MOTIVIC MULTIPLICATIVE MCKAY CORRESPONDENCE FOR SMOOTH PROJECTIVE SURFACES

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ABSTRACT. We revisit the classical 2-dimensional McKay correspondence. Here are the main differences between the classical correspondence and our version: (1) the main point is that we take care of the multiplicative structure given by the orbifold product ; (2) instead of using cohomology, we deal with the Chow motives ; (3) our correspondence is global: it works for all Gorenstein quotients of smooth projective surfaces while the previous works often treat only the 'local' case of affine plane quotient by a finite subgroup of $\mathrm{SL}_2(\mathbf{C})$. More precisely, we prove that for any smooth projective surface endowed with a faithful action of a finite group such that the quotient surface is Gorenstein, then there is an isomorphism of algebra objects, in the category of complex Chow motives, between the motive of the minimal resolution and the orbifold motive of the quotient surface. In particular, the complex Chow ring (*resp.* Grothendieck ring) of the minimal resolution is isomorphic to the complex orbifold Chow ring (*resp.* Grothendieck ring) of the quotient surface. This confirms the two-dimensional *Motivic Crepant / HyperKähler Resolution Conjecture*.

1. INTRODUCTION

Finite subgroups of $\mathrm{SL}_2(\mathbf{C})$ are classically studied by Klein [19] and Du Val [9]. A complete classification (up to conjugacy) is available : cyclic, binary dihedral, binary tetrahedral, binary octahedral and binary icosahedral. The last three types correspond to the groups of symmetries of Platonic solides[1] as the names indicate. Let $G \subset \mathrm{SL}_2(\mathbf{C})$ be such a (non-trivial) finite subgroup acting naturally on the vector space $V := \mathbf{C}^2$. The quotient $X := V/G$ has a unique rational double point[2]. Let $f : Y \to X$ be the minimal resolution of singularities:

$$\begin{array}{ccc} & & V \\ & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{f} & X \end{array}$$

which is a crepant resolution, that is, $K_Y = f^*K_X$. The exceptional divisor, denoted by E, consists of a bunch of (-2)-curves[3].

The classical McKay correspondence ([20], *cf.* also [22]) establishes an bijection between the set $\mathrm{Irr}'(G)$ of non-trivial irreducible representations of G on the one hand and the set $\mathrm{Irr}(E)$ of

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[1] *i.e.* regular polyhedrons in $\mathbf{R}^3$.

[2] Such (isolated) surface singularities are also known as Klein, Du Val, Gorenstein, canonical, simple or ADE singularities according to different points of view.

[3] *i.e.* smooth rational curve with self-intesection equal to -2.

irreducible components of E on the other hand :

$$\begin{array}{ccc} \mathrm{Irr}'(G) & \simeq & \mathrm{Irr}(E) \\ \rho & \mapsto & E_\rho. \end{array}$$

Thus $E = \bigcup_{\rho\in\mathrm{Irr}'(G)} E_\rho$. Moreover, this bijection respects the 'incidence relations' : precisely, for any $\rho_1 \neq \rho_2 \in \mathrm{Irr}'(G)$, the intersection number $(E_{\rho_1} \cdot E_{\rho_2})$, which is 0 or 1, is equal to the multiplicity of ρ_2 in $\rho_1 \otimes V$ (hence is also equal to the multiplicity of ρ_1 in $\rho_2 \otimes V$), where V is the 2-dimensional natural representation via $G \subset \mathrm{SL}(V)$. All these informations can be encoded into Dynkin diagrams of A-D-E type, which is on the one hand the dual graph of the exceptional divisor E and on the other hand the *McKay graph* of the non-trivial irreducible representations of G, with respect to the prefered representation V. Apart from the original observation of McKay, there are many approaches to construct this correspondence geometrically and to extend it to higher dimensions : K-theory of sheaves [14], G-Hilbert schemes [21], [17], [16], [15], motivic integration [2], [3], [7], [8] and derived categories [4] *etc.* We refer the reader to Reid's note of his Bourbaki talk [22] for more details and history.

Following Reid [21], one can recast the above McKay correspondence (the bijection) as follows: *the isomorphism classes of irreducible representations index a basis of the homology of the resolution Y*. This is of course equivalent to say that *the conjugacy classes of G index a basis of the cohomology of Y*. We remark that, which is the starting point of this paper, the quotient $X = V/G$ has a natural orbifold structure, meaning that X underlies the smooth Deligne-Mumford stack $\mathcal{X} := [V/G]$, and the (co)homology of the coarse moduli space $|I\mathcal{X}|$ of its *inertia stack* $I\mathcal{X}$ has a basis indexed by the conjugacy classes of G. Thus Reid's McKay correspondence can be stated as an isomorphism of vector spaces:

$$H^*(Y) \simeq H^*(|I\mathcal{X}|).$$

Chen and Ruan defined in [6] the *orbifold* cohomology $H^*_{orb}([V/G])$, whose underlying vector space is exactly the cohomology of $|I\mathcal{X}|$; the supplementary ingredient is that they can put a highly non-trivial (associative and commutative) ring structure, the so-called *orbifold product*, on this orbifold cohomology. See Definition 2.1 for a down-to-earth construction. Therefore it is natural to ask whether there is a multiplicative isomorphism (of algebras)

$$H^*(Y) \simeq H^*_{orb}([V/G]).$$

However none of aforementioned beautiful theories takes care of the multiplicative structures. Nevertheless, the existence of such an isomorphism of algebras is known. For example, it is a baby case of the result of Ginzburg-Kaledin [13] on symplectic resolutions of symplectic quotient singularities. An explicit formula is proposed by Bryan-Graber-Pandharipande in [5], which is used there to prove (a stronger version of) the $\mathbf{C}^2/\mathbf{Z}_3$ case. We will also use this formula to construct our multiplicative isomorphism.

