Motivic MZV's and the cyclic insertion conjecture

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17 January 2018
HIM Periods Trimester, Bonn

Outline

1. Definitions and conjectures
2. Motivic MZV's and algebraic tools
3. Alternating block decomposition
4. Extra material (time permitting)

Definitions and conjectures

Mutiple zeta values

Definition (MZV)

Multiple zeta value $\zeta(s_1, s_2, \ldots, s_k)$ is defined by

$$\zeta(s_1, s_2, \ldots, s_k) \coloneqq \sum_{0<n_1<n_2<\cdots<n_k} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}}$$

- Where $s_i \geq 1 \in \mathbb{Z}$
- For convergence $s_k \geq 2$

Also define

- Weight is sum $s_1 + \cdots + s_k$ of arguments
- Depth is number k of arguments

MZV relations

MZV satisfy *lots* of relations

- Duality relations
- Associator relations
- Derivation relations
- (Extended) Double shuffle relations
- ...

Not always clear how to prove *explicit* relations from these.

Theme: progress towards and generalisation of some *explicit* conjectural families of identities

Zagier-Broadhurst Identity

Theorem (Zagier-Broadhurst, BBBL 2001)

For $n \geq 0 \in \mathbb{Z}$, have

$$\zeta(\{1,3\}^n) = \frac{1}{2n+1}\frac{\pi^{4n}}{(4n+1)!}$$

Proof (Sketch).

- Generalise to single variable *multiple polylogarithms*.
- Generating series satisfies a differential equation.
- Explicit solution in terms of ${}_2F_1$. Compare coefficients.

Combinatorial proofs have also been given. □

"Dressed with 2's"

Theorem (BBBL, 1998)

Let $n \geq 0 \in \mathbb{Z}$*, write*

$$I = \{ \text{ all } 2n+1 \text{ ways of inserting 2 into } \{1,3\}^n \ \} \ .$$

Then

$$\sum_{\mathbf{s} \in I} \zeta(\mathbf{s}) = \frac{\pi^{4n+2}}{(4n+3)!}$$

Example

For $n = 2$, have

$$\zeta(2,1,3,1,3) + \zeta(1,2,3,1,3) + \zeta(1,3,2,1,3) + \\ \zeta(1,3,1,2,3) + \zeta(1,3,1,3,2) = \frac{\pi^{10}}{11!}$$

Cyclic insertion conjecture

Numerical experimentation lead to conjectural generalisation.

Notation

Let $a_1, \ldots, a_{2n+1} \in \mathbb{Z}_{\geq 0}$. Write

$$Z(a_1, \ldots, a_{2n+1}) = \zeta(\{2\}^{a_1}, 1, \{2\}^{a_2}, 3, \ldots, 1, \{2\}^{a_{2n}}, 3, \{2\}^{a_{2n+1}})$$

Conjecture (Cyclic insertion - BBBL, 1998)

$$\sum_{\sigma \in \mathbb{Z}/n\mathbb{Z}} Z(a_{\sigma(1)}, \ldots, a_{\sigma(2n+1)}) \stackrel{?}{=} \frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!}$$

Shorthand: "wt" is weight of MZV's on the LHS

Special cases

- $n = 0$

$$\zeta(\{2\}^{a_1}) = \frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!} \qquad \checkmark$$

- $a_1 = \cdots = a_{2n+1} = 0$

$$(2n+1)\zeta(\{1,3\}^n) = \frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!} \qquad \checkmark$$

- $a_1 = 1, a_2 = \cdots = a_{2n+1} = 0$

Zagier-Broahurst dressed with 2's ✓

- $a_1 = \cdots = a_{2n+1} = m$

$$(2n+1)\zeta(\{\{2\}^m, 1, \{2\}^m, 3\}^n, \{2\}^m) \stackrel{?}{=} \frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!} \qquad ?$$

Previously conjectured by BBB (1997).

Bowman-Bradley

Best result so far is

Theorem (Bowman-Bradley, 2002)

Let $n, t \geq 0 \in \mathbb{Z}$, *then*

$$\sum_{\substack{a_1+\cdots+a_{2n+1}=t \\ a_i \geq 0}} Z(a_1, \ldots, a_{2n+1}) = \frac{1}{2n+1} \binom{t+2n}{t} \frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!}$$

Remark

Compatible with cyclic insertion: Any permutation of a composition $a_1 + \cdots + a_{2n+1} = t$ is still a composition.

Will use the motivic MZV framework to improve on this, up to $\mathbb{Q}$.

