Chapter 7

Borel subgroups

7.1 Borel fixed point Theorem

7.1.1 Reminder on complete varieties

Definition 7.1.1 (i) Let \( \phi : X \to Y \) be a morphism, the \( \phi \) is called proper if \( \phi \) is universally closed i.e. for any \( Z \), the morphism \( \phi \times \text{Id}_Z : X \times Z \to Y \times Z \) is closed.

(ii) A variety \( X \) over \( k \) is called proper or complete if the morphism \( X \to \text{Spec}(k) \) is proper.

Example 7.1.2 The variety \( \mathbb{A}^1_k \) is not proper. The point \( \text{Spec}(k) \) is proper.

Theorem 7.1.3 The projective spaces are proper varieties.

Proposition 7.1.4 Let \( \phi : X \to Y \) and \( \psi : Y \to Z \) be morphisms. If \( \phi \) and \( \psi \) are proper, then so is \( \psi \circ \phi \).

Proof. Exercise. \( \square \)

Proposition 7.1.5 Let \( X \) be a proper variety.

(i) If \( Y \) is closed in \( X \), then \( Y \) is proper.

(ii) If \( Y \) is proper, then so is \( X \times Y \).

(iii) If \( \phi : X \to Y \) is a surjective morphism, then \( Y \) is proper.

(iv) If \( \phi : X \to Y \) is a morphism, then \( \phi(X) \) is closed in \( Y \) and proper.

(v) If \( X \) is connected, then \( k[X] = 1 \).

Proof. Exercise. \( \square \)

Corollary 7.1.6 (i) Any projective variety is proper.

(ii) Any proper quasi-projective variety is projective.

Corollary 7.1.7 If \( X \) is affine and proper, then \( X = \text{Spec}(k) \).

Proof. Indeed, we have \( k[X] = k \). \( \square \)

Remark 7.1.8 There exists proper non projective varieties.
Corollary 7.1.9 Let $\phi : X \to Y$ be a $G$-equivariant morphism between $G$-homogeneous spaces. Assume that $\phi$ is bijective, then if $Y$ is propre, so is $X$.

Proof. Let $Z$ be a variety and consider the diagram

$$
\begin{array}{ccc}
X \times Z & \xrightarrow{\phi \times \text{Id}_Z} & Y \times Z \\
\downarrow{p_X} & & \downarrow{p_Y} \\
Z & \xrightarrow{} & Z.
\end{array}
$$

Let $W$ be a closed subset in $X \times Z$ and let $W' = p_X(W)$ be its image under the left vertical map. We have the equality $W' = p_X(W) = (\phi \times \text{Id}_Z) \circ p_Y(W)$ and because $p_Y$ is closed we only need to prove that $\phi \times \text{Id}_Z(W)$ is closed i.e. $\phi \times \text{Id}_Z$ is a closed morphism. But $\phi$ is universally open (because $G$-equivariant between homogeneous $G$-spaces) thus $\phi \times \text{Id}_Z$ is open. It is bijective thus a homeomorphism. Therefore it is closed. □

7.1.2 Borel fixed point Theorem

Lemma 7.1.10 Let $X$ be a variety and $G$ acting on $X$. Then $X^G = \{x \in X / gx = x \text{ for all } g \in G\}$ is closed in $X$.

Proof. Let $g \in G$, then the set $X^g = \{x \in X / gx = x\}$ is the inverse image of the diagonal $\Delta_X$ in $X \times X$ under the morphism $X \to X \times X$ defined by $x \mapsto (x,gx)$. Therefore it is closed. The set $X^G$ is the intersection of all $X^g$ and thus is also closed. □

Theorem 7.1.11 Let $G$ be a connected solvable group acting on $X$ a non empty proper variety. Then $G$ has a fixed point in $X$.

Proof. We proceed by induction on $\dim G$. For $\dim G = 0$, this is obvious since $G = \{e\}$. Otherwise, the group $D(G)$ is a proper subgroup in $G$ therefore $X^{D(G)}$ is non empty. This subset is closed thus proper. We claim that it is $G$-stable. Indeed, for $x \in X^{D(G)}$, $g \in G$ and $g' \in D(G)$, we have $g'gx = gg^{-1}g'gx = gx$ because $D(G)$ is normal and thus we have the inclusion $g^{-1}g'g \in D(G)$.