This isomorphism fits perfectly into Ruan's more general Cohomological Crepant Resolution Conjecture (CCRC), which we state here only in the case of global quotient by a finite group:

Conjecture 1.1 (CCRC [23])**.** *Let M be a smooth projective variety endowed with a faithful action of a finite group G. Assume that the quotient $X := M/G$ is Gorenstein then for any crepant resolution $Y \to X$, there is an isomorphism of graded* **C***-algebras:*

$$H^*_{qc}(Y, \mathbf{C}) \simeq H^*_{orb}([M/G], \mathbf{C})\,. \tag{1}$$

Here the left hand side is the *quantum corrected* cohomology algebra, whose underlying graded vector space is just $H^*(Y, \mathbf{C})$, endowed with the cup product with quantum corrections related to Gromov-Witten invariants with curve classes contracted by the crepant resolution, as defined in [23]. Since we only consider in this paper the 2-dimensional situation, the Gromov-Witten invariants always vanish hence there is no quantum corrections involved. See Lemma 2.3 for this vanishing.

Conjecture 1.1 suggests that one should consider the existence of such multiplicative McKay correspondence in the global situation (instead of a quotient of a vector space), that is, a Gorenstein quotient of a surface by a finite group action. Our main result is the following, which also pushes the (surface) McKay correspondence to the motivic level:

Theorem 1.2 (Motivic multiplicative global McKay correspondence)**.** *Let S be a smooth projective surface and G be a finite group acting faithfully on S such that the quotient X has Du Val singularities. Let $Y \to X$ be the minimal resolution. Then we have an isomorphism of algebra objects in the category* $\mathrm{CHM}_{\mathbf{C}}$ *of Chow motives with complex coefficients:*

$$\mathfrak{h}(Y)_{\mathbf{C}} \simeq \mathfrak{h}_{orb}\left([S/G]\right)_{\mathbf{C}}. \tag{2}$$

In particular, one has an isomorphism of $\mathbf{C}$*-algebras:*

$$\begin{aligned} \mathrm{CH}^*(Y)_{\mathbf{C}} &\simeq \mathrm{CH}^*_{orb}\left([S/G]\right)_{\mathbf{C}};\\ H^*(Y,\mathbf{C}) &\simeq H^*_{orb}\left([S/G],\mathbf{C}\right);\\ K(Y)_{\mathbf{C}} &\simeq K_{orb}\left([S/G]\right)_{\mathbf{C}};\\ K^{top}(Y)_{\mathbf{C}} &\simeq K^{top}_{orb}\left([S/G]\right)_{\mathbf{C}}. \end{aligned}$$

This result confirms the 2-dimensional case of the so-called *Motivic HyperKähler Resolution Conjecture* studied in [12] abd [11].

Convention : CHM is the category of Chow motives with rational coefficients. $\mathfrak{h}: \mathrm{SmProj}^{op} \to \mathrm{CHM}$ is the (contravariant) functor that associates a smooth projective variety its Chow motive and a morphism its graph as correspondence.

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2. Crepant resolution conjecture

We first recall briefly the construction of the *orbifold Chow motive algebra* and the *orbifold Chow ring* associated to a global Gorenstein quotient by a finite group. We refer to our previous work [12] (joint with Charles Vial), [11] as well as the original sources (for cohomology and Chow rings) [6], [10], [1], [18] for more details.

Let M be a smooth projective variety and G be a finite group acting faithfully on M. Assume that G preserves locally the canonical bundle: for any $x \in M$ fixed by $g \in G$, the differential $Dg \in \mathrm{SL}(T_xM)$. This is equivalent to require that the quotient $X := M/G$ has only Gorenstein singularities. Denote by $M^g = \{x \in M \mid gx = x\}$ the fixed locus of $g \in G$, $M^{\langle g,h\rangle} = M^g \cap M^h$ (with the reduced structure) and $\mathcal{X} := [M/G]$ the quotient smooth Deligne-Mumford stack.

Definition 2.1 (Orbifold theories)**.** We define an auxiliary algebra object $\mathfrak{h}(M,G)$ in CHM with G-action, and the orbifold motive $\mathfrak{h}([M/G])$ will be its subalgebra of invariants. The definitions for Chow rings and cohomology are similar.

