Remark 6.11. Let $m \in \mathbb{N}$, and $N \in \mathbb{N}$ be at first fixed. Notice that by Thm. 6.10, a sufficient condition for $\delta \in (0, 1)$, s.t. $\mathcal{I}$ contains a PWS of size $2^m$, where $\mathcal{I} \subset [N]$ is a subset, having the density $|\mathcal{I}|/N$ in $[N]$ bigger, i.e. $|\mathcal{I}| \geq \delta N$, is that:

$$\delta N \geq 3 \left( \frac{2}{\delta} \right)^{2^m-1}.$$  \text{(29)}

since the left side is monotone increasing, and the right side monotone decreasing with $\delta$, it follows that there exists $\delta_N \in (0, 1)$, s.t. we have equality in (29), with $\delta = \delta_N$. Some elementary computation yields:

$$\delta_N = 2 \left( \frac{3}{2N} \right)^{\frac{1}{2^m}}.$$  \text{(30)}

Of course, $m$ and $N$ has to be chosen appropriately, s.t. $\delta_N \in (0, 1)$. In case that this issue has been considered, it is obvious that for all subsets $\mathcal{I} \subset [N]$ having density $\delta$ in $[N]$ bigger that $\delta_N$, $\mathcal{I}$ contains a PWS of size $2^m$.

Let $N \in \mathbb{N}$, and $p \in [0, 1]$. $[N]_p$ denotes again the random subset of $[N]$, in which each element is chosen independently from $[N]$ by $p$. By means of Thm. 6.10 and Remark 6.11, the following statement, which gives a probabilistic construction of a sparse set $\mathcal{I}$, with density zero (a.s.) in $N$, for which $\mathcal{I}$ contains a PWS of an arbitrary size, can be established:

**Theorem 6.12.** Let be $m \in \mathbb{N}$. Then there is a sequence $\{p_N\}$, with $N$ large enough, in $(0, 1]$ tending to zero, for which it holds:

$$\lim_{N \to \infty} \mathbb{P}[|N|_{p_N} \text{ contains a PWS of size } 2^m] = 1$$

*Proof.* For $m \in \mathbb{N}$, and $\tau > 1$, choose $N \in \mathbb{N}$ large enough, s.t.:

$$p_N := \tau \delta_N \in (0, 1)$$

where $\delta_N$ is given by (30). Note that $|[N]_{p_N}|$ is binominal distributed. Correspondingly, we have $\mathbb{E}[|N|_{p_N}] = \tau \delta_N N$. We may give the estimate:

$$\mathbb{P}[|N|_{p_N} < \delta_N N] = \mathbb{P}\left[|N|_{p_N} < \frac{1}{\tau} \mathbb{E}[|N|_{p_N}]\right] \leq \exp\left(-\frac{(\tau-1)^2 \mathbb{E}[|N|_{p_N}]}{2}\right)$$

$$= \exp\left(-CN^{2m-1}\right) \xrightarrow{N \to \infty} 0, \quad C := \left(\frac{3}{2}\right)^{\pi^2} \frac{(\tau-1)^2}{\tau},$$  \text{(31)}

where the estimation is based on Chernoff bound. Thus, we have:

$$\mathbb{P}[|N|_{p_N} \text{ contains a PWS of size } 2^m] = \mathbb{P}[|N|_{p_N} \geq \delta_N N]$$

$$= 1 - \mathbb{P}[|N|_{p_N} < \delta_N N] \xrightarrow{n \to \infty} 1,$$

by (31). Clearly, $p_N$ tends to zero as $N \to \infty$, as desired. 

