Chapter 4

Representations

In this chapter we give the very first definition and properties of representations of a Lie algebra. We shall study in more details the representations of semisimple Lie algebras later on in the text.

4.1 Definition

Definition 4.1.1 Let $\mathfrak{g}$ be a Lie algebra and $V$ be a vector space. A representation of $\mathfrak{g}$ in $V$ is a Lie algebra morphism $\mathfrak{g} \rightarrow \mathfrak{gl}(V)$.

Definition 4.1.2 A injective representation $\mathfrak{g} \rightarrow \mathfrak{gl}(V)$ is called faithful. The dimension of $V$ is called the dimension of the representation.

Example 4.1.3 The adjoint representation $\mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ is a representation.

Example 4.1.4 Let $G$ be a Lie group and $G \rightarrow GL(V)$ be a representation of $G$, the differential of this map at identity is a representation of $\text{Lie}(G)$ in $V$.

Remark 4.1.5 A representation of $\mathfrak{g}$ is a linear map $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ such that for $x$ and $y$ in $\mathfrak{g}$ and for $v$ in $V$, we have $\rho([x,y])(v) = \rho(x)\rho(y)(v) - \rho(y)\rho(x)(v)$.

In particular this map is an $\alpha$-map and induces an algebra morphism $\rho_U : U(\mathfrak{g}) \rightarrow \mathfrak{gl}(V)$ such that $\rho = \rho_U \circ f_\mathfrak{g}$. This means that $V$ is a $U(\mathfrak{g})$-module with the action given by $x \cdot v = \rho_U(x)(v)$ for $x \in U(\mathfrak{g})$ and $v \in V$.

Conversely, if $V$ is an $U(\mathfrak{g})$-module whose multiplication is determined by an algebra morphism $\rho_U : U(\mathfrak{g}) \rightarrow \mathfrak{gl}(V)$, then by composition with $f_\mathfrak{g}$ we obtain a representation of $\mathfrak{g}$ in $V$.

There is therefore a one to one correspondence between representations of the Lie algebra $\mathfrak{g}$ and the $U(\mathfrak{g})$-modules.

We transpose the usual notions like, isomorphism, direct sums from $U(\mathfrak{g})$-modules to representations of $\mathfrak{g}$. Let us state more precisely some of these definitions.

Definition 4.1.6 (i) A representation of $\mathfrak{g}$ in $V$ is called simple if $V$ is a simple $U(\mathfrak{g})$-module i.e. if the is no non trivial submodule.

(ii) A representation $V$ is called reducible if there is a decomposition $V = V_1 \oplus V_2$ where $V_i$ are subrepresentations of $V$ for $i \in \{1,2\}$. If the representation is not reducible then we call it irreducible.

(iii) A representation of $\mathfrak{g}$ in $V$ is called semisimple or completely reducible if $V$ is a semisimple $U(\mathfrak{g})$-module i.e. if it is isomorphic to a direct sum of simple modules.

(iv) A representation $\mathfrak{g}$ in $W$ is a subrepresentation of $V$ if $W$ is an $U(\mathfrak{g})$-submodule of $V$. A representation $\mathfrak{g}$ in $W$ is a quotient representation of $V$ if $W$ is an $U(\mathfrak{g})$ quotient module of $V$. 

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Definition 4.1.7 For $V$ a representation of the Lie algebra $\mathfrak{g}$ and for $x$ in $\mathfrak{g}$, we shall denote by $x_V$ the endomorphism of $V$ induced by $x$.

Lemma 4.1.8 Let $V$ be a representation of $\mathfrak{g}$ and $v \in V$, then the subset $\mathfrak{g}_v = \{ x \in \mathfrak{g} / x_V \cdot v = 0 \}$ of $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{g}$.

