# Defects and adjunctions in Landau-Ginzburg models

based on work with **Daniel Murfet** and **Ingo Runkel**

**Nils Carqueville**

LMU München

# Bigger picture

**Landau-Ginzburg models**

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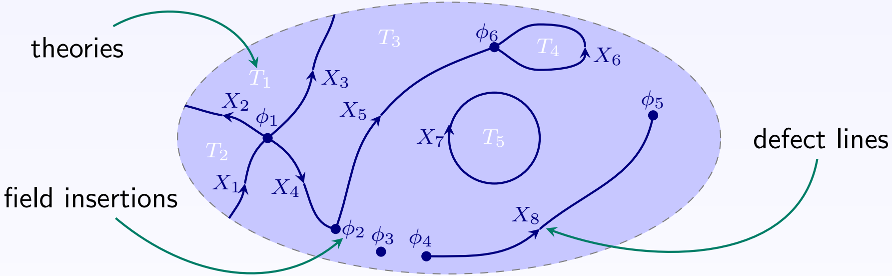
**Applications.**

- understand open/closed TFT *universally* from within bicategory
- compute any correlator as a 2-morphism
- new proof of Cardy condition
- ...
- **generalised orbifolds**

# 2d TFTs with defects

**Worldsheet**, partitioned into domains:

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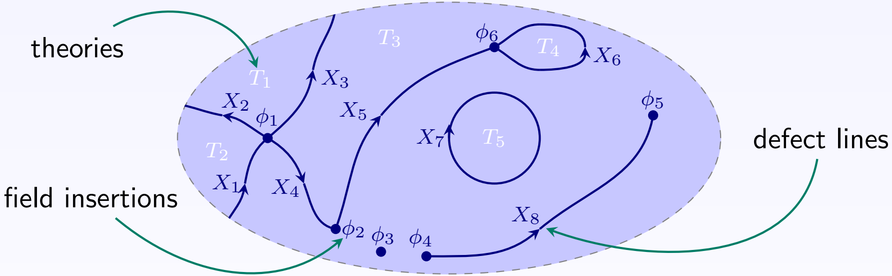
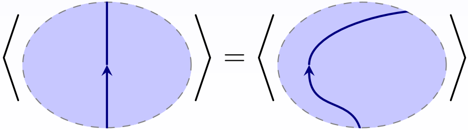



A **TFT** assigns a number $\langle \ldots \rangle$, the **correlator**, to any worldsheet, depending only on isotopy class of defect lines:



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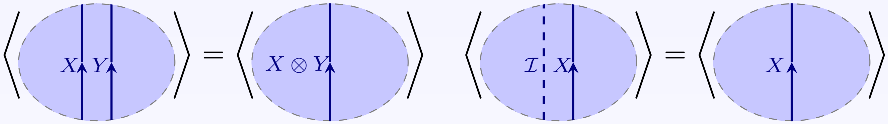

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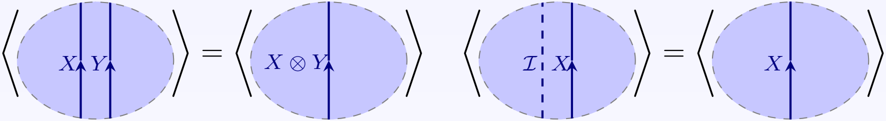
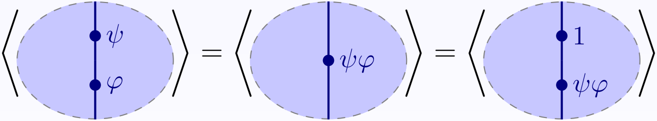



**operator product** of fields, unit = identity field



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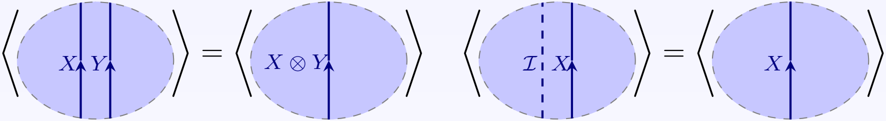


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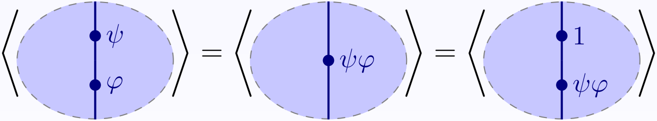


**Claim.** 2d TFTs with defects give **bicategory**:

- objects (domains) = theories
- 1-morphisms (lines) = defects
- 2-morphisms (points) = fields

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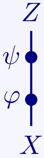


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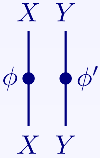


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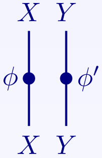


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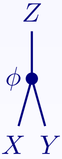


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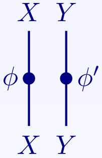


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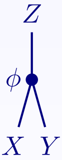


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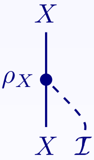

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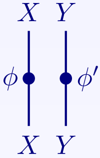


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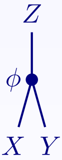


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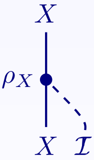


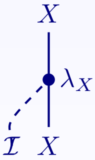

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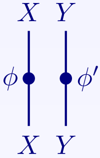


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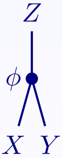


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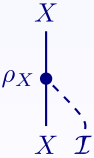