Let $Gx$ be a minimal orbit of $G$ in $X^{D(G)}$. It has to be closed therefore proper. Let $G_x$ be the stabiliser of $x$. We have a bijective morphism $G/G_x \to Gx$ between $G$-equivariant homogeneous $G$-spaces. Therefore as $Gx$ is proper, so is $G/G_x$. But $G_x$ contains $D(G)$ therefore $G_x$ is a normal subgroup in $G$ and the quotient $G/G_x$ is affine. Being connected, proper and affine the quotient $G/G_x$ is a point and so is the orbit $Gx$. Therefore $x$ is a fixed point for the action of $G$. □

We may recover the Lie-Kolchin’s Theorem from the above result.

Theorem 7.1.12 Let $G$ be a connected solvable group and let $\rho : G \to \text{GL}(V)$ be a rational representation. Then there exists a basis of $V$ such that $\rho(G) \subset B_n$.

Proof. As usual, by induction on $\dim V$, we only need to prove that there exists a one dimensional subspace of $V$ stable under the action of $G$. This is equivalent to the existence of a fixed point in $\mathbb{P}(V)$ and follows from the former statement. □
7.2 Cartan Subgroups

7.2.1 Borel pairs

Definition 7.2.1 Any maximal closed solvable connected subgroup of $G$ is called a Borel subgroup of $G$.

Theorem 7.2.2 Let $G$ be a connected algebraic group. Then all Borel subgroups are conjugated and if $B$ is a Borel subgroup, then $G/B$ is projective.

Proof. Let $S$ be a Borel subgroup of maximal dimension. By Chevalley’s Theorem, there exists a representation $V$ of $G$ together with a line $V_1 \subset V$ such that $S = G_{V_1}$. We claim that we may assume $V$ to be faithful. Indeed, let $W$ be a faithful representation of $G$ and consider the representation $V \oplus W$. Then $G_{V_1} = S$ also for this representation.

So we assume $V$ to be faithful. By Lie-Kolchin’s Theorem, there exists a complete flag $V_\bullet = V_1 \subset V_2 \subset \cdots \subset V_n = V$ stable under $S$. We have $S \subset G_{V_1} \subset G_{V_2} = S$ thus $S = G_{V_\bullet}$. We thus have a bijective morphism

$$G/S \to G_{V_\bullet} \subset \mathcal{F}$$

where $\mathcal{F}$ is the variety of all flags. Let us prove that $G_{V_\bullet}$ is a minimal orbit therefore closed. Indeed, let $V'_\bullet$ be another complete flag in $V$ and let $G_{V'_1}$ be its stabiliser. The elements in $G_{V'_1}$ are upper triangular matrices for a basis compatible with $V'_\bullet$ thus $G_{V'_1}$ is connected. By assumption, we have $\dim G_{V'_1} = \dim G_{V_1} \leq \dim S$. Therefore $\dim G_{V'_\bullet} \geq \dim G_{V_\bullet}$ proving the minimality.

But $\mathcal{F}$ is a closed subset in the product of all grassmannians therefore it is projective. In particular the orbit $G_{V_\bullet}$ is proper. We deduce that $G/S$ is proper. Being quasi-projective, it is projective.

Let $B$ be any Borel subgroup. Then it acts on $G/S$ by left multiplication. It has a fixed point $gS$ i.e. $Bg \subset gS$. Thus $B \subset gSg^{-1}$. By maximality we must have equality. □

Definition 7.2.3 A couple $(B,T)$ with $B$ a Borel subgroup and $T$ a maximal torus of $G$ contained in $B$ is called a Borel pair.

Corollary 7.2.4 (i) Any maximal torus $T$ of $G$ is contained in a Borel subgroup $B$. Furthermore the Borel pairs are conjugated.

(ii) The maximal closed connected unipotent subgroups of $G$ are all connected and of the form $B_u$ for some Borel subgroup $B$ of $G$.

