Two-block Springer fibers and Springer representations in types C & D

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$GL_m(\mathbb{C})$-conjugacy classes of nilpotent endomorphisms of $\mathbb{C}^m$

partitions λ of m

1:1

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Definition (Springer fiber of type A)

$x\colon \mathbb{C}^m \to \mathbb{C}^m$ nilpotent endomorphism of Jordan type λ

$$\mathcal{B}^\lambda_{GL_m} = \{\{0\} \subsetneq F_1 \subsetneq F_2 \subsetneq \ldots \subsetneq F_m = \mathbb{C}^m \mid xF_i \subseteq F_{i-1}\}$$

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Theorem (Springer, 1978)

There exists a graded S_m-action on $H^*(\mathcal{B}^\lambda_{GL_m}, \mathbb{C})$ such that $H^{\mathrm{top}}(\mathcal{B}^\lambda_{GL_m}, \mathbb{C})$ is the irreducible S_m-representation labeled by λ. This yields a correspondence

$$\mathrm{Irr}^{\mathrm{f.d.}}_{\mathbb{C}}(S_m) \xrightarrow{1:1} \{\text{nilpotent endomorphisms of } \mathbb{C}^m\} \Big/ GL_m(\mathbb{C}).$$

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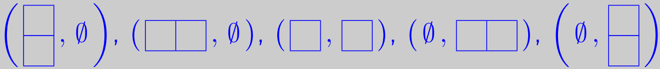

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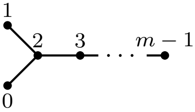
relations: $(s_i s_j)^{m_{ij}} = e$

$$m_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 3 & \text{if } i \text{ — } j \text{ in} \\ 2 & \text{else.} \end{cases}$$



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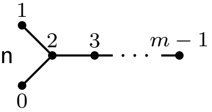

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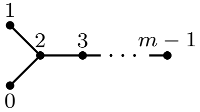

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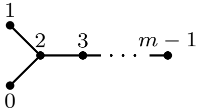

Theorem (Lusztig, 2004)

There exists an isomorphism of $\mathbb{C}[\mathcal{W}_{D_m}]$-modules

$$\underbrace{H^*(\mathcal{B}^{m,m}_{\mathrm{SO}_{2m}}, \mathbb{C})}_{\text{Springer representation}} \cong \underbrace{\mathbb{C} \otimes_{\mathbb{C}[S_m]} \mathbb{C}[\mathcal{W}_{D_m}]}_{\text{induced trivial module}}.$$

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- **Infinite-dimensional representation theory of Lie algebras.**

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Can we explicitly compute the $\alpha_{\lambda,\mu}$?

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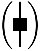
$\{\text{Kazhdan-Lusztig basis } \underline{b}_\mu\} \xrightarrow[\cong]{\psi}$ { cup diagrams on m vertices, $\#$ () $+ \#$ () even }

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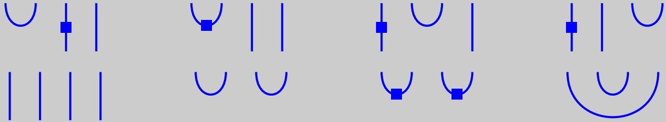

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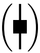

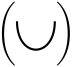

Example: (m=4)

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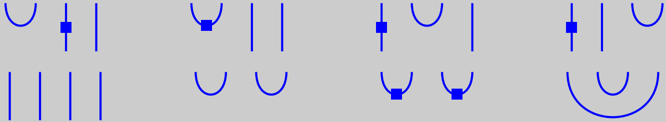
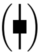
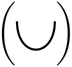


Theorem (Lejczyk–Stroppel, 2013)

$$\alpha_{\lambda,\mu} = \begin{cases} 1\,, & \text{if } \begin{array}{c} \phi(b_\lambda) \\ \psi(\underline{b}_\mu) \end{array} \text{ oriented,} \\ 0\,, & \text{else.} \end{cases}$$

$$\{\text{standard basis } b_\lambda\} \xrightarrow[\cong]{\phi} \left\{\begin{array}{l}\{\wedge,\vee\}\text{-sequences,}\\ \text{length } m,\ \#(\wedge) \text{ even}\end{array}\right\}$$

$$\underline{b}_\mu = \sum_\lambda \alpha_{\lambda,\mu} b_\lambda$$

$\{\text{Kazhdan-Lusztig basis } \underline{b}_\mu\} \xrightarrow[\cong]{\psi}$ { cup diagrams on m vertices, $\#$  $+\ \#$  even } $= C_{\mathrm{KL}}(m)$

