Homological mirror symmetry for not-so-simple singularities

Yankı Lekili

based on joint work with Kazushi Ueda

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History

Theorem. (Futaki-Ueda '2009) ¹ There is a quasi-equivalence of pre-triangulated categories

\[ \mathcal{W}(\mathbb{C}^2, \Lambda\{x^3+y^2=-\infty\}) \simeq D^b(A) \simeq \text{mf}(\mathbb{C}^2, \Gamma, x^3 + y^2) \]

\[ \mathcal{A} : \quad \bullet \xrightarrow{s} \bullet \quad |s| = 0 \]

\[ \Gamma := \{(t_1, t_2) \in \mathbb{G}_m^2 : t_1^3 = t_2^2\} \simeq \mathbb{G}_m \text{ acts by } t \cdot (x, y) = (t_1x, t_2y). \]

¹We stated a very special case of their result which applies to any Sebastiani-Thom sum of type A and type D polynomials. (ex. \(x^3 + xy^2 + z^2, \ x^2 + y^3 + z^7 + w^{42}\))

Special cases of the B-model were studied by Takahashi '2005, Ueda '2006
History

**Theorem. (L. - Perutz '2010)** We have a quasi-equivalence of pre-triangulated categories (over any commutative ring)

\[ \mathcal{F}(\tilde{x}^3 + \tilde{y}^2 = 1) \simeq \text{perf}(x^3 + y^2 + xyz = 0) \]

\[ \mathbb{C}^2 \subset \subset \mathbb{P}(2, 3, 1) \]

\[ \text{end}(a \oplus b) \simeq \mathcal{B} \simeq \text{end}(\mathcal{O} \oplus \mathcal{O}_{\{z=0\}}) \]
History

\[ A : \bullet \xrightarrow{a s} \bullet \quad |s| = 0 \]

\[ B : \bullet \xleftrightarrow{a s \ t} \bullet \quad |s| = 0, \ |t| = 1, \ sts = tst = 0 \]

\[ m_3(s, t, st) = -st \]
\[ m_3(t, s, ts) = ts \]
\[ m_3(t, s, t) = t \]
\[ m_k = 0 \text{ for } k \neq 2, 3. \]

\[ \mathcal{W} := B! \quad (A_\infty \text{ Koszul dual}) \]

\[ \mathcal{W}(\bar{x}^3 + \bar{y}^2 = 1) \simeq \text{perf} \mathcal{W} \simeq \text{coh}(x^3 + y^2 + xyz = 0) \]
In fact, we have a diagram of quasi-equivalences.

\[ \mathcal{W}(\mathbb{C}^2, \Lambda\{\tilde{x}^3 + \tilde{y}^2 = -\infty\}) \xrightarrow{\sim} \text{mf}(\mathbb{C}^2, \mathbb{G}_m, x^3 + y^2) \]

\[ \xrightarrow{\sim} \mathcal{F}(\tilde{x}^3 + \tilde{y}^2 = 1) \xrightarrow{\sim} \text{mf}_0(\mathbb{C}^3, \mathbb{G}_m, x^3 + y^2 + xyz) \]

\[ \xrightarrow{\sim} \mathcal{W}(\tilde{x}^3 + \tilde{y}^2 = 1) \xrightarrow{\sim} \text{mf}(\mathbb{C}^3, \mathbb{G}_m, x^3 + y^2 + xyz) \]

\[ \xrightarrow{\sim \text{Orlov}} \text{coh}(x^3 + y^2 + xyz = 0) \]
Invertible polynomials (Berglund-Hübsch)

A weighted homogeneous polynomial \( w \in \mathbb{C}[x_1, \ldots, x_{n+1}] \) with an isolated critical point at the origin is invertible if there is an integer matrix \( A = (a_{ij})_{i,j=1}^{n+1} \) with non-zero determinant such that

\[
w = \sum_{i=1}^{n+1} \prod_{j=1}^{n+1} x_j^{a_{ij}}. \tag{1}
\]

The transpose of \( w \) is defined as

\[
\tilde{w} = \sum_{i=1}^{n+1} \prod_{j=1}^{n+1} \tilde{x}_j^{a_{ji}}. \tag{2}
\]

For example, the transpose of

\[x^{n-1} + xy^2 + z^2 \]

is \( \tilde{x}^{n-1} \tilde{y} + \tilde{y}^2 + \tilde{z}^2 \).
Invertible polynomials
-HMS conjecture