(1°) For any $g \in G$, the *age function*, denoted by $\mathrm{age}(g)$, is a $\mathbf{Z}$-valued locally constant function on M^g, whose value on a connected component Z is

$$\mathrm{age}(g)|_Z := \sum_{j=0}^{r-1} \frac{j}{r} \mathrm{rank}(W_j),$$

where r is the order of g, W_j is the eigen-sub-bundle of the restricted tangent bundle $TM|_Z$, for the natural automorphism induced by g, with eigenvalue $e^{\frac{2\pi i}{r}j}$. The age function is invariant under conjugacy.

(2°) We endow the direct sums

$$\mathfrak{h}(M,G) := \bigoplus_{g \in G} \mathfrak{h}(M^g)(-\mathrm{age}(g))$$

$$\mathrm{CH}^*(M,G) := \bigoplus_{g \in G} \mathrm{CH}^{*-\mathrm{age}(g)}(M^g)$$

$$H^*(M,G) := \bigoplus_{g \in G} H^{*-2\,\mathrm{age}(g)}(M^g)$$

$$K(M,G) := \bigoplus_{g \in G} K(M^g)$$

with the natural G-action induced by the following action: for any $g, h \in G$,

$$\begin{array}{rcl} h : M^g & \xrightarrow{\simeq} & M^{hgh^{-1}} \\ x & \mapsto & hx. \end{array}$$

(3°) For any $g \in G$, define

$$V_g := \sum_{j=0}^{r-1} \frac{j}{r} [W_j] \in K_0(M^g)_{\mathbf{Q}},$$

whose virtual rank is $\mathrm{age}(g)$, where r and W_j 's are as in (1°).

(4°) For any $g_1, g_2 \in G$, let $g_3 = g_2^{-1} g_1^{-1}$, we define the (virtual class of) the *obstruction bundle* on the fixed locus $M^{\langle g_1, g_2 \rangle}$ by

$$F_{g_1,g_2} := V_{g_1}\big|_{M^{<g_1,g_2>}} + V_{g_2}\big|_{M^{<g_1,g_2>}} + V_{g_3}\big|_{M^{<g_1,g_2>}} + TM^{<g_1,g_2>} - TM|_{M^{<g_1,g_2>}} \in K_0\left(M^{<g_1,g_2>}\right)_{\mathbf{Q}}. \tag{3}$$

(5°) The *orbifold product* $\star_{orb}$ is defined as follows: given $g, h \in G$, let $\iota : M^{<g,h>} \hookrightarrow M$ be the natural inclusion.

- For cohomology:

$$\begin{array}{rcl} \star_{orb} : H^{i-2\,\mathrm{age}(g)}(M^g) \times H^{j-2\,\mathrm{age}(h)}(M^h) & \to & H^{i+j-2\,\mathrm{age}(gh)}(M^{gh}) \\ (\alpha, \beta) & \mapsto & \iota_*\left(\alpha|_{M^{<g,h>}} \smile \beta|_{M^{<g,h>}} \smile c_{top}(F_{g,h})\right) \end{array}$$

- For Chow groups:

$$\begin{array}{rcl} \star_{orb} : \mathrm{CH}^{i-\mathrm{age}(g)}(M^g) \times \mathrm{CH}^{j-\mathrm{age}(h)}(M^h) & \to & \mathrm{CH}^{i+j-\mathrm{age}(gh)}(M^{gh}) \\ (\alpha, \beta) & \mapsto & \iota_*\left(\alpha|_{M^{<g,h>}} \cdot \beta|_{M^{<g,h>}} \cdot c_{top}(F_{g,h})\right) \end{array}$$

- For K-theory:

$$\begin{array}{rcl} \star_{orb} : K(M^g) \times K(M^h) & \to & K(M^{gh}) \\ (\alpha, \beta) & \mapsto & \iota_!\left(\alpha|_{M^{<g,h>}} \cdot \beta|_{M^{<g,h>}} \cdot \lambda_{-1}(F_{g,h}^\vee)\right) \end{array}$$

- For motives: $\star_{orb} : \mathfrak{h}(M^g)(-\operatorname{age}(g)) \otimes \mathfrak{h}(M^h)(-\operatorname{age}(h)) \to \mathfrak{h}(M^{gh})(-\operatorname{age}(gh))$ is determined by the correspondence
$$\delta_*(c_{top}(F_{g,h})) \in \mathrm{CH}^{\dim M^g + \dim M^h + \operatorname{age}(g) + \operatorname{age}(h) - \operatorname{age}(gh)}(M^g \times M^h \times M^{gh}),$$
where $\delta : M^{<g,h>} \to M^g \times M^h \times M^{gh}$ is the natural morphism sending x to (x, x, x).

(6°) Finally, we take the subalgebra of invariants whose existence is guaranteed by the idempotent completeness of CHM :
$$\mathfrak{h}_{orb}([M/G]) := \mathfrak{h}(M, G)^G ;$$
$$\mathrm{CH}^*_{orb}([M/G]) := (\mathrm{CH}^*(M, G), \star_{orb})^G .$$
and similarly for other theories. These are commutative algebras and depend only on the stack $[M/G]$ (not the presentation).