Proof. (i) Let $T$ be a maximal torus. It is closed connected and solvable therefore contained in a maximal such group: a Borel subgroup. It is a maximal torus of $B$. Because any two Borel subgroups are conjugated and any two maximal tori in $B$ are conjugated, the result follows.

(ii) Let $U$ be unipoetne maximal. It is closed connected and solvable therefore contained in a maximal such group: a Borel subgroup. It is a maximal unipotent subgroup of $B$. But $B_u$ is such a group thus $U = B_u$. There are conjugated because Borel subgroups are and that $(gBg^{-1})_u = gB_u g^{-1}$. □

Definition 7.2.5 A closed subgroup $P$ of $G$ is called a parabolic subgroup if $G/P$ is complete (and therefore projective).

Proposition 7.2.6 Let $P$ be a closed subgroup of $G$. The following conditions are equivalent.

(i) The subgroup $P$ is a parabolic subgroup of $G$.

(ii) The subgroup $P$ contains a Borel subgroup.
Proof. If $P$ contains a Borel subgroup $B$, then we have a surjective morphism $G/B \to G/P$ thus $G/P$ is proper since $G/B$ is. Conversely, if $G/P$ is proper, then any Borel subgroup $B$ has a fixed point $gP$ in $G/P$ thus $Bg \subset gP$ and $g^{-1}Bg \subset P$. □

Corollary 7.2.7 A closed subgroup $B$ in $G$ is a Borel subgroup if and only if it is a connected solvable parabolic subgroup.

Theorem 7.2.8 Let $\phi : G \to G'$ be a surjective morphism of algebraic groups. Let $H$ be a closed subgroup of $G$. If $H$ is a parabolic subgroup, a Borel subgroup a maximal torus or a maximal unipotent subgroup, then so is $\phi(H)$. Furthermore, any such subgroup is obtained in that way.

Proof. Because the map $\phi$ is surjective, the morphism $\phi$ realises $G'$ as a homogeneous $G$-space. The morphism $G/H \to G'/\phi(H)$ induced by $\phi$ is surjective thus if $H$ is a parabolic subgroup, so is $\phi(H)$.

If $H$ is a Borel subgroup, then $\phi(H)$ is connected and solvable thus a Borel subgroup.

If $H$ is a maximal unipotent subgroup, then $H = B_u$ for some Borel subgroup and $\phi(H) = \phi(B_u) \subset \phi(B)_u$. Furthermore if $\phi(g) \in \phi(B)_u$, then there exists $b \in B$ such that $\phi(g) = \phi(b)$. Write $b = b_u b_u$ the Jordan decomposition. We get $\phi(g) = \phi(b) \phi(b)_u$ which is the Jordan decomposition of $\phi(g)$ thus $\phi(b_u) = e$ and $\phi(g) = \phi(b_u)$ i.e we have the equality $\phi(H) = \phi(B)_u$ is a maximal unipotent subgroup.

If $H$ is a maximal torus, let $B$ be a Borel subgroup containing $H$. Then $\phi(H)$ is again a torus of $\phi(B)$ a Borel subgroup in $G'$. Furthermore, we have $B = HB_u$ thus $\phi(B) = \phi(H) \phi(B)_u$ thus $\phi(H)$ is a maximal torus of $\phi(B)$ and thus a maximal torus of $G'$.

If $H'$ is a Borel subgroup, a maximal unipotent subgroup or a maximal torus of $G'$ and $H$ is of the same type. Then $\phi(H)$ is of the same type and there exists $g' = \phi(g)$ such that $H' = g' \phi(H) g'^{-1} = \phi(g H g^{-1})$. If $H'$ is a parabolic subgroup, then $H'$ contains a Borel subgroup $\phi(B)$ with $B$ a Borel subgroup of $G$. Then $H = \phi^{-1}(H')$ contains $B$ and is therefore a parabolic subgroup of $G$ with $\phi(H) = H'$.

□

7.2.2 Centraliser of Tori, Cartan subgroups

Lemma 7.2.9 (i) Let $G$ be a connected algebraic group and let $B$ be a Borel subgroup of $G$. Let $\phi \in \text{Aut}(G)$ with $\phi|_B = \text{Id}_B$, then $\phi = \text{Id}_G$.