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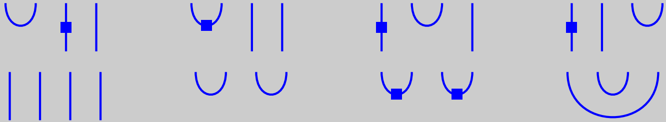
$\vee\vee\vee\vee$, $\vee\vee\wedge\wedge$, $\vee\wedge\vee\wedge$, $\vee\wedge\wedge\vee$, $\wedge\vee\vee\wedge$, $\wedge\vee\wedge\vee$, $\wedge\wedge\vee\vee$, $\wedge\wedge\wedge\wedge$



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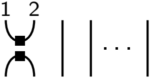
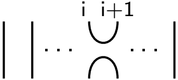
$e_0 = s_0 - 1 =$ 

$e_i = s_i - 1 =$ 

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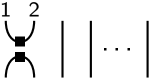
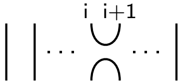
$e_0 = s_0 - 1 =$ 

$e_i = s_i - 1 =$ 

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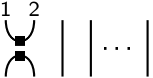
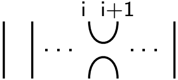
$e_0 = s_0 - 1 =$ 

$e_i = s_i - 1 =$ 

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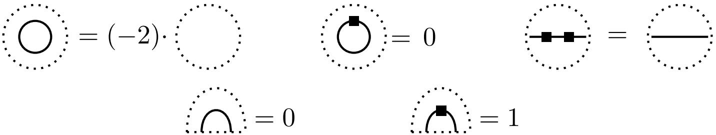

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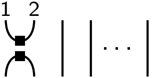
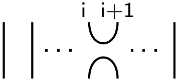
$e_0 = s_0 - 1 =$ 

$e_i = s_i - 1 =$ 

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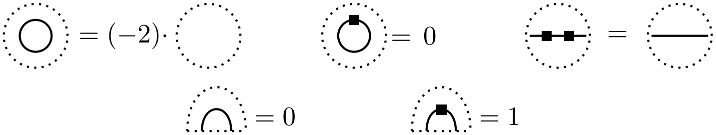
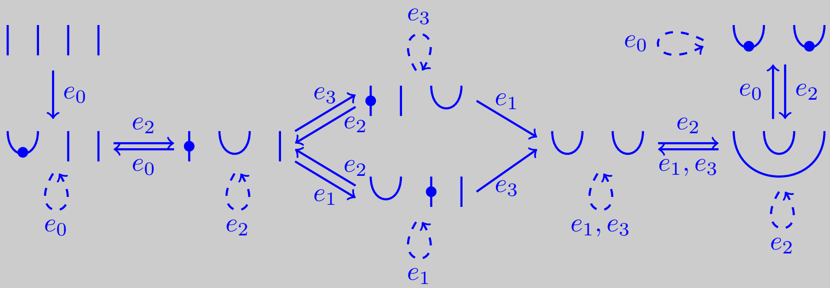


Example: $(m = 4)$



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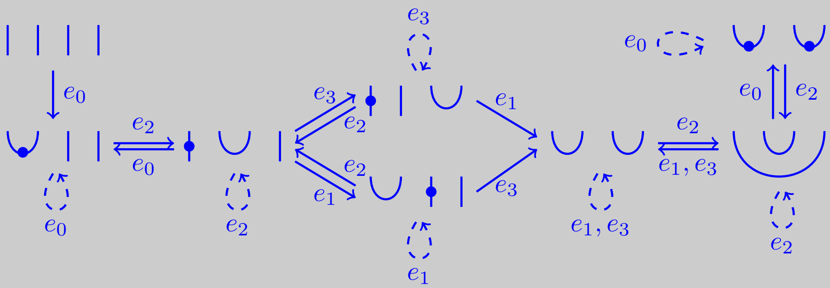