With orbifold theories being defined, we can speculate that a motivic or K-theoretic version of the Crepant Resolution Conjecture 1.1 should hold. But the problem is that in the definition of the *quantum corrections*, there is the subtle convergence property which is difficult to make sense in general for Chow groups / motives or for K-theory. Therefore, we will look at some cases that these quantum corrections vanish:

Case 1: HyperKahler resolution. The first one is when the resolution Y is holomorphic symplectic, which implies that all (Chow-theoretic, K-theoretic or cohomological) Gromov-Witten invariants vanish (see the proof of [11, Lemma 8.1]). In this case, we indeed have the following *Motivic HyperKähler Resolution Conjecture* (MHRC), proposed in [12]:

Conjecture 2.2 (MHRC [12], [11])**.** *Let M be a smooth projective holomorphic symplectic variety endowed with a faithful symplectic action of a finite group G. If quotient $X := M/G$ has a crepant resolution $Y \to X$, then there is an isomorphism of algebra object in* $\mathrm{CHM}_{\mathbf{C}}$*:*
$$\mathfrak{h}(Y) \simeq \mathfrak{h}_{orb}([M/G]).$$
In particular, we have an isomorphism of graded $\mathbf{C}$*-algebras:*
$$\mathrm{CH}^*(Y, \mathbf{C}) \simeq H^*_{orb}([M/G], \mathbf{C}) .$$

Thanks to the orbifold Chern character isomorphism constructed by Jarvis-Kaufmann-Kimura in [18], MHRC also implies the K-theoretic HyperKähler Resolution Conjecture in *loc.cit.* . Conjecture 2.2 is proved in our joint work with Charles Vial [12] for Hilbert schemes of abelian varieties and generalized Kummer varieties and in [11] for Hilbert schemes of K3 surfaces.

Case 2: Surface minimal resolution. The second one is the main purpose of the paper, namely the surface case, *i.e.* $\dim(Y) = 2$. In this case, the vanishing of quantum corrections is explained in the following lemma.

Lemma 2.3. *Let X be a surface with Du Val singularities and $\pi : Y \to X$ be the minimal resolution. Then the virtual fundamental class of $\overline{M_{0,3}}(Y, \beta)$ is rationally equivalent to zero for any curve class β which is contracted by π.*

Proof. Consider the forgetful-stablization morphism
$$f : \overline{M_{0,3}}(Y, \beta) \to \overline{M_{0,0}}(Y, \beta) .$$
By the general theory, the virtual fundamental class of $\overline{M_{0,3}}(Y, \beta)$ is the pull-back of the virtual fundamental class of $\overline{M_{0,0}}(Y, \beta)$. However, the virtual dimension of $\overline{M_{0,0}}(Y, \beta)$ is $(\beta \cdot K_Y) + (\dim Y - 3) =$

-1 since π is crepant. Therefore, both moduli spaces have zero virtual fundamental class in Chow group, cohomology or K-theory. □

Thanks to the vanishing of quantum corrections, the motivic version of the Crepant Resolution Conjecture 1.1 for surfaces is exactly the content of our main Theorem 1.2 (but only for global quotient by a finite group). See Introduction for the precise statement.

3. Proof of Theorem 1.2

Let us recall the setting: S is a smooth projective surface, G is a finite group acting faithfully on S such that the canonical bundle is locally preserved (Gorenstein condition), $X := S/G$ is the quotient surface (with Du Val singularities) and $Y \to X$ is the minimal (crepant) resolution. We denote by $\mathbb{L} := \mathbb{1}(-1)$ the Lefschetz motive in CHM.

For any $x \in S$, let

$$G_x := \{g \in G \mid gx = x\}$$

be the stabilizer. Let $\mathrm{Irr}(G_x)$ be the set of isomorphism classes of irreducible representations of G_x. We remark that by assumption, there are only finitely many points of S with non-trivial stabilizer.

3.1. **Resolution side.** We first compute the Chow motive algebra (or Chow ring) of the minimal resolution Y.

For any $x \in S$, we denote by $\overline{x}$ its image in S/G. The Chow motive of Y has the decomposition in CHM

$$\mathfrak{h}(Y) \simeq \mathfrak{h}(S)^G \oplus \bigoplus_{\overline{x} \in S/G} \bigoplus_{\rho \in \mathrm{Irr}'(G_x)} \mathbb{L}_{\overline{x},\rho} \simeq \left(\mathfrak{h}(S) \oplus \bigoplus_{x \in S} \bigoplus_{\rho \in \mathrm{Irr}'(G_x)} \mathbb{L}_{x,\rho} \right)^G, \tag{4}$$

where $\mathbb{L}_{x,\rho}$ is the Lefschetz motive corresponding to the irreducible component of the exceptional divisor over x, indexed by the non-trivial irreducible representation ρ of G_x via the classical McKay correspondence. The product structure is determined as follows via the above decomposition, which is also part of the classical McKay correspondence. Let $i_x : \{x\} \hookrightarrow S$ be the natural inclusion.