(ii) As a consequence, if $g \in G$ centralises $B$, then $g \in Z(G)$ i.e. $C_G(B) \subset Z(G)$.

(iii) In particular $Z(B) \subset Z(G)$.

Proof. Let $\phi : G \to G$ be such an automorphism. It is constant on $B$ therefore it can be factorised through the quotient $G/B$ i.e. there exists a morphism $\psi : G/B \to G$ such that $\phi = \psi \circ \pi$ with $\pi : G \to G/B$ the quotient map. But $G/B$ is proper thus $\psi(G/B)$ is proper and closed in $G$ therefore affine. This implies that $\psi(G/B)$ is anopt and the result follows. □

Proposition 7.2.10 Let $G$ be an algebraic group and let $B$ be a Borel subgroup of $G$. If $B$ is nilpotent, then $G^0 = B$.

Proof. We may assume that $G$ is connected. We proceed by induction on $\dim G$. If $B = \{e\}$, then $G = G/B$ is affine and proper thus $G = \{e\}$. If not, let $n$ be such that $C^n(B)$ is non trivial but $C^{n+1}(B) = \{e\}$. The group $C^n(B)$ is central in $B$ therefore it is central in $G$. We may thus look at the quotients $B/C^n(B) \subset G/C^n(B)$. By induction we have $G/C^n(B) = B/C^n(B)$ and the result follows. □
Corollary 7.2.11 Let $G$ be a connected group of dimension at most 2, then $G$ is solvable.

Proof. Let $B$ be a Borel subgroup of $G$. We want to prove that $G = B$. Let us write $B = TB_u$ with $T$ a maximal torus of $G$. If dim $B = 1$, then $B = T$ or $B = B_u$ thus $B$ nilpotent and the result follows from the previous proposition. If dim $B = 2$, then $B = G$ because $G$ is connected therefore irreducible. □

Corollary 7.2.12 Let $G$ be a connected algebraic group.

(i) If $G = G_s$, then $G$ is a torus.

(ii) If $G_u$ is a subgroup, then $G$ is solvable.

(iii) If $G_s$ is a subgroup, then $G$ is nilpotent.

Proof. (i) Let $B$ be a Borel subgroup of $G$. Then $B = TB_u = T$ thus $B$ is nilpotent and $G = B = T$.

(ii) The subgroup $G_u$ is normal since for $g \in G$ and $g_u \in G_u$ we have $gg_u g^{-1} \in G_u$. We may consider the quotient $G/G_u$ whose elements are all semisimple thus $G/G_u$ is a torus. Therefore $G$ is an extension of $T$ and $G_u$ both of which are solvable thus $G$ is solvable.

(iii) Let $B$ be a Borel subgroup. The subgroup $B_s = B \cap G_s$ is commutative by the structure Theorem on solvable groups. Thus we may embed $B$ in $\text{GL}_n$ such that $B_s = D_n \cap B$ therefore $B_s$ is a closed subgroup of $B$. This subgroup is normal in $B$ (because the conjugate of a semisimple element is again semisimple) and thus it is central: $B = N_B(B_s) = C_B(B_s)$. This implies by the characterisation of nilpotent groups that $B$ is nilpotent. By the above proposition $G = B$. □

Proposition 7.2.13 Let $T$ be a maximal torus in $G$, then $C = C_G(T)^0$ is nilpotent and $C = N_G(C)^0$.

Proof. Let $g \in C_s$ an element which is semisimple. Then $gt = tg$ for all $t \in T$. Let $H$ be the closed subgroup spanned by $T$ and $g$. Then $H$ is commutative all its elements are semisimple therefore it is a torus and $T \subset H$ thus $T = H$ and $g \in T$. This proves that $C_s = T$ is a subgroup thus $C$ is nilpotent.

Another proof: let $B$ be a Borel subgroup of $C$, then $T$ is a maximal torus of $B$ and is central in $B$ thus $B$ is nilpotent and thus $C$ is also nilpotent.