Remark

There exists a filtration

$$\{0\} \subseteq \mathbb{C}[C_{\mathrm{KL}}(m)]_{\lfloor \frac{m}{2} \rfloor} \subseteq \ldots \subseteq \mathbb{C}[C_{\mathrm{KL}}(m)]_n \subseteq \ldots \subseteq \mathbb{C}[C_{\mathrm{KL}}(m)]_0 = \mathbb{C}[C_{\mathrm{KL}}(m)]$$

of $\mathbb{C}[\mathcal{W}_{D_m}]$-modules, where

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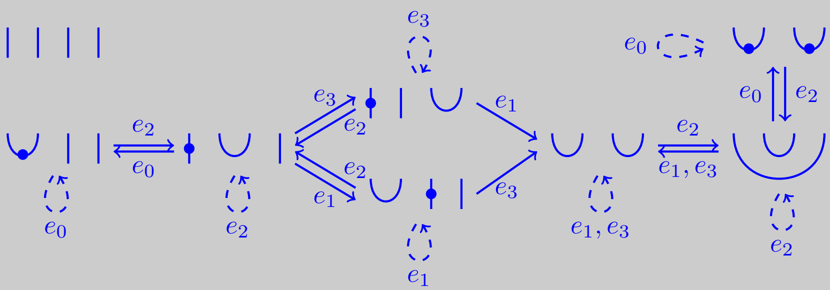

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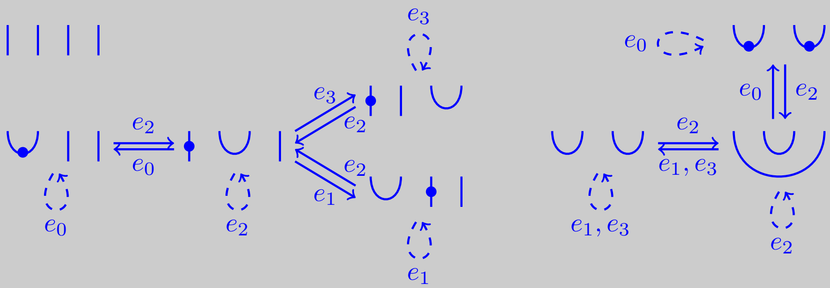

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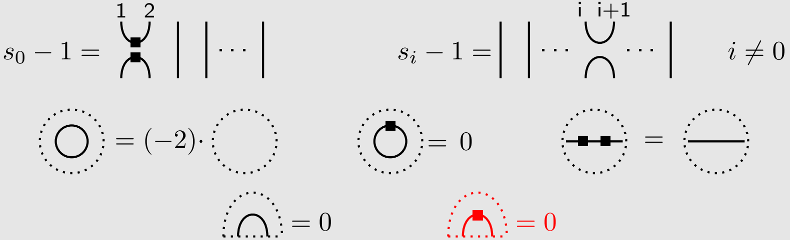
The subquotients

$$\mathbb{C}[C_{\mathrm{KL}}(m)]_n \Big/ \mathbb{C}[C_{\mathrm{KL}}(m)]_{n+1} = \mathrm{span}_{\mathbb{C}}\{[\mathbf{a}] \mid \mathbf{a} \in C_{\mathrm{KL}}(m)\,,\ \#(\text{cups}) = n\}$$

are irreducible $\mathbb{C}[\mathcal{W}_{D_m}]$-modules with $\mathbb{C}[\mathcal{W}_{D_m}]$-action given by:



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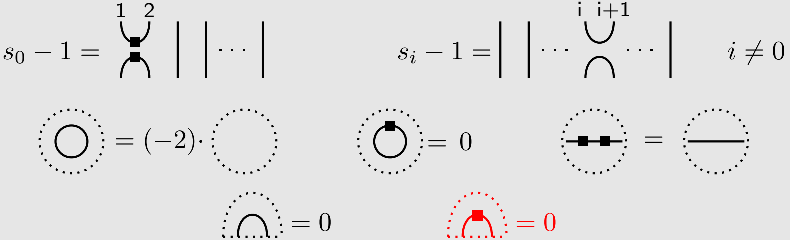