Hoffman's conjecture

Separate conjecture, with a similar flavour

Conjecture (Hoffman, MZV Infopage, 2000)

For $m \geq 0 \in \mathbb{Z}$,

$$2\,\zeta(3,3,\{2\}^m) - \zeta(3,\{2\}^m,1,2) \stackrel{?}{=} -\zeta(\{2\}^{m+3}) = -\frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!}$$

Remark

Verified up to weight 22, $m = 8$ using MZV datamine, Vermaseren (2009).

Will prove this up to $\mathbb{Q}$, using the motivic MZV framework.

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Unification and generalisation

Goal

Cyclic insertion and Hoffman are *special cases* of a more general (conjectural) family.

Can produce many new (conjectural) identities.

Example

$$\zeta(\{2\}^m,1,3,3,1,2)+\zeta(3,1,2,1,\{2\}^m,3)-\zeta(1,2,1,\{2\}^m,3,1,2)+$$
$$+\zeta(1,2,1,3,3,\{2\}^m)-\zeta(3,\{2\}^m,1,3,3)\overset{?}{=}\frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!}$$

(For above: $\in \pi^{\mathrm{wt}}\mathbb{Q}$ holds. Generally can use motivic MZV's to prove certain *symmetrised* versions, up to $\mathbb{Q}$.)

Motivic MZV's and algebraic tools

MZV's as iterated integrals

$$\zeta(s_1,\ldots,s_r) = (-1)^r I(0;\underbrace{1,0,\ldots,0}_{s_1},\ldots,\underbrace{1,0,\ldots,0}_{s_r};1)$$

where

$$I(a_0;a_1,\ldots,a_N;a_{N+1}) = \int_{a_0\leq t_1<\cdots<t_N\leq a_{N+1}} \frac{\mathrm{d}t_1}{t_1-a_1}\cdots\frac{\mathrm{d}t_N}{t_N-a_N}$$

Convergent if $a_1 \neq a_0$ and $a_N \neq a_{N+1}$

Properties

- $I(0;a_1,\ldots,a_N;0) = 0$ for $N \geq 1$ (Equal boundaries)
- $I(a_0;a_1,\ldots,a_N;a_{N+1}) = I(1-a_0;1-a_1\ldots,1-a_N;1-a_{N+1})$ (Functoriality)
- $I(a_0;a_1,\ldots,a_N;a_{N+1}) = (-1)^N I(a_{N+1};a_N,\ldots,a_1;a_0)$ (Reversal of paths)
- $I(a;w;b)I(a;v;b) = I(a;w ⧢ v;b)$ (Shuffle product)

Brown's motivic MZV's (See Winter school)

- Algebra $\mathcal{H}$ of motivic MZV's

$$\zeta^{\mathfrak{m}}(s_1, \ldots, s_r) := [\mathcal{O}(\pi_1^{\mathrm{un}}(\mathbb{P}^1 \backslash \{ 0, 1, \infty \}, \vec{1}_0, -\vec{1}_1)), \overbrace{\mathrm{dch}}^{\text{straight line}}, \underbrace{\Omega}_{\text{integrand}}]^{\mathfrak{m}} .$$

Contains all motivic iterated integrals

$$I^{\mathfrak{m}}(a_0; a_1, \ldots, a_N; a_{N+1}), a_i \in \{ 0, 1 \}$$

- Projection to algebra $\mathcal{A}$ of de Rham motivic MZV's

$$\zeta^{\mathfrak{a}}(s_1, \ldots, s_r) := [\mathcal{O}(\pi_1^{\mathrm{un}}(\mathbb{P}^1 \setminus \{ 0, 1, \infty \}, \vec{1}_0, -\vec{1}_1)), \underbrace{\varepsilon}_{\text{augmentation ideal}}, \Omega]^{\mathfrak{m}} ,$$

kernel generated by $\zeta^{\mathfrak{m}}(2)$.

- Coaction

$$\Delta \colon \mathcal{H} \to \mathcal{A} \otimes_{\mathbb{Q}} \mathcal{H}$$

lifts Goncharov's 'semicircular' coproduct on $\mathcal{A}$. $\mathcal{H}$ Hopf algebra comodule over $\mathcal{A}$.