The group

$$\Gamma_w := \{(t_0, t_1, \ldots, t_{n+1}) \in (\mathbb{G}_m)^{n+2} | t_1^{a_1,1} \cdots t_{n+1}^{a_1,n+1} = \cdots = t_1^{a_{n+1,1}} \cdots t_{n+1}^{a_{n+1,n+1}} = t_0 t_1 \cdots t_{n+1}\}$$

acts naturally on $$\mathbb{C}^{n+2} := \text{Spec} \mathbb{C}[x_0, \ldots, x_{n+1}]$$.

$$\text{mf}(\mathbb{C}^{n+2}, \Gamma_w, w + x_0 \cdots x_{n+1})$$ denote the idempotent completion of the dg category of $$\Gamma_w$$-equivariant coherent matrix factorizations of $$w + x_0 \cdots x_{n+1}$$ on $$\mathbb{C}^{n+2}$$.
Conjectures (L.-Ueda ’2018)

For any invertible polynomial $w$, one has a diagram of quasi-equivalences:

$$
\begin{align*}
&\mathcal{W}(\mathbb{C}^{n+1}, \Lambda\{\tilde{w}^{-1} = -\infty\}) \quad \sim \quad m\mathfrak{f}(\mathbb{C}^{n+1}, \Gamma_w, w) \\
&\quad \downarrow \partial \\
&\quad \mathcal{F}(\tilde{w} = 1) \quad \sim \quad m\mathfrak{f}_0(\mathbb{C}^{n+2}, \Gamma_w, w + x_0 \cdots x_{n+1}) \\
&\quad \downarrow \partial \\
&\quad \mathcal{W}(\tilde{w} = 1) \quad \sim \quad m\mathfrak{f}(\mathbb{C}^{n+2}, \Gamma_w, w + x_0 \cdots x_{n+1})
\end{align*}
$$

$n = 1$ case is known by (J.)Smith-Habermann ’2019, Habermann ’2020.
Matrix factorizations

-ex ample

By a matrix factorization, we mean a pair \((f, g)\) of matrices with entries in \(\mathbb{C}[x_0, x_1, \ldots, x_n]\) such that

\[
f \cdot g = (w + x_0x_1 \ldots x_{n+1})Id \quad \text{and} \quad g \cdot f = (w + x_0x_1 \ldots x_{n+1})Id
\]

The matrix factorization associated with the structure sheaf of the critical locus of \(f(x_0, x_1, x_2, x_3) = x_1^{n+1} + x_2^2 + x_3^2 + x_0x_1x_2x_3\) can be computed as

\[
f = \begin{pmatrix}
x_1^n & -x_2 & x_0x_1x_2 + x_3 & 0 \\
-x_2 & -x_1 & 0 & x_0x_1x_2 + x_3 \\
x_3 & 0 & -x_1 & x_2 \\
0 & x_3 & x_2 & x_1^n
\end{pmatrix},
\]

\[
g = \begin{pmatrix}
x_1 & -x_2 & x_0x_1x_2 + x_3 & 0 \\
-x_2 & -x_1^n & 0 & x_0x_1x_2 + x_3 \\
x_3 & 0 & -x_1^n & x_2 \\
0 & x_3 & x_2 & x_1
\end{pmatrix}.
\]
Problems

- How do we compute $F(\tilde{w} = 1)$?
- How do we compute $W(\tilde{w} = 1)$?
One (our) approach

- How do we compute $\mathcal{F}(\check{w} = 1)$?

  Moduli of $A_\infty$ structures - introduced to HMS in L.-Perutz, generalized substantially by Polishchuk, L.-Polishchuk in relation to moduli of curves. On a more abstract level (deformation theory) goes back to Kontsevich, Seidel, ...

- How do we compute $\mathcal{W}(\check{w} = 1)$?

  Koszul duality for Fukaya categories - introduced in Etg"u-L., generalized in Ekholm-L. On a more abstract level goes back to Keller, ...