\qed
In analogy to Thm. 5.4, and as a tightening of the above Thm., we have the following statement:

**Theorem 6.13.** Let \( m \in \mathbb{N} \), and \( \delta \in (0,1) \). Then there is a sequence \( \{p_N\} \), with \( N \) large enough, in \( (0,1] \), tending to zero, for which it holds:
\[
\lim_{N \to \infty} \mathbb{P} [A_{N,m,\delta}] = 1,
\]
where \( A_{N,m,\delta} \) denotes the event:
\[
A_{N,m,\delta} := \{ \forall I \subset [N]_{p_N}, \ |I| \geq \delta \cdot |[N]_{p_N}| : \text{I contains a PWS of size } 2^m \}.
\]

**Proof.** For an \( p_N \in (0,1] \), define another event \( A_{N,m,\delta} \) by:
\[
A_{N,m,\delta} := \left\{ |[N]_{p_N}| \geq \frac{\delta_N N}{\delta} \right\},
\]
where \( \delta_N \) is given by (30). Obviously, we have by Rem. 6.11, \( A_{N,m,\delta} \subset A_{N,m,\delta} \), which gives:
\[
\mathbb{P} [A_{N,m,\delta}] \geq \mathbb{P} [A_{N,m,\delta}] . \tag{32}
\]

Now, for certain \( m \in \mathbb{N} \), \( \delta > 0 \), and arbitrary \( \tau > 1 \), choose \( N \in \mathbb{N} \) sufficiently large, s.t.:
\[
p_N := \frac{\tau \delta_N}{\delta} \in (0,1)
\]
\(|[N]_{p_N}|\) is binomial distributed, with expectation \( \tau (\delta_N/\delta)N \), and by computation similar to (31), we have:
\[
\mathbb{P} [A_{N,m,\delta}] \leq \exp \left( -CN \frac{2^m-1}{\tau} \right) \xrightarrow{N \to \infty} 0, \quad C := \left( \frac{3}{2} \right)^{\frac{1}{2}} \frac{(\tau-1)^2}{\tau} \frac{1}{\delta}.
\]
Together with (32), the desired statement is shown. \( \square \)

Now, we are ready to give the desired applications of previous Theorems to the solvability of PAPR reduction problems:

**Theorem 6.14.** Let be \( m \in \mathbb{N} \). Given an extension constant \( C_{Ex} > 0 \), with \( C_{Ex} < 2^m \). Then there exists a sequence \( \{p_N\} \) in \( (0,1] \), with \( N \) large enough, tending to zero, s.t.:
\[
\lim_{N \to \infty} \mathbb{P} \left[ \text{The PAPR problem is not solvable for } (\{w_n\}_{n \in \mathbb{N}}, [N]_{p_N}) \text{ with } C_{Ex} \right] = 1
\]

**Proof.** Choose \( m \in \mathbb{N} \). Let \( C_{Ex} \) be given, with \( 2^m > C_{Ex} \). Further, construct the sequence \( \{p_N\}_{N \in \mathbb{N}} \) in \( (0,1] \) tending to zero s.t. the statement Thm. 6.12 holds. Thus, we know that the probability of \( [N]_{p_N} \) containing a PWS of size \( 2^m \) tends to 1 as \( N \) goes to infinity. Notice that for each of such events, a corresponding PWS \( f \) has the norms \( \| f \|_{L^1([0,1])} = 1 \) and \( \| f \|_{L^2([0,1])} = 2^m \). By the assumption on \( C_{Ex} \), we have a function in \( \mathfrak{F}^1([N]_{p_N}) \), for which (13) does not hold. In this case, Thm. 4.10 asserts that the PAPR reduction problem is not solvable, as desired. \( \square \)
Theorem 6.15. Let be $m \in \mathbb{N}$. Given an extension constant $C_{Ex} > 0$, with $C_{Ex} < 2^{\frac{m}{2}}$, and $\delta > 0$. Then there exists a sequence $\{p_N\}_{N \in \mathbb{N}}$ in $[0, 1]$ tending to 0, for which it holds:

$$\lim_{N \to \infty} \mathbb{P}[B_{N, \delta}] = 1,$$

where $B_{N, \delta}$ denotes the event:

"The PAPR problem is not solvable for all $(\{w_n\}_{n \in \mathbb{N}}, I)$ with $C_{Ex}$, where $I \subset [N]_{p_N}$, $|I| \geq \delta |[N]_{p_N}|$.