Proof. It is a subspace and for $x$ and $y$ in $\mathfrak{g}_v$, we have $[x,y]_V \cdot v = x_V \cdot (y_V \cdot v) - y_V \cdot (x_V \cdot v) = 0$. □

4.2 Tensor product of representations

Taking tensor products is a natural operation on representations. Indeed, let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two Lie algebras and let $V_i$ for $i \in \{1, 2\}$ be a representation of $\mathfrak{g}_i$. By the last remark, the vector space $V_i$ for each $i \in \{1, 2\}$ is an $U(\mathfrak{g}_i)$-module and therefore $V_1 \otimes V_2$ is an $U(\mathfrak{g}_1) \otimes U(\mathfrak{g}_2)$-module. But we have seen in Proposition 3.2.2 that $U(\mathfrak{g}_1 \times \mathfrak{g}_2) = U(\mathfrak{g}_1) \otimes U(\mathfrak{g}_2)$ therefore $V_1 \otimes V_2$ is a $\mathfrak{g}_1 \times \mathfrak{g}_2$-representation whose action is given by:

$$(x_1, x_2)_V \cdot (v_1 \otimes v_2) = (f_{\mathfrak{g}_1}(x_1) \otimes 1 + 1 \otimes f_{\mathfrak{g}_2}(x_2)) \cdot v_1 \otimes v_2$$

$$= (x_1)_V \cdot v_1 \otimes v_2 + v_1 \otimes (x_2)_V \cdot v_2.$$

Definition 4.2.1 If $\mathfrak{g} = \mathfrak{g}_1 = \mathfrak{g}_2$, and composing with the inclusion $\mathfrak{g} \rightarrow \mathfrak{g} \times \mathfrak{g}$ of Lie algebras given by $x \mapsto (x,x)$ we obtain, for $V_1$ and $V_2$ two representations of $\mathfrak{g}$ a representation of $\mathfrak{g}$ in $V_1 \otimes V_2$ called the tensor product representation. The action is given by

$$x_V \cdot (v_1 \otimes v_2) = x_V \cdot v_1 \otimes v_2 + v_1 \otimes x_V \cdot v_2.$$

By induction we get

Proposition 4.2.2 Let $(V_i)_{i \in \{1,n\}}$ be representations of the Lie algebra $\mathfrak{g}$, then the tensor product $V = \otimes_{i=1}^n V_i$ is a representation of $\mathfrak{g}$ with action given by the following formula:

$$x_V \cdot (v_1 \otimes \cdots \otimes v_n) = \sum_{i=1}^n v_1 \otimes \cdots \otimes x_V \cdot v_i \otimes \cdots \otimes v_n.$$

4.3 Representations in the space of morphisms

As in the previous section, let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two Lie algebras and let $V_i$ for $i \in \{1, 2\}$ be a representation of $\mathfrak{g}_i$. The vector space $V_i$ for each $i \in \{1, 2\}$ is an $U(\mathfrak{g}_i)$-module and therefore $\text{Hom}(V_1, V_2)$ is an $U(\mathfrak{g}_1)^{\text{op}} \otimes U(\mathfrak{g}_2)$-module. But we have seen in Proposition 3.3.2 that $U(\mathfrak{g}_1)^{\text{op}} = U(\mathfrak{g}_1^{\text{op}})$ therefore $\text{Hom}(V_1, V_2)$ is a $\mathfrak{g}_1^{\text{op}} \times \mathfrak{g}_2$-representation whose action is given, for $\phi \in \text{Hom}(V_1, V_2)$ and $v_1 \in V_1$, by:

$$((x_1, x_2)_V \cdot \phi)(v_1) = ((f_{\mathfrak{g}_1}^{\text{op}}(x_1) \otimes 1 + 1 \otimes f_{\mathfrak{g}_2}(x_2)) \cdot \phi)(v_1)$$

$$= \phi((x_1)_V \cdot v_1) + (x_2)_V \cdot \phi(v_1)$$

$$= ((x_2)_V \circ \phi + \phi \circ (x_1)_V)(v_1).$$

Definition 4.3.1 If $\mathfrak{g} = \mathfrak{g}_1 = \mathfrak{g}_2$, and composing with the inclusion $\mathfrak{g} \rightarrow \mathfrak{g}^{\text{op}} \times \mathfrak{g}$ of Lie algebras given by $x \mapsto (-x, x)$ we obtain, for $V_1$ and $V_2$ two representations of $\mathfrak{g}$ a representation of $\mathfrak{g}$ in $\text{Hom}(V_1, V_2)$. The action is given by