- $\mathfrak{h}(S) \otimes \mathfrak{h}(S) \xrightarrow{\delta_S} \mathfrak{h}(S)$ is the usual product induced by the small diagonal of S^3.
- For any x with nontrivial stabilizer G_x and any $\rho \in \mathrm{Irr}'(G_x)$,
$$\mathfrak{h}(S) \otimes \mathbb{L}_{x,\rho} \xrightarrow{i_x^*} \mathbb{L}_{x,\rho}$$
is determined by the class $x \in \mathrm{CH}^2(S) = \mathrm{Hom}(\mathfrak{h}(S) \otimes \mathbb{L}, \mathbb{L})$.
- For any $\rho \in \mathrm{Irr}'(G_x)$ as above,
$$\mathbb{L}_{x,\rho} \otimes \mathbb{L}_{x,\rho} \xrightarrow{-2i_{x,*}} \mathfrak{h}(S),$$
is determined by $-2x \in \mathrm{CH}^2(S)$.
- For any $\rho_1 \neq \rho_2 \in \mathrm{Irr}'(G_x)$,
  - If they are *adjacent*, that is, ρ_1 appears (with multiplicity 1) in the G_x-module $\rho_2 \otimes T_xS$, then
  $$\mathbb{L}_{x,\rho_1} \otimes \mathbb{L}_{x,\rho_2} \xrightarrow{i_{x,*}} \mathfrak{h}(S),$$
  is determined by $x \in \mathrm{CH}^2(S)$.
  - If they are not adjacent, then $\mathbb{L}_{x,\rho_1} \otimes \mathbb{L}_{x,\rho_2} \xrightarrow{0} \mathfrak{h}(S)$ is the zero map.

- The other multiplication maps are zero.

The G-action on (4) is as follows:

- The G-action of $\mathfrak{h}(S)$ is induced by the original action on S.
- For any $h \in G$, it maps for any $x \in S$ and $\rho \in \mathrm{Irr}'(G_x)$, the Lefschetz motive $\mathbb{L}_{x,\rho}$ isomorphically to $\mathbb{L}_{hx,h\rho}$, where $h\rho \in \mathrm{Irr}'(G_{hx})$ is the representation which makes the following diagram commutes:

$$\begin{array}{ccc} G_x & \xrightarrow[\simeq]{g \mapsto hgh^{-1}} & G_{hx} \\ & {\scriptstyle\rho}\searrow \quad \swarrow{\scriptstyle h\rho} & \\ & V_\rho & \end{array} \tag{5}$$

3.2. **Orbifold side.** Now we compute the orbifold Chow motive algebra (or Chow ring) of the quotient stack $[S/G]$. The compute is quite straight-forward. Here $\mathbb{L} := \mathbb{1}(-1)$ is the Lefschetz motive.

First of all, it is easy to see that $\mathrm{age}(g) = 1$ for any element $g \neq \mathrm{id}$ of G, and $\mathrm{age}(\mathrm{id}) = 0$. By Definition 2.1,

$$\mathfrak{h}(S, G) = \mathfrak{h}(S) \oplus \bigoplus_{\substack{g \in G \\ g \neq \mathrm{id}}} \bigoplus_{x \in S^g} \mathbb{L}_{x,g} = \mathfrak{h}(S) \oplus \bigoplus_{x \in S} \bigoplus_{\substack{g \in G_x \\ g \neq \mathrm{id}}} \mathbb{L}_{x,g}, \tag{6}$$

where $\mathbb{L}_{x,g}$ is the Lefschetz motive $\mathbb{1}(-1)$ indexed by the fixed point x of g.

Lemma 3.1 (Obstruction class). *For any $g, h \in G$ different from* id*, the obstruction class is*

$$c_{g,h} = \begin{cases} 1 & \text{if } g = h^{-1} \\ 0 & \text{if } g \neq h^{-1} \end{cases}$$

Proof. For any $g \neq \mathrm{id}$ and any $x \in S^g$, the action of g on T_xS is diagonalizable with a pair of conjugate eigenvalues, therefore V_g in Definition 2.1 is a trivial vector bundle of rank one on S^g. Hence for any $g, h \in G$ different from id and $x \in S$ fixed by g and h, the dimension of the fiber of the obstruction bundle $F_{g,h}$ at x is

$$\dim F_{g,h}(x) = \dim V_g(x) + \dim V_h(x) + \dim V_{(gh)^{-1}}(x) - \dim T_xS,$$

which is 1 if $g \neq h^{-1}$ and is 0 if $g = h^{-1}$. The computation of $c_{g,h}$ follows. □

Once the obstruction classes are computed, we can write down explicitly the orbifold product from Definition 2.1, which is summarized in the following proposition.

Proposition 3.2. *The orbifold product on $\mathfrak{h}(S, G)$ is given as follows via the decomposition (6):*

$$\begin{aligned} \mathfrak{h}(S) \otimes \mathfrak{h}(S) &\xrightarrow{\delta_S} \mathfrak{h}(S); \\ \mathfrak{h}(S) \otimes \mathbb{L}_{x,g} &\xrightarrow{i_x^*} \mathbb{L}_{x,g} \ \forall gx = x; \\ \mathbb{L}_{x,g} \otimes \mathbb{L}_{x,g^{-1}} &\xrightarrow{i_{x,*}} \mathfrak{h}(S). \end{aligned}$$

where the first morphism is the usual product given by small diagonal; the second and the third morphisms are given by the class $x \in \mathrm{CH}^2(S)$ and $i_x : \{x\} \hookrightarrow S$ is the natural inclusion; all the other possible maps are zero.