Moreover, we have an isomorphism of $\mathbb{C}[\mathcal{W}_{D_m}]$-modules

$$H^*(\mathcal{B}^{m,m}_{\mathrm{SO}_{2m}}, \mathbb{C}) \cong \bigoplus_n \mathbb{C}[C_{\mathrm{KL}}(m)]_n \Big/ \mathbb{C}[C_{\mathrm{KL}}(m)]_{n+1}\,.$$

Question

Is the cup diagram combinatorics describing $H^*(\mathcal{B}^{m,m}_{SO_{2m}}, \mathbb{C})$ already visible on the space $\mathcal{B}^{m,m}_{SO_{2m}}$? Does this tell us anything about the topology of $\mathcal{B}^{m,m}_{SO_{2m}}$ (or even more generally $\mathcal{B}^{2m-k,k}_{SO_{2m}}$)?

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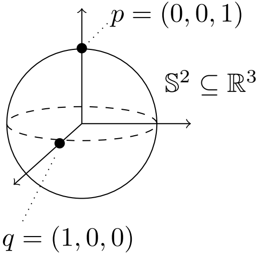

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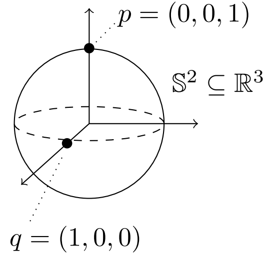

$C_{\mathrm{KL}}(m)$

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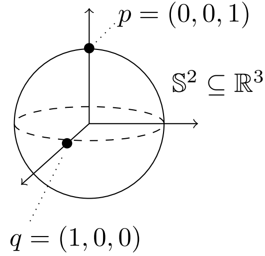

$$\underbrace{C_{2m-k,k}(m)}_{\text{diagrams with } \lfloor \frac{k}{2} \rfloor \text{ cups}} \subseteq C_{\mathrm{KL}}(m)$$

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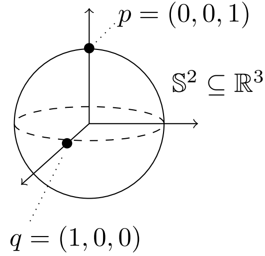

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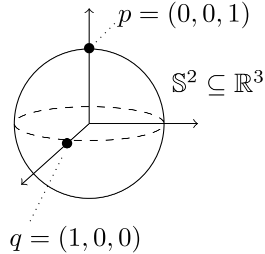

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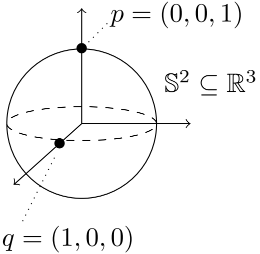

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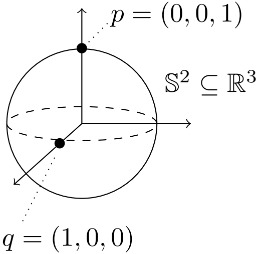

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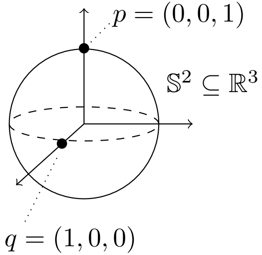

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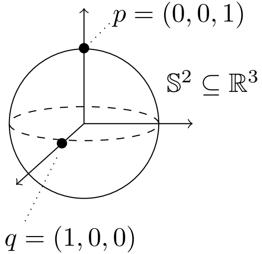

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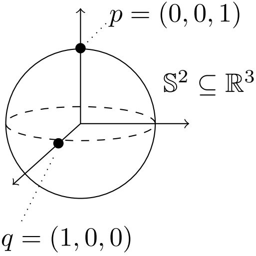

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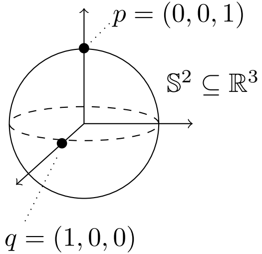

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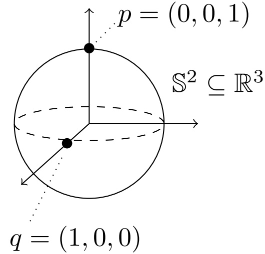

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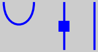
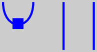
$\mathbf{a} =$ 