$$x_V \cdot \phi = x_V \circ \phi - \phi \circ x_{V_1}.$$
4.4. INVARIANTS

Combining with Proposition 4.2.2 we get the

**Proposition 4.3.2** Let \((V_i)_{i \in \{1,n+1\}}\) be representations of the Lie algebra \(\mathfrak{g}\), then the space of multi-linear maps \(\text{Hom}(\otimes_{i=1}^{n+1} V_i, V_{n+1})\) is a representation of \(\mathfrak{g}\) with action given by the following formula:

\[
(x_V \cdot \phi)(v_1 \otimes \cdots \otimes v_n) = -\sum_{i=1}^{n} \phi(v_1 \otimes \cdots \otimes x_V \cdot v_i \otimes \cdots \otimes v_n) + (x_{V_{n+1}}) \cdot \phi(v_1 \otimes \cdots \otimes v_n).
\]

**Definition 4.3.3** Let \(V\) be a representation of \(\mathfrak{g}\), then \(V^\vee = \text{Hom}(V, k)\) is called the dual representation of \(V\).

Let \(V\) be a representation of \(\mathfrak{g}\) and consider the representation of \(\mathfrak{g}\) in \(B = \text{Hom}(V \otimes V, k)\) the space of bilinear forms. Let \(b \in \text{Hom}(V \otimes V, k)\), then, by Lemma 4.1.8, the set \(\mathfrak{g}_b = \{x \in \mathfrak{g} / x_B \cdot b = 0\}\) is a Lie subalgebra of \(\mathfrak{g}\). The condition \(x_B \cdot b = 0\) translates here in

\[
b(x_V \cdot v, v') + b(v, x_V \cdot v') = 0 \text{ for all } v \text{ and } v'.
\]

**Example 4.3.4** Let \(V = k^n\), \(\mathfrak{g} = \mathfrak{gl}(V)\) acting on \(V\) by the identity map \(\mathfrak{g} = \mathfrak{gl}(V) \rightarrow \mathfrak{gl}(V)\). We identify \(\mathfrak{g}\) with the vector space of \(n \times n\)-matrices. Choose \(b((x_i)_{i \in [1,n]}, (y_i)_{i \in [1,n]}) = \sum_{i=1}^{n} x_i y_i\) the standard symmetric bilinear form, then we have

\[
\mathfrak{g}_b = \{A \in \mathfrak{gl}(V) / A \text{ is antisymmetric}\} \text{ is a Lie algebra.}
\]

When \(k = R\), then this is the Lie algebra of the orthogonal group \(O(n, R)\).

**Example 4.3.5** Let \(V = k^{2n}\), \(\mathfrak{g} = \mathfrak{gl}(V)\) acting on \(V\) by the identity map \(\mathfrak{g} = \mathfrak{gl}(V) \rightarrow \mathfrak{gl}(V)\). We identify \(\mathfrak{g}\) with the vector space of \(2n \times 2n\)-matrices. Choose the following standard antisymmetric bilinear form \(b((x_i)_{i \in [1,2n]}, (y_i)_{i \in [1,2n]}) = \sum_{i=1}^{n} x_i y_{2n+1-i} - \sum_{i=1}^{n} x_{2n+1-i} y_i\), then we have

\[
\mathfrak{g}_b = \left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{gl}(V) \text{ with } A, B, C \text{ and } D \text{ a } n \times n\text{-matrices } / D = -t A, B = t B \text{ and } C = t C \right\}
\]

is a Lie algebra. When \(k = R\), then this is the Lie algebra of the symplectic group \(\text{Sp}(n, R)\).

4.4 Invariants

**Definition 4.4.1** Let \(V\) be a representation of \(\mathfrak{g}\), an element \(v \in V\) is called invariant if \(\mathfrak{g}_v = \mathfrak{g}\) i.e. \(x_V \cdot v = 0\) for all \(x \in \mathfrak{g}\). The set of invariants of \(\mathfrak{g}\) in \(V\) is denoted by \(V^\mathfrak{g}\).