The G-action on (6) is as follows by Definition 2.1:

- The G-action on $\mathfrak{h}(S)$ is the original action.
- For any $h \in G$, it maps for any $x \in S$ and $g \neq \mathrm{id} \in G_x$, the Lefschetz motive $\mathbb{L}_{x,g}$ isomorphically to $\mathbb{L}_{hx,hgh^{-1}}$.

3.3. **The multiplicative correspondence.** With both sides of the correspondence computed, we can give the *multiplicative McKay correspondence* morphism, which is in the category $\mathrm{CHM}_{\mathbf{C}}$ of complex Chow motives.

$$\Phi : \mathfrak{h}(S) \oplus \bigoplus_{x\in S} \bigoplus_{\rho\in \mathrm{Irr}'(G_x)} \mathbb{L}_{x,\rho} \to \mathfrak{h}(S) \oplus \bigoplus_{x\in S} \bigoplus_{\substack{g\in G_x\\ g\neq \mathrm{id}}} \mathbb{L}_{x,g}, \tag{7}$$

which is given by the following 'matrix by blocs':

- $\mathrm{id} : \mathfrak{h}(S) \to \mathfrak{h}(S)$;
- For each $x \in S$ (with nontrivial stabilizer G_x), the morphism
$$\bigoplus_{\rho\in \mathrm{Irr}'(G_x)} \mathbb{L}_{x,\rho} \to \bigoplus_{\substack{g\in G_x\\ g\neq \mathrm{id}}} \mathbb{L}_{x,g}$$
is the 'matrix' with coefficient $\frac{1}{\sqrt{|G_x|}} \sqrt{\chi_{\rho_0}(g) - 2} \cdot \chi_\rho(g)$ at place $(\rho, g) \in \mathrm{Irr}'(G_x) \times (G_x \backslash \{\mathrm{id}\})$, where χ denotes the character, ρ_0 is the natural 2-dimensional representation T_xS of G_x. Note that $\rho_0(g)$ has determinant 1, hence its trace $\chi_{\rho_0}(g)$ is a real number.
- The other morphisms are zero.

To conclude the main theorem, one has to show three things: (*i*) Φ is compatible with the G-action; (*ii*) Φ is multiplicative and (*iii*) Φ induces an isomorphism Φ^G of complex Chow motives on G-invariants.

Lemma 3.3. *Φ is G-equivariant.*

Proof. The G-action on the first direct summand $\mathfrak{h}(S)$ is by definition the same, hence is preserved by $\Phi|_{\mathfrak{h}(S)} = \mathrm{id}$. For the other direct summands, since it is a matrix computation, we can treat the Lefschetz motives as 1-dimensional vector spaces: let $E_{x,\rho}$ be the 'generator' of $\mathbb{L}_{x,\rho}$ and $e_{x,g}$ be the 'generator' of $\mathbb{L}_{x,g}$. Then the G-actions computed in the previous subsections say that for any x and any $h \in G_x$,

$$h.E_{x,\rho} = E_{hx,h\rho} \text{ and } h.e_{x,g} = e_{hx,hgh^{-1}},$$

where $h\rho$ is defined in (5).

Therefore

$$\begin{aligned}
& \Phi(h.E_{x,\rho}) \\
= \ & \Phi(E_{hx,h\rho}) \\
= \ & \frac{1}{\sqrt{|G_{hx}|}} \sum_{g\in G_{hx}} \sqrt{\chi_{\rho_0}(g)-2}\,\chi_{h\rho}(g)\, e_{hx,g} \\
= \ & \frac{1}{\sqrt{|G_{x}|}} \sum_{g\in G_{x}} \sqrt{\chi_{\rho_0}(g)-2}\,\chi_{h\rho}(hgh^{-1})\, e_{hx,hgh^{-1}} \\
= \ & \frac{1}{\sqrt{|G_{x}|}} \sum_{g\in G_{x}} \sqrt{\chi_{\rho_0}(g)-2}\,\chi_{\rho}(g)\, e_{hx,hgh^{-1}} \\
= \ & \frac{1}{\sqrt{|G_{x}|}} \sum_{g\in G_{x}} \sqrt{\chi_{\rho_0}(g)-2}\,\chi_{\rho}(g)\, h.e_{x,g} \\
= \ & h.\Phi(E_{x,\rho}),
\end{aligned}$$

where the third equality is a change of variable: replace g by hgh^{-1}, the fourth equality follows from the definition of $h\rho$ in (5) □

Proposition 3.4 (Multiplicativity)**.** *Φ preserves the multiplication, i.e. Φ is a morphism of algebra objects in* $\mathrm{CHM}_{\mathbf{C}}$.