$\mathbf{b} =$ 

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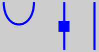
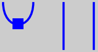
$\mathbf{a} =$  $\mathbf{b} =$  $\mathbf{c} =$  $\mathbf{d} =$ 

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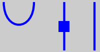
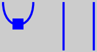
$\mathbf{a} =$  $\mathbf{b} =$  $\mathbf{c} =$  $\mathbf{d} =$ 

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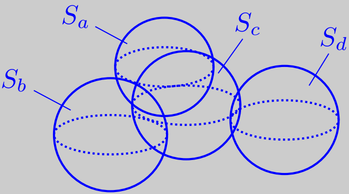
$\mathcal{S}^{5,3}_{SO_8} \cong$ 

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$$\begin{array}{ccccc} & & \text{"(Stroppel-Webster)"} & & \\ & \mathcal{P}erv\left(\Upsilon^D_m\right) & \dashrightarrow & D^b\left(\mathsf{Coh}(Y^D_m)\right) & \\ \text{(Ehrig-Stroppel)} & \uparrow\sim & & \sim\,\vdots\uparrow & \text{(Li)} \\ & \mathcal{P}erv\left(\Upsilon^B_{m-1}\right) & \dashrightarrow & D^b\left(\mathsf{Coh}(Y^C_{m-1})\right) & \end{array}$$

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$$H_*(\mathcal{S}^{m,m}_{SO_{2m}},\mathbb{C}) \circlearrowleft \mathbb{C}[\mathcal{W}_{D_m}]$$

(The component group for $\mathcal{B}^{m,m}_{SO_{2m}}$ is trivial.)

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Theorem (2016)

We have an isomorphism of $\mathbb{C}[\mathcal{W}_{D_m}]$-modules

$$H_{2n}(\mathcal{S}^{m,m}_{SO_{2m}},\mathbb{C}) \xrightarrow{\cong} \mathbb{C}[C_{\mathrm{KL}}(m)]_n \Big/ \mathbb{C}[C_{\mathrm{KL}}(m)]_{n+1}\,,\ [C_{\mathbf{a}}] \mapsto [\mathbf{a}].$$

In particular,

$$H_*(\mathcal{S}^{m,m}_{SO_{2m}},\mathbb{C}) \cong \mathbb{C}[C_{\mathrm{KL}}(m)] \cong \mathbb{C}\otimes_{\mathbb{C}[S_m]}\mathbb{C}[\mathcal{W}_{D_m}] \cong H^*(\mathcal{B}^{m,m}_{SO_{2m}},\mathbb{C}).$$

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$\mathbf{a} \in C_{\mathrm{KL}}(m)$, $1 < i$ connected by a cup, leftmost ray in $\mathbf{a}$ connected to vertex j

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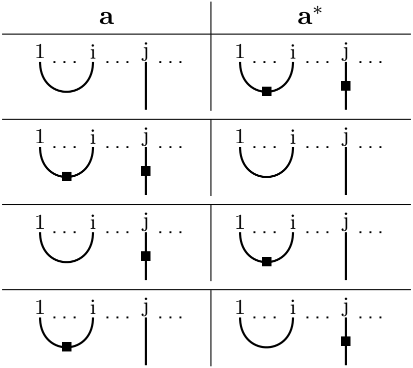

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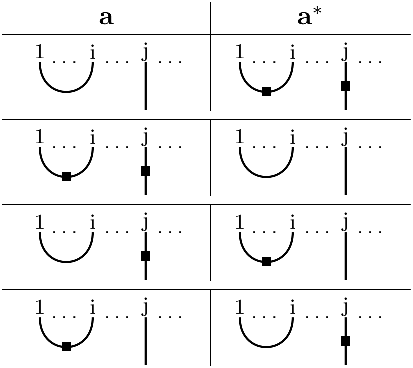

Theorem (2016)

The $\mathbb{Z}/2\mathbb{Z}$-action on $H_*(\mathcal{S}^{m,m}_{SO_{2m}}, \mathbb{C})$ (m odd) is given by

$$(-1) \cdot [C_{\mathbf{a}}] = \begin{cases} [C_{\mathbf{a}}] & \text{if } 1 \text{ is connected to a ray,} \\ [C_{\mathbf{a}^*}] & \text{if } 1 \text{ is conntected to a cup.} \end{cases}$$