There are other approaches. I am aware of at least three groups working on these conjectures. Gammage-(J.)Smith, Polishchuk-Varolg"unes, Cho-Choa-Jeong.
Simple singularities

Consider a Kleinian singularity

\[ \tilde{w}_Q(x, y, z) = \begin{cases} 
  x^{n+1} + y^2 + z^2 & Q = A_n \\
  x^{n-1} + xy^2 + z^2 & Q = D_n \\
  x^4 + y^3 + z^2 & Q = E_6 \\
  x^3 + xy^3 + z^2 & Q = E_7 \\
  x^5 + y^3 + z^2 & Q = E_8 
\end{cases} \]

Simple singularities are stabilizations of Kleinian singularities.
HMS for simple singularity

If $w$ is a polynomial defining a simple singularity ($n > 1$), then

$$w + x_0x_1 \ldots x_{n+1} \text{ and } w$$

are right-equivalent. By a theorem of Orlov, this implies

$$\text{mf}(\mathbb{C}^{n+2}, \Gamma_w, w + x_0 \cdots x_{n+1}) = \text{mf}(\mathbb{C}^{n+2}, \Gamma_w, w)$$

**Theorem. (L.-Ueda ’2020)** If $w$ is a polynomial for a simple singularity,

$$\mathcal{W}(\ddot{w} = 1) \simeq \text{Perf}\mathcal{G}_{Q,n} \simeq \text{mf}(\mathbb{C}^{n+2}, \Gamma_w, w)$$

Here $\mathcal{G}_{Q,n}$ is the $n$–Calabi-Yau completion of the path algebra of a Dynkin quiver $Q$. 
Q Dynkin quiver, $A_Q$ path algebra.

Futaki-Ueda:

$$\mathcal{W}(\mathbb{C}^{n+1}, \Lambda_{\{\tilde{w}_Q = -\infty\}}) \simeq D^b(A_Q) \simeq mf(\mathbb{C}^{n+1}, \Gamma_w, w)$$

Push-forward to generators on each side and get to generators of

$$\mathcal{F}(\tilde{w} = 1) \text{ and } mf_0(\mathbb{C}^{n+2}, \Gamma_w, w)$$

$$B_{Q,n} = A_Q \oplus A^\vee_Q[-n], \quad (n > 1).$$

$$(a, f) \cdot (b, g) = (ab, ag + fb)$$

There is no $A_\infty$ deformations possible by computation of $HH^*(B_{Q,n})$. Finally, we have the Koszul dual DG-algebra

$$\mathcal{G}_{Q,n} := B_{Q,n}^!$$

and we can deduce

$$\mathcal{W}(\tilde{w} = 1) \simeq Perf\mathcal{G}_{Q,n} \simeq mf(\mathbb{C}^{n+2}, \Gamma_w, w)$$

by proving Koszul duality on both A- and B-sides.
Matrix factorizations
-Hochschild Cohomology

\[ V := \mathbb{C}x_0 \oplus \mathbb{C}x_1 \oplus \cdots \oplus \mathbb{C}x_{n+1}. \]

Then (by Dyckerhoff, Ballard-Favero-Katzarkov,…) we have

\[ \text{HH}^t(\mathfrak{m}f(\mathbb{C}^{n+2}, \Gamma, w)) \text{ is isomorphic to} \]

\[ \bigg( \bigoplus_{\gamma \in \ker \chi, \ l \geq 0 \atop t - \dim N_\gamma = 2u} H^{-2l}(d\omega_\gamma) \otimes \chi^{\otimes (u+l)} \otimes \Lambda^{\dim N_{\gamma}} N_{\gamma}^V \bigg) \]

\[ \bigg( \bigoplus_{\gamma \in \ker \chi, \ l \geq 0 \atop t - \dim N_\gamma = 2u+1} H^{-2l-1}(d\omega_\gamma) \otimes \chi^{\otimes (u+l+1)} \otimes \Lambda^{\dim N_{\gamma}} N_{\gamma}^V \bigg)^\Gamma. \quad (3) \]
Here $H^i(dw_\gamma)$ is the $i$-th cohomology of the Koszul complex

\[ C^*(dw_\gamma) := \{ \cdots \to \Lambda^2 V_\gamma^\vee \otimes \chi \otimes (-2) \otimes S_\gamma \to V_\gamma^\vee \otimes \chi^\vee \otimes S_\gamma \to S_\gamma \}, \]

where the rightmost term $S_\gamma$ sits in cohomological degree 0, and the differential is the contraction with

\[ dw_\gamma \in (V_\gamma \otimes \chi \otimes S_\gamma)^\Gamma. \]

The vector space $V_\gamma$ is the subspace of $\gamma$-invariant elements in $V$, $S_\gamma$ is the symmetric algebra of $V_\gamma$, $w_\gamma$ is the restriction of $w$ to $\text{Spec} S_\gamma$, and $N_\gamma$ is the complement of $V_\gamma$ in $V$ so that $V \cong V_\gamma \oplus N_\gamma$ as a $\Gamma$-module.
Hochschild Cohomology

-Here is an example

Let $w = x_1^3 + x_2^2 + x_3^2$. $\text{Jac}_w = \mathbb{C}[x_1]/(x_1^2)$.