We know that $C = N_G(T)^0$. Let us prove the inclusion $N_G(C) \subset N_G(T)$. Note that $C$ being nilpotent, then $C_s$ is a closed subgroup containing $T$ and thus equal to $T$. But $C_s$ is stable under conjugation thus if $g \in N_G(C)$, then $gtg^{-1} = gc_s g^{-1} \subset C \cap G_s = C_s = T$ proving the result. □

7.2.3 Cartan subgroups

Definition 7.2.14 Let $G$ an algebraic group and let $T$ be a maximal torus. The group $C = C_G(T)^0$ is called a Cartan subgroup of $G$.

Remark 7.2.15 We shall prove later that $C_G(T)$ is connected therefore is a Cartan subgroup.

Lemma 7.2.16 Let $G$ be a connected algebraic group and let $H$ be a closed subgroup. Let us set

$$X = \bigcup_{g \in G} gHg^{-1}.$$

(i) The subset $X$ contains a dense open subset of its closure $\overline{X}$.

(ii) If $G/H$ is proper, then $X$ is closed.

(iii) If $N_G(H)/H$ is finite and there exists an element $g \in G$ which is contained in a finite number of conjugates of $H$, then $\overline{X} = G$. 

Proof. (i) Let $M = \{(x,y) \in G \times G \mid y \in xHx^{-1}\}$. This is a closed irreducible subvariety in $G \times G$. Indeed, it is the image of $G \times H$ under the isomorphism $G \times G \to G \times G$ given by $(x,y) \mapsto (x,xyx^{-1})$. The variety $X$ is the image of the second projection and the result follows since this image is constructible by (one of the many) Chevalley’s Theorem.

(ii) If $G/H$ is proper we simply factor the above map through $G/H \times G$. Indeed, the relation $y \in xHx^{-1}$ only depends on the class of $x$ in $G/H$. More precisely, we defined $N = \{(xH,y) \in G/H \times G \mid y \in xHx^{-1}\}$. We have a projection $\psi : G \times G \to G/H \times G$ whose restriction maps $M$ to $N$. We have $M = \psi^{-1}(N) = \psi^{-1}(\psi(M))$. In particular, because $\psi$ is open (the quotient map is universally open) we get that $N = \psi(M)$ is closed. But $G/H$ is proper so the projection $p : G/H \times G \to G$ is closed and $X = p(N)$ is closed.

(iii) Let $q$ be the projection of $N$ into $G/H$ and $p$ the projection to $G$. The map $q$ is surjective with fibers isomorphic to $H$. Therefore $\dim N = \dim G$. On the other hand, let $g \in G$ an element contained in finitely many conjugates of $H$, say $g \in x_iHx_i^{-1}$ for $i \in [1,n]$. We consider the fiber $p^{-1}(g) = \{xH \in G/H \mid g \in xHx^{-1}\}$. For $xH$ in the fiber we have $xHx^{-1} = x_iHx_i^{-1}$ thus $x^{-1}x_i \in N_G(H)$ thus $xH$ and $x_iH$ are equal modulo an element in $N_G(H)/H$. Therefore $p^{-1}(g)$ is finite thus $\dim p(N) = \dim N = \dim G$ and $G$ being connected we have the equalities $\overline{X} = p(N) = G$.

Theorem 7.2.17 Let $G$ be a connected algebraic group.

(i) The union of all Cartan subgroups (i.e. $\cup_{\text{torus } C_G(T)} C_G(T)^0$) contains a dense open subset of $G$.

(ii) The group $G$ is equal to the union of all Borel subgroups.

(iii) Any semisimple elements is contained in a maximal torus.

(iv) Any unipotent elements element of $G$ is contained in a maximal connected unipotent subgroup.

Proof. (i) Let $T$ be a maximal torus and let $C = C_G(T)^0$. We know that $C = N_G(C)^0$ thus $N_G(C)/C$ is finite. We also know that there exists $t \in T$ such that $C_G(T) = C_G(t)$. Let us prove that $t$ is in finitely many conjugate of $C$. If $t \in xCx^{-1}$, then $x^{-1}tx \in C$ thus $x^{-1}tx \in C_s = T$ (because $C$ is nilpotent therefore $C_s$ is a subgroup and thus the unique maximal torus of $C$). Therefore $C \subset C_G(x^{-1}tx) = x^{-1}C_G(t)x = x^{-1}C_G(T)x$. We get $C = x^{-1}Cx$. So $t$ is contained in only one conjugate of $C$: the group $C$ itself. By the previous lemma we get that the union of Cartan subgroups is dense and therefore contains a dense open.