Infinitesimal coproduct

Definition (Derivations D_k)

Let $\mathcal{L} := \mathcal{A}/(\mathcal{A}_{>0} \cdot \mathcal{A}_{>0})$, which kills products and $\zeta^{\mathfrak{m}}(2)$. For k odd define

$$\begin{aligned} D_k \colon \qquad & \mathcal{H} \to \mathcal{L}_k \otimes_{\mathbb{Q}} \mathcal{H} \\ & I^{\mathfrak{m}}(w) \mapsto (\pi \otimes \mathrm{id}) \circ (\Delta - 1 \otimes \mathrm{id})\, I^{\mathfrak{m}}(w) \end{aligned}$$

$$D_k I^{\mathfrak{m}}(a_0; a_1, \ldots, a_N; a_{N+1}) =$$

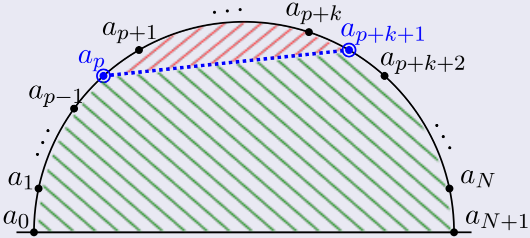
$$\sum_{p=0}^{N-k} I^{\mathfrak{L}}(a_p; a_{p+1}, \ldots, a_{p+k}; a_{p+k+1}) \otimes \qquad \rightsquigarrow \text{Subsequence}$$

$$I^{\mathfrak{m}}(a_0; a_1, \ldots, a_p, a_{p+k+1}, \ldots, a_N; a_N + 1) \qquad \rightsquigarrow \text{Quoient sequence}$$

Derivations D_k mnemonic

Mnemonic.

$$D_k I^{\mathfrak{m}}(\underbrace{w}_{(a;w';b)}) = \sum_{\substack{S \text{ subword } w,\\ \text{of length } k+2}} I^{\mathfrak{L}}(S) \otimes I^{\mathfrak{m}}(w - \text{interior } S)$$



$$\rightsquigarrow \quad I^{\mathfrak{L}}(a_p; a_{p+1}, \ldots, a_{p+k}; a_{p+k+1}) \otimes$$
$$I^{\mathfrak{m}}(a_0; a_1, \ldots, a_p, a_{p+k+1}, \ldots, a_N; a_N + 1)$$

Transcendental Galois Theory

Theorem (Brown, 2012)

Let $D_{<N} = \bigoplus_{1<2r+1<N} D_{2r+1}$. *In weight* N,

$$\ker D_{<N} = \zeta^{\mathfrak{m}}(N)\mathbb{Q}\,.$$

Example

Can show $\zeta^{\mathfrak{m}}(\{2\}^n) = \pm I^{\mathfrak{m}}(0; \underbrace{1,0,1,0,\ldots,1,0}_{n \text{ times}}; 1) \in \zeta^{\mathfrak{m}}(2n)\mathbb{Q}$

- Integral word alternates 0 and 1
- Odd length subsequence has same boundaries, vanishes
- Therefore all D_{2r+1} vanish

Conclude $\zeta^{\mathfrak{m}}(\{2\}^n) \in \ker D_{<2n} = \zeta^{\mathfrak{m}}(2n)\mathbb{Q}$.

$\zeta^{\mathfrak{m}}(\{1,3\}^n)$

More interesting: $\zeta^{\mathfrak{m}}(\{1,3\}^n) = I^{\mathfrak{m}}(0; (1100)^n; 1) \in \zeta^{\mathfrak{m}}(4n)\mathbb{Q}$

- Word has period 4, so length $1 \pmod 4$ subsequence vanish
- For length $3 \pmod 4$, look at starting position

$$1 \pmod 4: \qquad I^{\mathfrak{L}}(0; (1100)^a 1; 1) \otimes I^{\mathfrak{m}}((0110)^b 0 \mid 10(0110)^c 01)$$

$$2 \pmod 4: \qquad I^{\mathfrak{L}}(1; 1(0011)^a; 0) \otimes I^{\mathfrak{m}}((0110)^b 01 \mid 0(0110)^c 01)$$

- Cancel using reversal of paths in $I^{\mathfrak{L}}$. Similar for position $3, 4 \pmod 4$
- See cancellation as 'reversing' segments. Involution pairs up subsequences:

$$I^{\mathfrak{m}}(01 \mid 10 \mid \color{red}{0\,1 \mid 10 \mid \cdots \mid 10 \mid 01 \mid 1\,0} \mid 01)$$

Conclude $\zeta^{\mathfrak{m}}(\{1,3\}^n) \in \ker D_{<4n} = \zeta^{\mathfrak{m}}(4n)\mathbb{Q}$

Alternating block decomposition

Alternating blocks

Observation

In $\zeta^{\mathfrak{m}}(\{1,3\}^n)$ proof, points 00 and 11 in w are 'somehow' significant.

- Splitting here decomposes a word into *alternating blocks* $0101\cdots$ or $1010\cdots$.