Proof. Choose $m \in \mathbb{N}$. Let $C_{ex}$ be given, with $2^{\frac{m}{2}} > C_{Ex}$. Take the sequence $\{p_N\}_{N \in \mathbb{N}}$ in $(0, 1]$, for which the statement in Thm. 4.10 holds. For each $N \in \mathbb{N}$, consider the event $A_{N, m, \delta}$. For $[N]_{p_N}$ in this event, every subset $I \subset [N]_{p_N}$, having the density $|I| / |[N]_{p_N}|$ contains a PWS of size $2^{\frac{m}{2}}$. By similar argument made in the previous Thm., and by the assumption made for $m$, we find a function $f \in \mathbb{F}^I(\mathbb{Z})$ harming (13). Thus in this case, PAPR reduction is not solvable. As $p_N$ tending to 0 and $\mathbb{P}[A_{N, m, \delta}]$ tending to 1, the desired statement is obtained.

6.3. On the Perfect Walsh Sums and an Erdős Conjecture. We have already seen in this section, and in the Section 5, that the existence of a certain combinatorial object in the considered information set $I$, allows us to turn Thm. 4.10, which gives a necessary condition for the solvability of PAPR reduction problem, into another Theorems, which is easy to handle, such as Thm. 5.6 and Thm. 6.2. If the considered information set is infinite, we have already seen, that by probabilistic method, it is possible to construct the corresponding combinatorial object, having the desired property, i.e. sparsity in the natural number, for both Fourier - and Walsh case (Thm. 5.4, Thm. 6.12, and Thm. 6.15). However to find such combinatorial objects with sparsity constraint deterministically is not easy. Concerning to arithmetic progressions, one of the famous conjecture of Erdős, says that a set of positive integer $A$, satisfying a certain constraint on its size, contains arbitrarily long arithmetic progressions:

**Conjecture 6.16 (Erdős Conjecture on Arithmetic Progressions).** Let be $A \subset \mathbb{N}$, for which $\sum_{n \in A} \frac{1}{n} = \infty$. Then $A$ contains arbitrarily long arithmetic progressions. Specifically: For each $m \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$, such that $A \cap [n_0]$ contains an arithmetic progression of length $m$.

Thus, if that Conjecture holds true, a subset of $\mathbb{N}$ which is not too small but sparse contains arbitrarily long arithmetic progressions. The conjecture remains unsolved. However, it is due to Green and Tao, that the set of prime numbers contained arbitrarily long arithmetic progressions, of density 0 in $\mathbb{N}$ (see Section 5.1). Further, it is clear that the set of primes numbers satisfies the condition given in the previously mentioned Erdős conjecture. For recent reports concerning to recent problem concerning to the previously mentioned Erdős conjecture, we refer to [13].

For the combinatorial object, which is important for the Walsh case, i.e. the subsets of the natural numbers forming PWS, things turn out differently:
Theorem 6.17 (Solution of Erdős Problem for Walsh sums). Let be $I \subseteq \mathbb{N}$, for which it holds:
\[ \sum_{k \in I} \frac{1}{k} = \infty. \] (33)

Then $I$ contains a PWS of arbitrary size. Specifically: For each $m \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$, such that $I \cap [2^{n_0}]$ contains a PWS of size $m$.

To proof the above Thm., we need the following statement:

Lemma 6.18. For every $m \in \mathbb{N}$, there exists an $n_0 \in \mathbb{N}$, such that for every dyadic number $N$, i.e., $N = 2^\tilde{n}$, for a $\tilde{n} \in \mathbb{N}$, fulfilling $N \geq 2^{n_0}$, it follows that for every subset $I \subseteq [N]$, having the cardinality:
\[ |I| \geq \frac{N}{(\log N)^2}, \] (34)
$I$ contains a PWS of the size $2^m$.