Proof. The cases of multiplying $\mathfrak{h}(S)$ with itself or with a Lefschetz motive $\mathbb{L}_{x,\rho}$ are all obviously preserved by Φ. We only need to show that for any $x \in S$ with non-trivial stabilizer G_x, the morphism

$$\bigoplus_{\rho\in\mathrm{Irr}'(G_x)} \mathbb{L}_{x,\rho} \to \bigoplus_{\substack{g\in G_x\\ g\neq \mathrm{id}}} \mathbb{L}_{x,g}$$

given by the matrix with coefficient $\frac{1}{\sqrt{|G_x|}}\sqrt{\chi_{\rho_0}(g)-2}\cdot\chi_\rho(g)$ at place (ρ, g) is multiplicative (note that the result of the multiplication could go outside of these direct sums to $\mathfrak{h}(S)$). Since this is just a matrix computation, let us treat Lefschetz motives as 1-dimensional vector spaces (or equivalently, we are looking at the corresponding multiplicativity of the realization of Φ for Chow rings): let $E_{x,\rho}$ be the 'generator' of $\mathbb{L}_{x,\rho}$ and $e_{x,g}$ be the 'generator' of $\mathbb{L}_{x,g}$. Then the computations of the products in the previous two subsections say that:

$$E_{x,\rho_1}\cdot E_{x,\rho_2} = \begin{cases} -2x & \text{if } \rho_1=\rho_2; \\ x & \text{if } \rho_1,\rho_2 \text{ are adjacent;} \\ 0 & \text{if } \rho_1,\rho_2 \text{ are not adjacent;} \end{cases} \tag{8}$$

$$e_{x,g}\cdot e_{x,h} = \begin{cases} x & \text{if } g=h^{-1}; \\ 0 & \text{if } g\neq h^{-1}; \end{cases} \tag{9}$$

Therefore for any $\rho_1, \rho_2 \in \mathrm{Irr}'(G_x)$, we have

$$\begin{aligned}
& \Phi(E_{x,\rho_1}) \cdot \Phi(E_{x,\rho_2}) \\
= \; & \frac{1}{|G_x|} \sum_{g \in G_x} \sum_{h \in G_x} \sqrt{\chi_{\rho_0}(g) - 2} \sqrt{\chi_{\rho_0}(h) - 2}\, \chi_{\rho_1}(g) \chi_{\rho_2}(h)\, e_{x,g} \cdot e_{x,h} \\
= \; & \frac{1}{|G_x|} \sum_{g \in G_x} \sqrt{\chi_{\rho_0}(g) - 2} \sqrt{\chi_{\rho_0}(g^{-1}) - 2}\, \chi_{\rho_1}(g) \chi_{\rho_2}(g^{-1}) \cdot x \\
= \; & \frac{1}{|G_x|} \sum_{g \in G_x} (\chi_{\rho_0}(g) - 2)\, \chi_{\rho_1}(g) \overline{\chi_{\rho_2}(g)} \cdot x \\
= \; & \frac{1}{|G_x|} \left(\sum_{g \in G_x} \chi_{\rho_0 \otimes \rho_1}(g) \overline{\chi_{\rho_2}(g)} - 2 \sum_{g \in G_x} \chi_{\rho_1}(g) \overline{\chi_{\rho_2}(g)} \right) \cdot x \\
= \; & (\langle \rho_0 \otimes \rho_1, \rho_2 \rangle - 2 \langle \rho_1, \rho_2 \rangle) \cdot x \\
= \; & \Phi\left(E_{x,\rho_1} \cdot E_{x,\rho_2} \right)
\end{aligned}$$

where the first equality is the definition of Φ (and we add the non-existent $e_{x,1}$ with coefficient 0), the second equality uses (9) the orthogonality among $e_{x,g}$'s (*i.e.* $\mathbb{L}_{x,g}$'s), the third equality uses the fact that χ_{ρ_0} takes real value; the last equality uses all three cases of (8). □

Proposition 3.5 (Additive isomorphism)**.** *Taking G-invariants on both sides of (10), Φ^G is an isomorphism of complex Chow motives between $\mathfrak{h}(Y)$ and $\mathfrak{h}_{orb}([S/G])$.*

Proof. We should prove the following morphism is an isomorphism:

$$\Phi^G : \mathfrak{h}(S)^G \oplus \left(\bigoplus_{x \in S} \bigoplus_{\rho \in \mathrm{Irr}'(G_x)} \mathbb{L}_{x,\rho} \right)^G \to \mathfrak{h}(S)^G \oplus \left(\bigoplus_{x \in S} \bigoplus_{\substack{g \in G_x \\ g \neq \mathrm{id}}} \mathbb{L}_{x,g} \right)^G . \tag{10}$$

Since Φ is given by 'matrix by blocs', it amounts to show that for each $x \in S$ (with G_x non trivial), the following is an isomorphism :

$$\bigoplus_{\rho \in \mathrm{Irr}'(G_x)} \mathbb{L}_{x,\rho} \to \left(\bigoplus_{\substack{g \in G_x \\ g \neq \mathrm{id}}} \mathbb{L}_{x,g} \right)^{G_x} . \tag{11}$$

which is equivalent to say that the following square matrix is non-degenerated:

$$\left(\sqrt{\chi_{\rho_0}(g) - 2} \cdot \chi_\rho(g) \right)_{(\rho, [g])} , \tag{12}$$

where ρ runs over the set $\mathrm{Irr}'(G_x)$ of isomorphism classes of non-trivial irreducible representations and $[g]$ runs over the set of conjugacy classes of G_x different from id.