**Example 4.4.2** Let \(G\) be a Lie group and \(V\) be a representation of \(G\) i.e. we have a Lie group morphism \(G \rightarrow \text{GL}(V)\). Then the differential at the identity defines a representation \(\text{Lie}(G) \rightarrow \mathfrak{gl}(V)\) and if \(v \in V\) is \(G\)-invariant, then \(v\) is an invariant for \(\text{Lie}(G)\). Indeed, we have for \(\epsilon\) small and any \(x \in \text{Lie}(G)\) the equality \(v = (1 + \epsilon x) \cdot v = v + \epsilon(x_V \cdot v)\) and the result follows.

**Example 4.4.3** Let \(V\) and \(W\) be representations of \(\mathfrak{g}\) and let \(\phi \in \text{Hom}(V, W)\). Then \(\phi\) is invariant if and only if \(x_V \circ \phi = \phi \circ x_W\) or equivalently the morphism \(\phi\) is a Lie algebra morphism. In symbols:

\[
\text{Hom}(V, W)^\mathfrak{g} = \text{Hom}_\mathfrak{g}(V, W).
\]
Example 4.4.4 Let $V$ be a representation of $\mathfrak{g}$. There is always an invariant in $\text{Hom}(V, V)$, namely $\text{Id}_V$ (by the previous example). In particular, because $V^\vee \otimes V$ is isomorphic to $\text{Hom}(V, V)$ has a representation of $\mathfrak{g}$ (Exercice!) has an invariant element say $c_V^\otimes$. If $(v_i)_{i \in [1, n]}$ is a basis of $V$ and if $(v'_j)_{i \in [1, n]}$ is the dual basis in $V^\vee$, then
\[ c_V^\otimes = \sum_{i=1}^{n} u_i^\vee \otimes v_i. \]
Remark that the element $c_V^\otimes$ does not depend on the choice of the basis.

Example 4.4.5 Let $V$ be a representation of $\mathfrak{g}$ and let $b \in \text{Hom}(V \times V, k)$ a bilinear form. Then $b$ is an element in $\text{Hom}(V \otimes V, k) = \text{Hom}(V, V^\vee)$. Then $b$ is invariant if and only if the corresponding map $b : V \to V^\vee$ is a morphism of representations. In symbols:
\[ \text{Hom}(V \times V, k)^\theta = \text{Hom}_\mathfrak{g}(V, V^\vee). \]
In particular, if $V$ is finite dimensional and $b$ is non degenerate and invariant, then $b : V \to V^\vee$ is a morphism of representations. We therefore have an invariant $c_V \in V \otimes V$ corresponding to $c_V^\otimes \in V^\vee \otimes V$. If $(v_i)_{i \in [1, n]}$ is a basis of $V$ and if $(v'_j)_{i \in [1, n]}$ is the dual basis for $b$ defined by $b(v_i, v'_j) = \delta_{i,j}$, then
\[ c_V = \sum_{i=1}^{n} v'_i \otimes v_i. \]
Remark that the element $c_V$ does not depend on the choice of the basis.

Proposition 4.4.6 Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{a}$ an ideal in $\mathfrak{g}$. Let $V$ be a representation of $\mathfrak{g}$ and consider it as a representation of $\mathfrak{a}$. Then $V^\mathfrak{a}$ is a representation of $\mathfrak{g}$.

Proof. The subset $V^\mathfrak{a}$ is a subspace of $V$. Furthermore, for $x \in \mathfrak{g}$, $y \in \mathfrak{a}$ and $v \in V^\mathfrak{a}$, we have $y_V \cdot (x_V \cdot v) = [y, x]_V \cdot v + x_V \cdot (y_V \cdot v) = 0$ because $\mathfrak{a}$ is an ideal in $\mathfrak{g}$. \hfill $\square$

4.5 Invariant bilinear forms

Consider the special representation $V = \text{Hom}(\mathfrak{g} \times \mathfrak{g}, k)$ induced by the adjoint representation and the trivial representation.

Definition 4.5.1 A bilinear form $b$ on $\mathfrak{g}$ is called invariant if it is invariant as an element of the representation $\text{Hom}(V \times V, k)$.