Proof. Let $m \in \mathbb{N}$ be fixed, and $n_0$ the smallest natural number, for which the following holds:
\[ 0 < 2 \left( \frac{3}{2 \cdot 2^{n_0}} \right) \frac{1}{2^m} < 1 \quad \text{and} \quad \frac{2^{n_0}}{(\log 2^{n_0})^2} \geq 2 \left( \frac{3}{2 \cdot 2^{n_0}} \right) \frac{1}{2^m} \frac{2^{n_0}}{2^{n_0}}. \]

Notice that $N$ is always of the form $N = 2^\tilde{n}$, $\tilde{n} \in \mathbb{N}$. Since $N \geq 2^{n_0}$, it holds for $I \subseteq [N]$, fulfilling (34):
\[ |I| \geq \frac{N}{(\log N)^2} \geq \frac{2^{n_0}}{(\log 2^{n_0})^2} \geq 2 \left( \frac{3}{2 \cdot 2^{n_0}} \right) \frac{1}{2^m} \frac{2^{n_0}}{2^{n_0}}. \]

Accordingly by (30), $I$ contains an PWS of size $2^m$ as desired. \qed

So, now we are ready to give the corresponding proof:

Proof of Thm. 6.17. For $r \in \mathbb{N}$, define $\delta_I(r)$ by:
\[ \frac{|I \cap [r]|}{r} =: \delta_I(r), \]
and for $r = 0$, $\delta_I(0) = 0$. Notice that by means of $\delta_I$, we can write:
\[ \sum_{r \in I} \frac{1}{r} = \sum_{r=1}^{\infty} \frac{\delta_I(r-1)}{r}. \] (35)

We continue by the following computations:
\[ \sum_{r=1}^{\infty} \frac{\delta_I(r-1)}{r} = \sum_{r=2}^{\infty} \frac{\delta_I(r)}{r+1} = \sum_{l=1}^{2^{i+1}-1} \frac{\delta_I(r)}{r+1} \leq \sum_{l=1}^{\infty} \frac{1}{2^i} \sum_{r=2^i}^{2^{i+1}-1} \delta_I(r) \]
\[ = \sum_{l=1}^{\infty} \frac{1}{2^i} \sum_{r=2^i}^{2^{i+1}-1} \frac{|I \cap [r]|}{r} \leq \sum_{l=1}^{\infty} \frac{1}{2^i} \frac{|I \cap [2^i]|}{r} \sum_{r=2^i}^{2^{i+1}-1} \frac{1}{r}, \] (36)
where the second equality follows from the segmentation of the sum into parts. Using the estimation \( (1/r) < \int_{r-1}^{r}(1/x)dx \), we can compute:

\[
\sum_{r=2^l}^{2^{l+1}-1} \frac{1}{r} < \int_{2^l-1}^{2^{l+1}-1} \frac{dx}{x} = \log \left( \frac{2^{l+1}-1}{2^l-1} \right) = \log \left( 1 - \frac{1}{2^l} \right) \leq C_1,
\]

where \( C_1 > 0 \) is a constant which does not depend on \( l \). Setting the above estimation to (36), we have:

\[
\sum_{r=1}^{\infty} \frac{\delta_T(r-1)}{r} \leq C_1 \sum_{l=1}^{\infty} \frac{|I \cap [2^{l+1}]|}{2^l} = 2C_1 \sum_{l=1}^{\infty} \frac{|I \cap [2^{l+1}]|}{2^{l+1}} = 2C_1 \sum_{l=1}^{\infty} \delta_T(2^{l+1}).
\]

Now, we claim that there exist infinitely many numbers \( t_1, t \in \mathbb{N} \), for which it holds:

\[
\delta_T(2^{l+1}) > \frac{1}{(\log(2^{l+1}))^2}. \tag{37}
\]

Otherwise, we have \( \delta_T(2^{l+1}) > (1/(\log(2^{l+1}))^2) \), for all \( t \in \mathbb{N} \), which gives the contradiction:

\[
\infty = \sum_{r=1}^{\infty} \frac{\delta_T(r-1)}{r} \leq 2C_1 \sum_{l=1}^{\infty} \frac{1}{(l+1)^2(\log 2)^2} < \infty,
\]

where the equality follows from the assumption (33) and (35), and the last inequality follows from the finiteness of \( C_1 \) and the usual convergence of geometric series. Thus (37) holds.