We have $\Gamma = \{(t_0, t_1, t_2, t_3) : t_1^3 = t_2^2 = t_3^2 = t_0 t_1 t_2 t_3\}$.

$\chi = t_1^3 = t_2^2 = t_3^2 = t_0 t_1 t_2 t_3$.

We compute the summands of the formula (3) for each $\gamma \in \text{Ker} \chi$ and check directly that the only contributions are (for $m \in \mathbb{N}$)

$(1, 1, 1, 1) : \quad x_0^{6m}, \quad x_0^{4+6m} x_1 \in \text{HH}^{-4m}, \text{HH}^{-4m-2}$

$x_0^\vee x_0^{6m+1}, \quad x_0^\vee x_0^{5+6m} x_1 \in \text{HH}^{-4m+1}, \text{HH}^{-4m-1}$

$(1, 1, -1, -1) : \quad x_0^{3+6m} x_2 x_3^\vee, \quad x_0^{1+6m} x_1 x_2 x_3^\vee \in \text{HH}^{-4m-2}, \text{HH}^{-4m}$

$x_0^\vee x_0^{4+6m} x_2 x_3^\vee, \quad x_0^\vee x_0^{2+6m} x_1 x_2 x_3^\vee \in \text{HH}^{-4m-1}, \text{HH}^{-4m+1}$

$(e^{2\pi i/3}, e^{-2\pi i/3}, -1, -1) : \quad x_0^\vee x_1^\vee x_2^\vee x_3^\vee \in \text{HH}^2$

$(e^{-2\pi i/3}, e^{2\pi i/3} - 1, -1) : \quad x_0^\vee x_1^\vee x_2^\vee x_3^\vee \in \text{HH}^2$
Not-so-simple case

\[ \tilde{w} = x_1^{n+1} + x_2^{n+1} + \ldots + x_n^{n+1} \]

\[ A_{\tilde{w}} := \text{end}_{\mathcal{W}(\mathbb{C}^{n+1}, \Lambda_{\tilde{w}})}(\tilde{G}) \]

\[ \simeq \text{end}_{\mathcal{Mj}(\mathbb{C}^{n+1}, \Gamma_w, w)}(G) \]

\[ n = 4 \]
Moduli of $A_\infty$ structures

Let $B_{\tilde{w}} = \text{end}_{\mathcal{F}(\tilde{w}=1)}(\partial \tilde{G})$, and $B_{\tilde{w}} = H^*(B_{\tilde{w}})$.

Form a moduli space

$$U_\infty(B) = \{(B, m_\bullet, \iota) : m_1 = 0, \quad H^* \mathcal{B} \overset{\iota}{\sim} B\}/ \sim$$

$U_\infty(B)$ is represented by an affine scheme if $HH^1(B)_{<0} = 0$ and finite-type affine scheme if $\dim HH^2(B)_{<0} < \infty$.

On the B-side, let $U$ be the positive part of the base space of the semiuniversal unfolding of $w$ that respects $\Gamma$ action.

In our example, $U = \text{Spec}\mathbb{C}[u_1, u_{n+1}]$.

$$w_u = x_1^{n+1} + x_2^{n+1} + \cdots + x_n^{n+1} + u_1x_0x_1 \cdots x_{n+1} + u_{n+1}x_0^n$$
Moduli of $A_\infty$ structures

**Theorem. (L.-Ueda, 2018')** Let $w$ be an invertible polynomial (with some mild assumptions)

\[ U \rightarrow U_\infty(B) \]

\[ u \mapsto \text{end}_{\text{mf}}(\mathbb{C}^{n+2}, \Gamma, w_u)(\partial G) \]

is a $\mathbb{G}_m$-equivariant isomorphism of affine schemes, sending 0 to the formal $A_\infty$ structure.

This reduces determining the mirror of $B_{\tilde{w}}$ to a finite-dimensional problem. Locate the correct mirror by proving a conjecture of Seidel and then using a spectral sequence of McLean to determine explicitly

\[ HH^*(\mathcal{F}(\tilde{w} = 1)) \simeq SH^*(\tilde{w} = 1) \]

It turns out in the example considered, there is a unique $u$, namely $(u_1, u_{n+1}) = (1, 0)$ that matches $HH^*(\mathcal{F}(\tilde{w} = 1))$. 
An ordinary person forgets examples or is drowned in examples.
- Shihoko Ishii