Remark 4.5.2 A bilinear form $b$ is invariant if and only if $b(x_\mathfrak{g} \cdot y, z) + b(y, x_\mathfrak{g} \cdot z) = 0$ i.e. iff
\[ b([x, y], z) + b(x, [y, z]) = 0 \]
for all $x$, $y$ and $z$ in $\mathfrak{g}$.

Definition 4.5.3 A bilinear form $b$ on $\mathfrak{g}$ is called fully invariant if for any derivation $D$ in $\text{Der}(\mathfrak{g})$, we have $b(Dx, y) + b(x, Dy) = 0$ for all $x$ and $y$ in $\mathfrak{g}$.
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Proposition 4.5.4 Let \( \mathfrak{g} \) be a Lie algebra and \( \mathfrak{a} \) an ideal of \( \mathfrak{g} \). Let \( \mathfrak{b} \) be an invariant symmetric bilinear form on \( \mathfrak{g} \).

(i) The orthogonal \( \mathfrak{b} \) of \( \mathfrak{a} \) for \( \mathfrak{b} \) is an ideal of \( \mathfrak{g} \).

(ii) If \( \mathfrak{a} \) is a characteristic ideal and \( \mathfrak{b} \) is fully invariant, then \( \mathfrak{b} \) is also a characteristic ideal.

(iii) If \( \mathfrak{b} \) is non degenerate, then \( \mathfrak{a} \cap \mathfrak{b} \) is commutative.

Proof. For (i) and (ii), let \( D \) be an inner (resp. any derivation of \( \mathfrak{g} \)). Let \( x \) be in \( \mathfrak{b} \) and \( y \in \mathfrak{a} \). We have \( b(Dx, y) = -b(x, Dy) = 0 \) and the result follows.

(iii) Let \( x \) and \( y \) in \( \mathfrak{a} \cap \mathfrak{b} \). We have \( b([x, y], z) = b(x, [y, z]) = 0 \) because \( x \in \mathfrak{a} \) and \( [y, z] \in \mathfrak{b} \) are orthogonal for \( \mathfrak{b} \). This is true for any \( z \) thus \([x, y] = 0 \) because \( \mathfrak{b} \) is non degenerate.

\[ \square \]

Definition 4.5.5 Let \( V \) be a finite dimensional representation of \( \mathfrak{g} \). The bilinear form associated to the representation \( V \) is the symmetric bilinear form defined by

\[ (x, y) \mapsto \text{Tr}(xVyV). \]

If \( V \) is the adjoint representation, the the associated bilinear form is called the Killing form. We denote it by \( \kappa_\mathfrak{g} \).

Proposition 4.5.6 Let \( V \) be a finite dimensional representation of \( \mathfrak{g} \), then the associated bilinear form is invariant.

Proof. Let \( x, y \) and \( z \) in \( \mathfrak{g} \). We compute

\[ \text{Tr}([x, y]VzV) = \text{Tr}(xVyzV) - \text{Tr}(yVxVzV) = \text{Tr}(xVyzV) - \text{Tr}(xVzVyV) = \text{Tr}(xV[y, z]V) \]

and the result follows. \( \square \)

Proposition 4.5.7 Let \( \mathfrak{g} \) be a finite dimensional Lie algebra and \( \mathfrak{h} \) an ideal of \( \mathfrak{g} \), then if \( \kappa_\mathfrak{g} \) is the Killing form of \( \mathfrak{g} \) and \( \kappa_\mathfrak{h} \) the Killing form of \( \mathfrak{h} \) we have \( \kappa_\mathfrak{h} = \kappa_\mathfrak{g}\mathfrak{h} \).

Proof. Let \( x \) and \( y \) be elements in \( \mathfrak{h} \), we want to compute the trace of \( \text{ad} x \text{ad} y \) as an endomorphism of \( \mathfrak{h} \) and of \( \mathfrak{g} \). Let call \( u \) the corresponding endomorphism of \( \mathfrak{g} \). As \( \mathfrak{h} \) is an ideal, the image of \( u \) is \( \mathfrak{h} \) and \( u \) induces an endomorphism \( u_\mathfrak{h} \) of \( \mathfrak{h} \) and \( u_\mathfrak{g}/\mathfrak{h} \) of \( \mathfrak{g}/\mathfrak{h} \). This last endomorphism vanishes and therefore \( \text{Tr} u = \text{Tr} u_\mathfrak{h} \). \( \square \)

Proposition 4.5.8 Let \( \mathfrak{g} \) be a finite dimensional Lie algebra, then the Killing form \( \kappa_\mathfrak{g} \) is fully invariant.