For the last step of the proof, let \( m \in \mathbb{N} \) be arbitrary. Let \( n_0 = n_0(m) \in \mathbb{N} \) be the number, s.t. for every dyadic \( N \) with \( N \geq 2^{n_0} \), every subset \( A \subset [N] \) having the cardinality \( |A| \geq N/(\log N)^2 \) contains a PWS of size \( 2^n \), as asserted by Lemma 6.18. By (37), we can find an \( t_0 \in \mathbb{N} \), with \( t_0 \geq n_0 \), for which \( |I \cap [2^t]| > N/(\log N)^2 \), which gives the remaining clue, that \( I \cap [2^n] \) contains a PWS of size \( 2^n \), as desired.

As an immediate consequence of Theorem 6.17, we have the following statement, which is analogous to Green and Tao’s Thm. on the existence of arithmetic progressions of arbitrarily long in the set of prime numbers:

**Corollary 6.19.** Let \( \mathcal{P} \subset \mathbb{N} \) denotes the set of prime numbers. Then, \( \mathcal{P} \) contains an PWS of arbitrary length, i.e. for every \( m \in \mathbb{N} \), there exists \( n_0 \in \mathbb{N} \), s.t. \( \mathcal{P} \cap [2^{n_0}] \) contains a PWS of size \( 2^n \).

**Proof.** Since \( \sum_{k \in \mathcal{P}} k^{-1} = \infty \), Thm. 6.17 gives the remaining argument. \( \square \)

### 6.4 Further Discussion and Outlooks

From Thm. 6.10, one can also answer the question, whether the PAPR reduction problem is solvable for \( \{w_n\}_{n \in \mathbb{N}} \) with \( C_{\text{Ex}} > 0 \), where \( I \subset \mathbb{N} \) is infinite: Let \( \{k_l\}_{l \in \mathbb{N}} \) be an enumeration of \( I \), \( m \in \mathbb{N} \) be arbitrary but firstly fixed. Further choose \( N \in \mathbb{N} \) large enough, s.t. \( \delta_N \) given in
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(30) takes value in $(0, 1)$, and correspondingly compute the value $\lceil \delta_N N \rceil$. Now by Rem. 6.11, we can find a PWS of size $2^m$ in $\{k_l\}_{l=1}^{\lceil \delta_N N \rceil}$, since $\{k_l\}_{l=1}^{\lceil \delta_N N \rceil} \geq \delta_N N$.

Previous observation asserts immediately, that there exists a PWS of size $2^m$ in $I$, since $\{k_l\}_{l=1}^{\lceil \delta_N N \rceil} \subset I$. Now, for an $C_{\text{Ex}}>0$, choose $m \in \mathbb{N}$ large enough, s.t. $2^m > C_{\text{Ex}}$, and observe that by the procedure given previously, there is a PWS of size $2^m$ in $\mathfrak{B}(I)$. By the choice of $m$, the existence of a PWS of size $2^m$. Lemma 6.8, and Thm. 4.10, it follows immediately that the PAPR reduction is not solvable for $(\{w_n\}_{n \in \mathbb{N}}, I)$ for arbitrarily chosen $C_{\text{Ex}}$, and hence for any $C_{\text{Ex}}$.

In case that the information set $I$ are dyadic integers, i.e. $I = \{2^k\}_{k=1}^K$, for a $K \in \mathbb{N}$, one can expect in some sense a positive result: It was shown in [7] (Thm. 4.13), there is some constant $C_{\text{Ex}}$, s.t. the PAPR reduction problem is solvable for $(\{w_n\}_{n \in \mathbb{N}}, \{2^k\}_{k=1}^K)$. In this case one can even find a finite compensation set, explicitly $\{2^K\} \backslash \{2^k\}_{k=1}^K$. A possible extension constant, is the constant $B_1$, for which the upper Khintchine’s Inequality [35] (I.B.8) holds:

$$\left\| \sum_{k=1}^{K} a_k r_k \right\|_{L^2([0,1])} \leq B_1 \left\| \sum_{k=1}^{K} a_k r_k \right\|_{L^1([0,1])},$$

for all sequences $\{a_k\}_{k=1}^K$.

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