Definition (Block decomposition)

Let w be a word starting with $\varepsilon_1 \in \{0, 1\}$. Write w as alternating blocks, with lengths $\ell_1, \ldots, \ell_k$. The block decomposition of w is

$$\mathrm{bl}(w) = (\varepsilon_1; \ell_1, \ldots, \ell_k) \, .$$

Example

$$\mathrm{bl}(\underbrace{0}_{1} \mid \underbrace{01}_{2} \mid \underbrace{10}_{2} \mid \underbrace{01010}_{5} \mid \underbrace{0}_{1} \mid \underbrace{01}_{2}) = (0; 1, 2, 2, 5, 1, 2)$$

Alternating blocks

Can recover w from $(\varepsilon_1; \ell_1, \ldots, \ell_k)$: blocks arise from $00 \to 0 \mid 0$ or $11 \to 1 \mid 1$.

Notation

Write $I_{\mathrm{bl}}(\varepsilon_1; \ell_1, \ldots, \ell_k) = I(\mathrm{bl}^{-1}(\varepsilon_1; \ell_1, \ldots, \ell_k))$. If $\varepsilon_1 = 0$, just write $(\ell_1, \ldots, \ell_k)$.

- Weight of $I_{\mathrm{bl}}(\varepsilon_1; \ell_1, \ldots, \ell_k)$ is $-2 + \sum_i \ell_i$. (Bounds of integration are counted!)
- If $\mathrm{wt} \equiv k \pmod 2$ then $I_{\mathrm{bl}} = 0$. (End points are equal!)
- I_{bl} is divergent iff $\ell_1 = 1$ or $\ell_k = 1$.

Example

$$I_{\mathrm{bl}}(1, 2, 2, 5, 1, 2) = I(0; 01100101000; 1)$$

Block structure of BBBL conjecture

- Write the BBBL identity as iterated integrals

$$\sum_{\text{cycle } a_i} \zeta(\{2\}^{a_1}, 1, \{2\}^{a_2}, 3, \ldots, 1, \{2\}^{a_{2n}}, 3, \{2\}^{a_{2n+1}})$$

$$\rightsquigarrow \pm \sum_{\text{cycle } a_i} I(0(10)^{a_1} 1 (10)^{a_2} 100 \cdots 01 (10)^{a_{2n}} 100 (10)^{a_{2n+1}} 1)$$

- Split into 'alternating blocks' at $00 \to 0 \mid 0$ or $11 \to 1 \mid 1$

$$= \pm \sum_{\text{cycle } a_i} I(0(10)^{a_1} 1 \mid (10)^{a_2} 10 \mid 0 \cdots 01 \mid (10)^{a_{2n}} 10 \mid 0(10)^{a_{2n+1}} 1)$$

- Record lengths of the blocks

$$= \pm \sum_{\text{cycle } a_i} I_{\mathrm{bl}}(2a_1 + 2, 2a_2 + 2, \ldots, 2a_{2n+1} + 2)$$

- Right hand side is $\zeta(\{2\}^{\mathrm{wt}/2}) = \pm I_{\mathrm{bl}}(\mathrm{wt} + 2)$.

Block structure of Hoffman's conjecture

- Write Hoffman's identity as iterated integrals

$$\begin{aligned} & 2\zeta(3,3,\{2\}^n) && -\zeta(3,\{2\}^n,1,2) \\ = & \zeta(3,3,\{2\}^n) && -\zeta(3,\{2\}^n,1,2) && +\zeta(\{2\}^n,1,2,1,2) \end{aligned}$$

$$\rightsquigarrow \pm \left(I(0100100(10)^n1) + I(0100(10)^n1101) + I(0(10)^n1101101) \right)$$

- Split into 'alternating blocks' at $00 \to 0 \mid 0$ or $11 \to 1 \mid 1$

$$\begin{aligned} = \pm \big(& I(010 \mid 010 \mid 0(10)^n1) + I(010 \mid 0(10)^n1 \mid 101) \\ & + I(0(10)^n1 \mid 101 \mid 101) \big) \end{aligned}$$

- Record lengths of the blocks

$$= \pm \left(I_{\mathrm{bl}}(3,3,2n+2) + I_{\mathrm{bl}}(3,2n+2,3) + I_{\mathrm{bl}}(2n+2,3,3) \right)$$

- Right hand side is $-\zeta(\{2\}^{n+3}) = \pm I_{\mathrm{bl}}(\mathrm{wt}+2)$

Common structure and generalisation

Both conjectures have same structure: cyclic permutations of block lengths ℓ_i.