(ii) The group $C$ being connected and nilpotent, it is contained in some Borel subgroup $B$ of $G$. Therefore the union of all Borel subgroups is dense but because $G/B$ is proper it is also closed by the previous lemma and the result follows.

(iii) Let $s$ be a semisimple element in $G$. It is in a Borel subgroup $B$ and by the structure theorem of Borel subgroups it is in a maximal torus of $B$ which is also a maximal torus of $G$.

(iv) Let $u$ be unipotent, it is contained in some Borel $B$ and thus in $B_u$, this is the result.

Corollary 7.2.18 Let $G$ be connected and assume that there exists a normal Borel subgroup, then $G = B$ i.e. the group $G$ is solvable.

Proof. First proof, the quotient $G/B$ is affine and proper. It is connected thus it is a point.

Second proof, the group $G$ is the union of the conjugates of $B$, there is a unique such conjugate $B$ itself.

Corollary 7.2.19 Let $G$ be connected, then we have the equality $Z(G) = Z(B)$ for any Borel subgroup $B$. 
**7.2. CARTAN SUBGROUPS**

Proof. We already know the inclusion $Z(B) \subset Z(G)$. Let $g \in Z(G)$, then there exists a Borel subgroup $B$ such that $g \in B$ and thus $g \in Z(B)$. Furthermore, if $xBx^{-1}$ is another Borel subgroup, then $g = xgx^{-1} \in xBx^{-1}$ and the result follows. \qed

**Lemma 7.2.20** Let $G$ be connected and $S$ be a connected solvable subgroup. Let $x \in C_G(S)$. Then there exists a Borel subgroup containing $S$ and $x$.

Proof. Let $B$ be a Borel subgroup containing $x$. In particular the variety $(G/B)^{\phi}$ contains $eB$. Let $S$ act on $G/B$, it stabilises $(G/B)^{\phi}$ which is proper thus there is a fixed point $gB$. We have $Sg \subset gB$ thus $S \subset gBg^{-1}$ and $xgB = gB$ thus $x \in gBg^{-1}$.

**Theorem 7.2.21** Let $G$ be connected and $S$ be a torus in $G$.

(i) Then $C_G(S)$ is connected.

(ii) If $B$ is a Borel subgroup of $G$ containing $S$, then $B \cap C_G(S)$ is a Borel subgroup of $C_G(S)$.

(iii) Furthermore any Borel subgroup of $C_G(S)$ is obtained in this way.

Proof. (i) Let $x \in C_G(S)$, then $x$ and $S$ are contained in some Borel subgroup $B$. Then $x \in C_B(S)$ which is connected by the structure Theorem on connected solvable groups. Therefore $x \in C_G(S)^0$ and $C_G(S) = C_G(S)^0$.

(ii) Set $C = C_G(S)$. It is enough to prove that $C/C \cap B$ is proper therefore it is enough to prove that $C(eB) \subset G/B$ is closed. Because the map $\pi : G \to G/B$ is open, it is enough to prove that $\pi^{-1}(C(eB)) = CB$ is closed. Note that this variety is irreducible as the image of $C \times B$ by multiplication.

For $y = cb \in CB$ with $c \in C$ and $b \in B$, we have $y^{-1}Sy = b^{-1}c^{-1}Scb = b^{-1}Sb \subset B$ because $S \subset B$. Therefore for any $y \in CB$ we also have $y^{-1}Sy \subset B$.

Let $T$ be a maximal torus of $B$ containing $S$ and let $\phi : B \to B/B_u$ be the quotient map. It realises an isomorphism from $T$ to $B/B_u$. We may consider the morphism $\psi : CB \times S \to B/B_u$ defined by $(y,s) \mapsto \phi(y^{-1}sy)$. By the rigidity lemma we have that $\psi$ does not depend on $y$ (we need $CB$ to be affine, we need that $S$ and $B/B_u$ are diagonalisable and that $\psi_y$ is a group morphism).