Proof. Let \( D \) be a derivation of \( \mathfrak{g} \) and \( x \) and \( y \) elements of \( \mathfrak{g} \). We start with the following

Lemma 4.5.9 Let \( D \) be a derivation of \( \mathfrak{g} \), then there exists a Lie algebra \( \mathfrak{g}' = \mathfrak{g} \oplus kx_0 \) such that for \( x \in \mathfrak{g} \), we have \( Dx = [x_0, x] \) and such that \( \mathfrak{g} \) is an ideal in \( \mathfrak{g}' \).

Proof. Indeed, the derivation \( D \) gives a map \( k \rightarrow \text{det}(\mathfrak{g}) \) defined by \( \lambda \mapsto \lambda D \). As we have already seen, we may then define \( \mathfrak{g}' \) the semidirect product of \( k \) and \( \mathfrak{g} \) which is an extention of \( k \) by \( \mathfrak{g} \) therefore \( \mathfrak{g} \) is an ideal in \( \mathfrak{g}' \). The Lie bracket \([x_0, x]\) is by definition \( Dx \). \( \square \)

By the previous proposition and the above lemma, we have, \( \kappa_\mathfrak{g}(Dx, y) = \kappa_\mathfrak{g}'(Dx, y) = \kappa_\mathfrak{g}'([x_0, x], y) \) but the Killing form \( \kappa_\mathfrak{g}' \) being invariant, we have \( \kappa_\mathfrak{g}'([x_0, x], y) = -\kappa_\mathfrak{g}'(x, [x_0, y]) = -\kappa_\mathfrak{g}(x, Dy) \) and the result follows. \( \square \)
4.6 Casimir element

In this section we construct a very useful element in the enveloping algebra $U(\mathfrak{g})$. Let us first remark that the envelopping algebra $(\mathfrak{g})$ is a representation of the Lie algebra $\mathfrak{g}$. Indeed, the adjoint representation gives a representation of $\mathfrak{g}$ in itself and by looking at the tensor product and direct sum representation, we get that $T(\mathfrak{g})$, the tensor algebra, is a representation of $\mathfrak{g}$, the action being given by

$$x_{T(\mathfrak{g})} \cdot (x_1 \otimes \cdots \otimes x_n) = \sum_{i=1}^{n} (x_1 \otimes \cdots \otimes [x, x_i] \otimes \cdots \otimes x_n).$$

**Lemma 4.6.1** The ideal $J$ generated by the elements of the form $x \otimes y - y \otimes x - [x, y]$ is a subrepresentation of $T(\mathfrak{g})$.

**Proof.** Let $x$, $y$ and $z$ in $\mathfrak{g}$, we need to prove that $x_{T(\mathfrak{g})}$ maps $y \otimes z - z \otimes y - [y, z]$ to an element of $J$. We compute

$$x_{T(\mathfrak{g})} \cdot (y \otimes z - z \otimes y - [y, z]) = [x, y] \otimes z + y \otimes [x, z] - [x, z] \otimes y - z \otimes [x, y] - [x, [y, z]],$$

and the terms in the last sum are in $J$, the result follows. □

**Corollary 4.6.2** The above action of $\mathfrak{g}$ on $T(\mathfrak{g})$ induces an action on $U(\mathfrak{g})$ which is therefore a representation of $\mathfrak{g}$.