Conjecture (Cyclic insertion, C., 2017, arXiv 1703.03784)

For any $(\ell_1, \ldots, \ell_k)$ *with all* $\ell_i > 1$,

$$\sum_{\text{cycle } \ell_i} I_{\text{bl}}(\ell_1, \ldots, \ell_k) \stackrel{?}{=} I_{\text{bl}}(\text{wt} + 2) = \begin{cases} \frac{\pi^{\text{wt}}}{(\text{wt}+1)!} & \text{wt } \textit{even} \\ 0 & \text{wt } \textit{odd} \end{cases}$$

- Numerically tested all cases weight ≤ 18, to 500 decimal places
- Can prove a symmetrised version, up to $\mathbb{Q}$
- Can prove *some* special cases, up to $\mathbb{Q}$

Examples

Example

Let $(\ell_1, \ldots, \ell_k) = (2m+2, 2, 3, 2, 3)$, then we obtain

$$\zeta(\{2\}^m, 1, 3, 3, 1, 2) + \zeta(3, 1, 2, 1, \{2\}^m, 3) - \zeta(1, 2, 1, \{2\}^m, 3, 1, 2) + \\ + \zeta(1, 2, 1, 3, 3, \{2\}^m) - \zeta(3, \{2\}^m, 1, 3, 3) \stackrel{?}{=} \frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!}$$

Proposition (C., 2017, arXiv 1703.03784)

The above identity holds up to $\mathbb{Q}$

Proof (Sketch).

Lift the identity to $\zeta^{\mathfrak{m}}$, and compute $D_{<2m+10}$. A (tedious) calculation shows $D_{<2m+10}$ vanishes. □

Progress and results

Theorem (Symmetric insertion, C., 2017, arXiv 1703.03784)

For any $(\ell_1, \ldots, \ell_k)$, *with even weight,*

$$\sum_{\text{permute } \ell_i} I_{\mathrm{bl}}(\ell_1, \ldots, \ell_k) \in I_{\mathrm{bl}}(\mathrm{wt} + 2)\mathbb{Q}$$

(Odd weight holds trivially, by duality)

Proof (Strategy).

- Lift to motivic version $I^{\mathfrak{m}}$.
- Define a reflection $\mathcal{R}$ on non-trivial subsequences
- Use $\mathcal{R}$ to cancel terms in $D_{<N}$
- Conclude $\in \zeta^{\mathfrak{m}}(\mathrm{wt})\mathbb{Q} = I^{\mathfrak{m}}_{\mathrm{bl}}(\mathrm{wt} + 2)\mathbb{Q}$ using Brown.

Progress and results

Proof (Details).

$$\mathcal{R}\colon I^{\mathfrak{m}}_{\mathrm{bl}}(\ell_1, \ldots, \overbrace{\underbrace{\ell_s}_{\text{start at position } \alpha}, \ldots, \underbrace{\ell_t}_{\text{end at position } \beta}}^{\text{non-trivial subsequence } S}, \ldots, \ell_k)$$

$$\mapsto I^{\mathfrak{m}}_{\mathrm{bl}}(\ell_1, \ldots, \overbrace{\underbrace{\ell_t}_{\text{start at position } \beta}, \ldots, \underbrace{\ell_s}_{\text{end at position } \alpha}}^{\text{reflection } \mathcal{R}S}, \ldots, \ell_k)$$

- Get permutation of ℓ_i.
- Both quotients are $I^{\mathfrak{L}}_{\mathrm{bl}}(\ell_1, \ldots, \ell_{s-1}, \alpha + \beta, \ell_{t+1}, \ldots, \ell_k)$
- Subsequences are

  $$I^{\mathfrak{m}}_{\mathrm{bl}}(\varepsilon; \ell_s - \alpha, \ell_{s+1}, \ldots, \ell_{t-1}, \ell_t - \beta) \text{ , and}$$
  $$I^{\mathfrak{m}}_{\mathrm{bl}}(\varepsilon'; \ell_t - \beta, \ell_{t-1}, \ldots, \ell_{s+1}, \ell_s - \alpha)$$
- Reverses or duals, differ by $(-1)^{\text{length}} = -1$. Cancel in $D_{<N}$ □

Corollaries of symmetric insertion

Corollary (Generalisation of Hoffman, up to $\mathbb{Q}$)

For $(\ell_1, \ell_2, \ell_3) = (2a+3, 2b+3, 2c+2)$, *we obtain*

$$\mathrm{Sym}_{a,b}\left(\zeta(\{2\}^a, 3, \{2\}^b, 3, \{2\}^c) - \zeta(\{2\}^b, 3, \{2\}^c, 1, 2, \{2\}^a)\right.$$
$$\left. + \zeta(\{2\}^c, 1, 2, \{2\}^a, 1, 2, \{2\}^b)\right) \in \pi^{\mathrm{wt}}\mathbb{Q}$$

Duality shows cyclic insertion already holds up to $\mathbb{Q}$

$$\zeta(\{2\}^a, 3, \{2\}^b, 3, \{2\}^c) - \zeta(\{2\}^b, 3, \{2\}^c, 1, 2, \{2\}^a)$$
$$+ \zeta(\{2\}^c, 1, 2, \{2\}^a, 1, 2, \{2\}^b)) \in \pi^{\mathrm{wt}}\mathbb{Q}$$

In particular, $a = b = 0$ is Hoffman's identity up to $\mathbb{Q}$.