Now let $y \in CB$, we have $y^{-1}Sy$ is a torus in $B$ thus there exists $u \in C^\infty(B) \subset B_u$ such that $u^{-1}y^{-1}Syu \subset T$. Furthermore, for any $s \in S$ we have $\psi(u^{-1}y^{-1}syu) = \psi(yu,s) = \psi(s) = \pi(s)$ (for this note that because $CB$ is stable by right multiplication by elements in $B$, so is $CB$ thus $yu \in CB$). But $\pi$ is injective on $T$ and $u^{-1}y^{-1}syu$ and $s$ are in $T$ thus $s = u^{-1}y^{-1}syu$ for all $s \in S$ thus $yu \in C$ and $y \in CB$. Thus $CB$ is closed proving the result.

(iii) Let $B'$ be a Borel subgroup of $C = C_G(S)$. Let $B$ be a Borel subgroup of $G$ containing $S$. Then there exists $c \in C$ such that $B' = c(B \cap C)c^{-1}$. But $cCc^{-1} = C$ and $B = cBc^{-1} \cap C$. This is what we wanted. \qed

**Corollary 7.2.22** Let $G$ be a connected group and $T$ a maximal torus. Let $C = C_G(T)$.

(i) The group $C$ is connected, nilpotent and equal to $N_G(C)^0$ (thus the quotient $N_G(C)/C$ is finite).

(ii) Any Borel subgroup $B$ containing $T$ contains $C$.

Proof. (i) The previous theorem implies the connectedness and we already proved that $C$ is nilpotent and equal to $N_G(C)^0$.

(ii) If $B$ contains $T$, then $B \cap C$ is a Borel subgroup of $C$ and is nilpotent as a subgroup of $C$. Thus we must have $C = C \cap B$. \qed
7.3 Normalisers of Borel subgroups

Theorem 7.3.1 (Chevalley) Let $G$ be a connected group.

(i) For any Borel subgroup, we have the equality $B = N_G(B)$.

(ii) For any parabolic subgroup, we have the equalities $N_G(P) = P = P^0$.

(iii) For any Borel subgroup we have the equality $B = N_G(B_u)$.

Proof. (i) We proceed by induction on $\dim G$. If $\dim G \leq 2$, then $G$ is solvable thus $G = B$ and the result follows.

Set $N = N_G(B)$ and let $n \in N$. Let $T$ be a maximal torus of $G$ contained in $B$. Then $nTn^{-1}$ is again a maximal torus contained in $B$. Therefore there exists $b \in B$ with $bnT(bn)^{-1} = T$. Replacing $n$ by $nb$ we may assume that $n \in N_G(T)$.

Consider the morphism $\psi : T \to T$ defined by $\psi(t) = ntn^{-1}t^{-1}$. This is a morphism of algebraic groups. Let $S = (\ker \psi)^0$ which is a subtorus of $T$. Then $n$ lies in $C_G(S)$.

Assume first that $S$ is not trivial. Then $n$ normalises $B \cap C_G(S)$ which is a Borel subgroup of $S$ thus if $C_G(S) \neq G$, we get by induction that $n$ lies in $B \cap C_G(S)/n \in B$. If $C_G(S) = G$, then $S$ is central in $G$ thus the quotient $G/S$ is an algebraic group and $B/S$ is a Borel subgroup. The element $nS$ is in $N_{B/S}(G/S)$ and by induction again we have $nS \subset B$ thus $n \in B$.

Assume now that $S$ is trivial. Then $\psi$ is surjective (because its image is a closed connected subgroup of the same dimension as $T$). Let $V$ be a representation of $G$ such that $N = G_U$ for some subspace of dimension 1 in $V$. Then $N$ acts on $U$ via a character $\chi \in X^*(N)$. This character has to be trivial on $B_u$ because it maps unipotent elements to unipotent elements in $\mathbb{G}_m$. It also has to be trivial on $T$ because any element of $T$ is a commutator. Therefore $B$ acts trivially on $U$ and if $u$ is a non trivial vector in $U$, the morphism $G \to V$ defined by $g \mapsto gu$ factors through $G/B$. But $G/B$ is proper thus the image is proper and closed in $V$ thus affine. Therefore the image is constant and $G$ acts trivially on $u$. We get $B = G = N$.