As this is about a matrix, it is enough to look at the realization of (11):

$$\bigoplus_{\rho \in \mathrm{Irr}'(G_x)} E_{x,\rho} \to \left(\bigoplus_{\substack{g \in G_x \\ g \neq \mathrm{id}}} e_{x,g} \right)^{G_x} ,$$

where both sides come equipped with non-degenerate quadratic forms given by intersection numbers and degrees of the orbifold product respectively. More precisely, by (8) and (9):

$$(E_{x,\rho_1} \cdot E_{x,\rho_2}) = \begin{cases} -2 & \text{if } \rho_1 = \rho_2; \\ 1 & \text{if } \rho_1, \rho_2 \text{ are adjacent;} \\ 0 & \text{if } \rho_1, \rho_2 \text{ are not adjacent;} \end{cases}$$

$$(e_{x,g} \cdot e_{x,h}) = \begin{cases} 1 & \text{if } g = h^{-1}; \\ 0 & \text{if } g \neq h^{-1}; \end{cases}$$

which are both clearly non-degenerate. Now Proposition 3.4 shows that our matrix (12) respects the non-degenerate quadratic forms on both sides, therefore it is non-degenerate.

Let us note here also an elemetary proof which does not use the orbifold product. We first remark that for any $g \neq \mathrm{id}$, $\rho_0(g) \in \mathrm{SL}_2(\mathbf{C})$ which is of finite order and different from the identity, hence its trace $\chi_0(g) \neq 2$. Therefore the non-degeneratedness of the matrix (12) is equivalent to the non-degeneratedness of the matrix

$$\left(\chi_\rho(g)\right)_{(\rho,[g])},$$

which is obtained from the character table of the finite group G_x by removing the first row (corresponding to the trivial representation) and the first column (corresponding to $\mathrm{id} \in G_x$). The non-degeneratedness of this matrix is a completely general fact, which holds for all finite groups. We will give a proof in Lemma 3.6 at the end of this section. □

The combination of Lemma 3.3, Proposition 3.4 and Proposition 3.5 proves the isomorphism of algebra objects (2) in the main Theorem 1.2. For the isomorphisms for the Chow rings and cohomology rings, it is enough to apply realization functors. For the isomorphisms for the K-theory and topological K-theory, it suffices to invoke the construction of *orbifold Chern characters* in [18] which induce isomorphisms of algebras from (orbifold) K-theory to (orbifold) Chow ring as well as from (orbifold) topological K-theory to (orbifold) cohomology ring. The proof of Theorem 1.2 is now complete. □

The following lemma is used in the second proof of Proposition 3.5. The elegant proof below is due to Cédric Bonnafé. We thank him for allowing us to use it. Recall that for a finite group G, its *character table* is a square matrix whose rows are indexed by isomorphism classes of irreducible complex representations of G and columns are indexed by conjugacy classes of G.

Lemma 3.6. *Let G be any finite group. Then the matrix obtained from the character table by removing the first row corresponding to the trivial representation and the first column corresponding to the identity element, is non-degenerated.*

Proof. Denote by $\mathbb{1}$ the trivial representation and by $\rho_1, \cdots, \rho_n$ the set of isomorphism classes of non-trivial representations of G. Suppose we have a linear combination $\sum_{i=1}^n c_i\chi_{\rho_i}$, with $c_i \in \mathbf{C}$, which vanishes for all non-identity conjugacy class, hence for all non-identity elements of G:

$$\sum_{i=1}^{n} c_i\chi_{\rho_i}(g) = 0, \ \forall g \neq \mathrm{id} \in G. \tag{13}$$

Set

$$c_0 := -\frac{1}{|G|}\sum_{i=1}^{n} c_i \dim(\rho_i),$$

and denote by χ_{reg} be the character of the regular representation, then (13) implies that the following linear combination vanishes for all $g \in G$:

$$c_0\chi_{reg} + \sum_{i=1}^{n} c_i\chi_{\rho_i} = 0.$$

If $c_0 \neq 0$, it contradicts to the fact that the trivial representation should appear (with multiplicity 1) in the regular representation.
Hence we have $c_0 = 0$. Then by the linear independency among the characters of irreducible representations, we must have $c_1 = \cdots = c_n = 0$. □

4. Final remarks

We note here another problem on multiplicative structures pointed out by Reid in [22, P. 69]: the irreducible factors and their multiplicities in the tensor product $\rho_1 \otimes \rho_2$ of two irreducible representations $\rho_1, \rho_2 \in \mathrm{Irr}'(G)$ are not encoded in the intersection number $(E_{\rho_1} \cdot E_{\rho_2})$. Whence is the question:

Question : How can we recover the multiplication table of the representation ring of a Klein group G ? Is it encoded in the orbifold product ?

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