Corollaries of symmetric insertion

Corollary (Improvement of Bowman-Bradley, up to $\mathbb{Q}$)

For $\ell_i = 2a_i + 2$, obtain

$$\sum_{\text{permute } a_i} \zeta(\{2\}^{a_1}, 1, \{2\}^{a_2}, 3, \ldots, 1, \{2\}^{a_{2n}}, 3, \{2\}^{a_{2n+1}}) \in \pi^{\mathrm{wt}}\mathbb{Q}$$

"Only need permutations of a single composition."

In particular, for $a_1 = \cdots = a_n = m$

Corollary (Evaluable MZV)

The following MZV is evaluable

$$\zeta(\{\{2\}^m, 1, \{2\}^m, 3\}^n, \{2\}^m) \in \pi^{\mathrm{wt}}\mathbb{Q}$$

Further progress?

Complete motivic proof of cyclic insertion is not (yet?) possible

- Cyclic insertion has a stability under D_k
- Odd weight implies $D_{<N}$(even weight) $= 0$
- Problem: Must fix rational multiple of $\zeta^{\mathfrak{m}}(\mathrm{wt})$ somehow
  $\rightsquigarrow$ analytically or numerically...
- $D_{<N}$(odd weight) involves $I^{\mathfrak{L}}$ explicitly

$$D_7 \sum_{\text{cycle}} I^{\mathfrak{m}}_{\mathrm{bl}}(2,10,3,2) = \underbrace{\left(I^{\mathfrak{L}}_{\mathrm{bl}}(6,3) + I^{\mathfrak{L}}_{\mathrm{bl}}(3,3,2,1) + I^{\mathfrak{L}}_{\mathrm{bl}}(2,3,2,3) + I^{\mathfrak{L}}_{\mathrm{bl}}(1,2,2,4)\right)}_{-\zeta^{\mathfrak{L}}(2)\zeta^{\mathfrak{L}}(2,3) - 2\zeta^{\mathfrak{L}}(2)\zeta^{\mathfrak{L}}(3,2) + 2\zeta^{\mathfrak{L}}(3)\zeta^{\mathfrak{L}}(2,2) = 0} \otimes I^{\mathfrak{m}}_{\mathrm{bl}}(10)$$

- In general only have

$$\text{odd weight} = \sum_k \alpha_k \zeta(2k+1)\zeta(\{2\}^{\mathrm{wt}/2-k}), \quad \alpha_k \in \mathbb{Q}$$

Recent work

Using iterated integrals over $\mathbb{P}^1 \setminus \{ \infty, 0, 1, z \}$ gives

Theorem (Hirose-Sato, 2017, arXiv 1704.06478)

The generalisation of Hoffman's identity holds exactly

$$\zeta(\{2\}^a, 3, \{2\}^b, 3, \{2\}^c) - \zeta(\{2\}^b, 3, \{2\}^c, 1, 2, \{2\}^a) + \zeta(\{2\}^c, 1, 2, \{2\}^a, 1, 2, \{2\}^b) = -\zeta(\{2\}^{a+b+c+3})$$

After the break:

- a further generalisation of cyclic insertion, and
- exact proofs!

Extra material (time permitting)

Full version of cyclic insertion

If some $\ell_i = 1$, the identity involves product term corrections.

$$\mathcal{L}_d = \{ (m_{d+1}, \ldots, m_k) \mid (\overbrace{1, \ldots, 1}^{d \text{ times}}, m_{d+1}, \ldots, m_k) \text{ is a cyclic permutation of } (\ell_1, \ldots, \ell_k) \}$$

"Take all cyclic permutations of $(\ell_1, \ldots, \ell_k)$ which start with d consecutive 1's. Then drop the initial 1's"

Conjecture (Cyclic insertion, C., 2017, arXiv 1703.03784)