(ii) Let $P$ be a parabolic subgroup and $B$ a Borel subgroup contained in $P$. We have $B \subset P^0$ because $B$ is connected. Let $n \in N_G(P)$, then $xBx^{-1}$ is again a Borel subgroup of $P^0$. Thus there exists $p \in P^0$ with $xBx^{-1} = pBp^{-1}$. Therefore $p^{-1}x \in N_G(B) = B$ thus $x \in P^0$ and the result follows.

(iii) Let $U = B_u$ and $N = N_G(U)$. We have $B \subset N$ thus $B$ is a Borel subgroup of $N^0$ (it has to be maximal). Therefore, any unipotent element in $N^0$ is conjugated to an element in $U$. But $U$ being normal in $N^0$, we have $U = (N^0)_u$. Therefore $N^0/U$ is a torus (connected and all elements are semisimple). Thus $N^0$ is solvable. Thus $N^0 = B$. Furthermore, because $N$ normalises $N^0$ we get $N \subset N_G(B) = B$ proving the result.

Corollary 7.3.2 Let $G$ be connected, let $B$ be a Borel subgroup and let $P$ and $Q$ be two parabolic subgroups containing $B$ and conjugated in $G$. Then $P = Q$.

Proof. We have $Q = gPg^{-1}$ thus $B$ and $gBg^{-1}$ are Borel subgroups of $Q$. Therefore there exists $qw \in Q$ with $qBg^{-1} = gBg^{-1}$. We get $gg^{-1} \in N_G(B) = B$ thus $g \in Q$ and $P = Q$. □

7.4 Reductive and semisimple algebraic groups

7.4.1 Radical and unipotent radical

Definition 7.4.1 Let $G$ be an affine algebraic group.

(i) We define the radical of $G$ to be the maximal closed connected solvable normal subgroup of $G$. We denote it by $R(G)$. 
We define the unipotent radical of $G$ to be the maximal closed connected unipotent normal subgroup of $G$. We denote it by $R_u(G)$.

Let us denote by $\mathcal{B}$ the set of all Borel subgroups of $G$.

**Proposition 7.4.2** We have the equalities

$$R(G) = \left( \bigcap_{B \in \mathcal{B}} B \right)^0 \text{ and } R_u(G) = \left( \bigcap_{B \in \mathcal{B}} B_u \right)^0 = R(G)_u.$$ 

**Proof.** The above intersection is obviously a closed connected solvable group of $G$. Furthermore since any two Borel subgroups are conjugated it is also normal. Therefore the intersection is contained in $R(G)$. Conversely, the group $R(G)$ being solvable and connected, it is contained in all Borel subgroup thus in the above intersection. Note that any automorphism of $G$ maps a Borel subgroup to a Borel subgroup thus the group $R(G)$ is even characteristic.

The same argument give the second equality. For the last one, because $R(G)$ is characteristic and solvable, we have that $R(G)_u$ is a normal subgroup of $R(G)$ and thus also normal of $G$. Being unipotent it is contained in $R_u(G)$. Now if $U$ is a normal unipotent connected subgroup of $G$, it is contained in $R(G)$ and thus in $R(G)_u$. □

### 7.4.2 Reductive and semisimple algebraic groups

**Definition 7.4.3** An algebraic group $G$ is called reductive if $R_u(G) = \{e\}$ and semisimple if $R(G) = \{e\}$.

**Lemma 7.4.4** Let $1 \rightarrow H \rightarrow G \rightarrow K \rightarrow 1$ be an exact sequence of algebraic groups. The group $G$ is unipotent if and only if $H$ and $K$ are also unipotent.

**Proof.** Exercise. □

**Proposition 7.4.5** The quotient $G/R_u(G)$ is reductive and the quotient $G/R(G)$ is semisimple.

**Proof.** Let $\pi : G \rightarrow G/R(G)$ be the quotient map and let $H$ be a connected closed normal solvable subgroup of $G/R(G)$. Then $\pi^{-1}(H)$ is also closed connected solvable and normal therefore contained in $R(G)$. The result follows. For the unipotent radical, the same proof works using the previous lemma. □