**Proposition 4.6.3** Let $\mathfrak{g}$ be a Lie algebra, let $\mathfrak{h}$ be an ideal of finite dimension $n$ and let $b$ be an invariant bilinear form on $\mathfrak{g}$ whose restriction to $\mathfrak{h}$ is non degenerate. Let $(h_i)_{i \in [1, n]}$ and $(h'_i)_{i \in [1, n]}$ be basis of $\mathfrak{h}$ such that $b(h_i, h'_j) = \delta_{i,j}$, then the element in $U(\mathfrak{g})$ defined by

$$c = \sum_{i=1}^{n} h_i h'_i$$

is invariant, lies in the center of $U(\mathfrak{g})$ and is independent of the choice of the basis.

**Proof.** We have already seen in Example 4.4.5 that the element

$$c_{\mathfrak{h}} = \sum_{i=1}^{n} h_i \otimes h'_i \in \mathfrak{h} \otimes \mathfrak{h}$$

does not depend on the basis and is invariant. The above element $c$ is the image of $c_{\mathfrak{h}}$ in $U(\mathfrak{g})$ and therefore is invariant and does not depend on the choice of the base. The element $c$ lies in the center by the next result. □

**Lemma 4.6.4** Let $c$ be an invariant element in $U(\mathfrak{g})$, then $c$ lies in the center of $U(\mathfrak{g})$. 
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Proof. First assume that \( c \) is the image of a pure tensor \( i.e. c = x_1 \cdots x_n \in T(g) \) with the \( x_i \) in \( g \). Then we have

\[
\begin{align*}
x T(g) \cdot c &= \sum_{i=1}^n x_1 \cdots [x_i, x_i] \cdots x_n \\
&= \sum_{i=1}^n x_1 \cdots x_{i-1} x x_i \cdots x_n - \sum_{i=1}^n x_1 \cdots x_i x x_{i+1} \cdots x_n \\
&= \sum_{i=1}^n x_1 \cdots x_{i-1} x x_i \cdots x_n - \sum_{i=2}^{n+1} x_1 \cdots x_{i-1} x x_i \cdots x_n \\
&= x x_1 \cdots x_n - x_1 \cdots x_n x.
\end{align*}
\]

Therefore if \( c \) is invariant we have \( xc = cx \) for all \( x \in g \). By linearity, the same is true a general invariant element \( c \). Now the result follows because \( g \) generates \( U(g) \).

\[\square\]

Definition 4.6.5 Let \( g \) be a Lie algebra and \( h \) an ideal of \( g \). Let \( V \) be a finite dimensional representation of \( g \) such that the bilinear form \( b^h_V \) on \( h \) associated to \( V \) is non degenerate. The element \( c_V^h \in Z(U(g)) \) is called the Casimir element of \( g \) associated to \( h \) and \( V \).

Proposition 4.6.6 Let \( g \) be a Lie algebra and \( h \) an ideal of \( g \) of dimension \( n \). Let \( V \) be a finite dimensional representation of \( g \) such that the bilinear form \( b^h_V \) on \( h \) associated to \( V \) is non degenerate. Let \( c = c_V^h \) be the Casimir element of \( g \) associated to \( h \) and \( V \).

(i) We have \( \text{Tr}(c_V) = n \).

(ii) If \( V \) is simple and \( n \) prime to \( \text{char} k \), then \( c_V \) is an automorphism of \( V \).

Proof. (i) By definition, for basis \( (h_i)_{i \in [1,n]} \) and \( (h'_i)_{i \in [1,n]} \) be basis of \( h \) such that \( b^h_V(h_i, h'_j) = \delta_{i,j} \), we have the equality \( c = \sum_{i=1}^n h_i h'_i \). We get

\[
\text{Tr}(c_V) = \sum_{i=1}^n \text{Tr}((h_i)_V(h'_i)_V) = \sum_{i=1}^n b^h_V(h_i, h'_i) = n.
\]

(ii) If \( n \) and \( \text{char} k \) are coprime, then \( \text{Tr}(c_V) \) does not vanish and \( c_V \) is not the zero map. But \( c \) is in the center of \( U(g) \) and therefore commutes with any \( x_V \) for \( x \in g \). In particular, \( \ker c_V \) is a \( g \)-invariant subspace of \( V \). The representation \( V \) being simple, we have \( \ker c_V = V \) or \( \ker c_V = 0 \). The first equality would imply \( c_V = 0 \), therefore \( c_V \) is injective and an automorphism. \[\square\]