For any $(\ell_1, \ldots, \ell_k)$ *of weight* N*,*

$$\sum_{\text{cycle } \ell_i} I_{\mathrm{bl}}(\ell_1, \ldots, \ell_k) \overset{?}{=} I_{\mathrm{bl}}(N+2) - \sum_{d=1}^{\lfloor k/2 \rfloor} \frac{2(2\pi \mathrm{i})^{2d}}{(2d+2)!} \sum_{\mathbf{m} \in \mathcal{L}_{2d}} I_{\mathrm{bl}}(\mathbf{m}) \,.$$

Full version of cyclic insertion

Example

With $(\ell_i) = (1,1,2,3)$, need only $\mathcal{L}_2 = \{\,(2,3)\,\}$. Get

$$I_{\mathrm{bl}}(1,1,2,3) + I_{\mathrm{bl}}(1,2,3,1) + I_{\mathrm{bl}}(2,3,1,1) + I_{\mathrm{bl}}(3,1,1,2)$$
$$\stackrel{?}{=} I_{\mathrm{bl}}(7) - \frac{2(2\pi \mathrm{i})^2}{4!} I_{\mathrm{bl}}(2,3)$$

Shuffle regularisation gives

$$\begin{aligned}&\big(3\zeta(1,1,3) + 2\zeta(1,2,2) + \zeta(2,1,2)\big) + \\ &\big(\zeta(2,3) - 6\zeta(1,1,3) - 4\zeta(1,2,2) - 2\zeta(2,1,2)\big) + \\ &\big(6\zeta(1,1,1,2)\big) + \big({-\zeta(5)}\big) \stackrel{?}{=} 0 + \zeta(2)\zeta(1,2)\end{aligned}$$ ✓

Another block decomposition conjecture

Conjecture (BBBL 1998, rewritten)

Let $a_1, a_2, a_3, b_1, b_2 \in \mathbb{Z}_{\geq 0}$. *Then*

$$\sum_{\sigma \in S_3} \operatorname{sgn}(\sigma) \zeta(\{2\}^{a_{\sigma(1)}}, 1, \{2\}^{b_1}, 3, \{2\}^{a_{\sigma(2)}}, 1, \{2\}^{b_2}, 3, \{2\}^{a_{\sigma(3)}}) \stackrel{?}{=} 0$$

Generalising the block decomposition structure leads to

Conjecture (Alt-odd, C., 2017, arXiv 1703.03784)

For any $(\ell_1, \ldots, \ell_{2k+1})$ *of even weight, with all* $\ell_i > 1$,

$$\mathrm{Alt}_{\{\ \ell_i \mid i \text{ odd } \}} I_{\mathrm{bl}}(\ell_1, \ldots, \ell_{2k+1}) \stackrel{?}{=} 0$$

"Alternating sum over odd-position blocks."

Remark

This conjecture is included in Hirose-Sato's generalisation too.

Another block decomposition conjecture

Example

For block lengths $\ell_i = 2a_i + 2$, $1 \leq i \leq 7$, get

$$\mathrm{Alt}_{a_1,a_3,a_5,a_7}\,\zeta(\{2\}^{a_1},1,\{2\}^{a_2},3,\{2\}^{a_3},1,\{2\}^{a_4},3,\{2\}^{a_5},1,\{2\}^{a_6},3,\{2\}^{a_7}) \stackrel{?}{=} 0$$

Example

For block lengths $(2a_1+3, 2a_2+3, 2a_3+3, 2a_4+2, 2a_5+3)$, get

$$\mathrm{Alt}_{a_1,a_3,a_5}\,\zeta(\{2\}^{a_1},3,\{2\}^{a_2},3,\{2\}^{a_3},3,\{2\}^{a_4},1,2,\{2\}^{a_5}) \stackrel{?}{=} 0$$

Analogue for Multiple Zeta Star Values

Definition (MZSV)

$$\zeta^\star(s_1, s_2, \ldots, s_k) \coloneqq \sum_{0<n_1\leq n_2\leq\cdots\leq n_k} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}}$$

Theorem (Yamamoto 2013, Conjectured by ITTW 2013)

$$\sum_{\sigma\in S_{2n}} \zeta^\star(1, \{2\}^{a_{\sigma(1)}}, 3, \{2\}^{a_{\sigma(2)}}, \ldots, 1, \{2\}^{a_{\sigma(2n-1)}}, 3, \{2\}^{a_{\sigma(2n)}}) \in \pi^{\mathrm{wt}}\mathbb{Q}$$

$$\sum_{\sigma\in S_{2n+1}} \zeta^\star(\{2\}^{a_{\sigma(1)}+1}, 1, \{2\}^{a_{\sigma(2)}}, 3, \{2\}^{a_{\sigma(3)}}, \ldots, 1, \{2\}^{a_{\sigma(2n)}}, 3, \{2\}^{a_{\sigma(2n+1)}}) \in \pi^{\mathrm{wt}}\mathbb{Q}$$

Analogue for MZSV's

Theorem (C., 2018)

For $\ell_i > 1$,

$$\sum_{\text{permute } \ell_i} \zeta^\star(\mathrm{bl}^{-1}(2 \circ \ell_1, \ell_2, \ldots, \ell_n)) = \sum_{\substack{\mathbf{r} \in \\ \mathrm{Part}_{\mathrm{odd}}(n)}} 2^{\#\mathbf{r}} \prod_i (\#r_i - 1)! \widehat{\zeta}\Big(\sum_{j \in r_i} \ell_j\Big)$$

Where

- $\zeta^\star(0\underbrace{10\cdots0}_{s_1}\cdots\underbrace{10\cdots0}_{s_k}1) = \zeta^\star(s_1, \ldots, s_k)$
- $\circ = \begin{cases} + & \mathrm{wt} \not\equiv k \pmod 2 \\ , & \mathrm{wt} \equiv k \pmod 2 \end{cases}$ *and* $\widehat{\zeta}(s) = \begin{cases} \zeta(s) & s \text{ odd} \\ \frac{1}{2}\zeta^\star(\{2\}^{s/2}) & s \text{ even} \end{cases}$
- $\mathrm{Part}_{\mathrm{odd}}(n) = \{$ *partitions of* $\{1, \ldots, n\}$ *into odd size parts* $\}$

"A polynomial in Riemann Zeta Values."

Analogue for MZSV's - Proof

Proof (Sketch).

- Apply Zhao's (generalised) 2-1 formula

$$\zeta^\star(\mathbf{s}) = \varepsilon(\mathbf{s}) \sum_{\mathbf{p} \in \Pi(\mathbf{s}^{(1)})} 2^{\#\mathbf{p}} \zeta(\mathbf{p})$$

- Show $\mathbf{s}^{(1)} = (\widetilde{\ell}_1, \ldots, \widetilde{\ell}_k)$ where

$$\widetilde{\ell}_j = \begin{cases} \ell_j & \ell_j \text{ odd} \\ \overline{\ell_j} & \ell_j \text{ even} \end{cases}$$

$\leftsquigarrow$ Alternating MZV's

- Apply (Zhao's generalisation of) the symmetric sum formula
- Use Zobilin's evaluation

$$\zeta(\overline{2n}) = -\tfrac{1}{2}\zeta^\star(\{2\}^n)$$

□

Analogue for MZSV's - Example

Example (Hoffman analogue)

For $(\ell_i) = (2a+3, 2b+3, 2c+2)$, have $\circ = +$, and

$$\mathrm{Part}_{\mathrm{odd}}(3) = \{\ \{1 \mid 2 \mid 3\}, \quad \{123\}\ \}\ .$$

Obtain

$$\begin{aligned}
&\zeta^\star(\{2\}^{a+1}, 3, \{2\}^b, 3, \{2\}^c) && + \zeta^\star(\{2\}^{b+1}, 3, \{2\}^a, 3, \{2\}^c) + \\
&+ \zeta^\star(\{2\}^{b+1}, 3, \{2\}^c, 1, 2, \{2\}^a) && + \zeta^\star(\{2\}^{a+1}, 3, \{2\}^c, 1, 2, \{2\}^b) + \\
&+ \zeta^\star(\{2\}^{c+1}, 1, 2, \{2\}^a, 1, 2, \{2\}^b) && + \zeta^\star(\{2\}^{c+1}, 1, 2, \{2\}^a, 1, 2, \{2\}^b)
\end{aligned}$$

$$= 2^3 (1-1)!^3 \zeta(2a+3)\zeta(2b+3) \cdot \tfrac{1}{2}\zeta^\star(\{2\}^{c+1}) + \qquad \leftsquigarrow \mathbf{r} = \{1 \mid 2 \mid 3\}$$

$$+ 2^1 (3-1)! \cdot \tfrac{1}{2}\zeta^\star(\{2\}^{a+b+c+4}) \qquad \leftsquigarrow \mathbf{r} = \{123\}$$

$$= 4\zeta(2a+3)\zeta(2b+3)\zeta^\star(\{2\}^{c+1}) + 2\zeta^\star(\{2\}^{a+b+c+4})$$

Summary

- Defined block decomposition of an iterated integral
- Used block decomposition to unify/generalise BBBL and Hoffman's conjectures
- Used motivic MZV's to prove a symmetrised version holds
- Improved Bowman-Bradley to only permutations, proved Hoffman, and other identities up to $\mathbb{Q}$