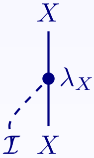


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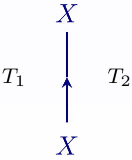

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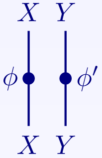

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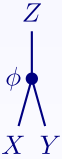

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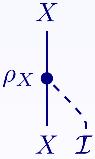


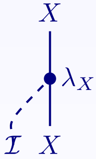


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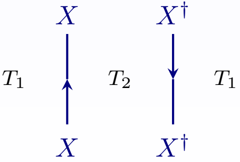

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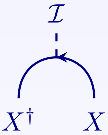


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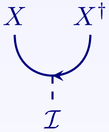


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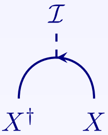

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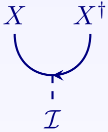

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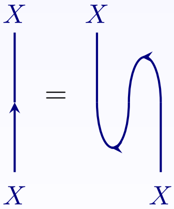


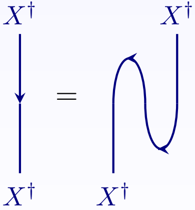

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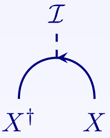


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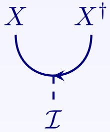


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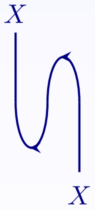


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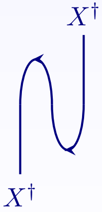

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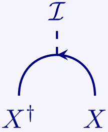


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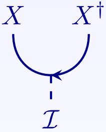


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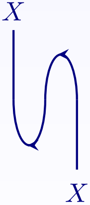


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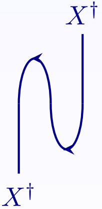


**Definition.** A bicategory **has adjoints** if for each 1-morphism $X$ there is a 1-morphism $X^\dagger$ with 2-morphisms $\mathrm{ev}_X, \mathrm{coev}_X$ such that the above *Zorro moves* hold.

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$$d_X = \begin{pmatrix} 0 & d_X^1 \\ d_X^0 & 0 \end{pmatrix} \in \mathrm{End}^1_{R\otimes S}(X)\,, \qquad d_X^2 = (V - W) \cdot 1_X$$

Kontsevich, Kapustin/Li 2002, Lazaroiu 2003, Brunner/Roggenkamp 2007

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- fields between $X$ and $Y$: **(BRST) cohomology** of

$$\mathrm{Hom}(X, Y) \ni \psi \longmapsto d_Y \psi - (-1)^{|\psi|} \psi d_X$$

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$$\mathcal{I}_W = \bigwedge\Big(\bigoplus_{i=1}^{n}(R\otimes R)\cdot\theta_i\Big)\,, \qquad d_{\mathcal{I}_W} = \sum_{i=1}^{n}\Big((x_i - y_i)\cdot\theta_i^* + \partial_{[i]}W\cdot\theta_i\Big)$$

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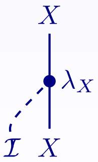


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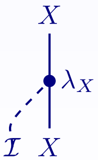


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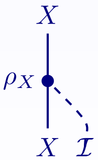


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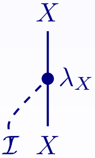
$$\mathcal{I}_W = (R\otimes R)^{\oplus 2}\,, \qquad d_{\mathcal{I}_W} = \begin{pmatrix} 0 & x-y \\ \frac{W(x)-W(y)}{x-y} & 0 \end{pmatrix}$$

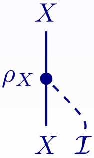
for $n = 1$, in general:

$$\mathcal{I}_W = \bigwedge\Big(\bigoplus_{i=1}^{n}(R\otimes R)\cdot\theta_i\Big)\,, \qquad d_{\mathcal{I}_W} = \sum_{i=1}^{n}\Big((x_i - y_i)\cdot\theta_i^* + \partial_{[i]}W\cdot\theta_i\Big)$$

**Fact.** $\mathrm{End}(\mathcal{I}_W) \cong R/(\partial W) =$ bulk space




$$: \mathcal{I}\otimes X \longrightarrow\!\!\!\!\!\rightarrow (R\otimes R)\otimes X \xrightarrow{\text{mult.}} X\,,$$




$$: X\otimes\mathcal{I} \longrightarrow X$$

- operator product: matrix multiplication

Brunner/Roggenkamp 2007, Kapustin/Rozansky 2004

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**Theorem.** Landau-Ginzburg models give a bicategory, called $\mathcal{LG}$.

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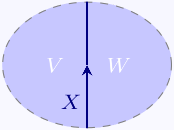


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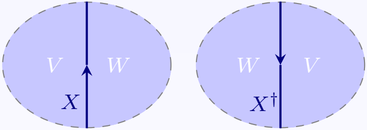


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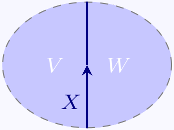


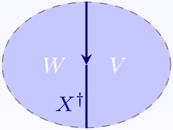


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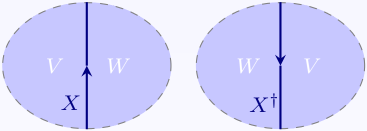


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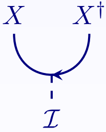


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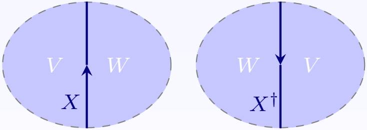


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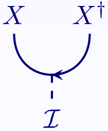


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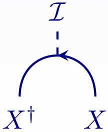


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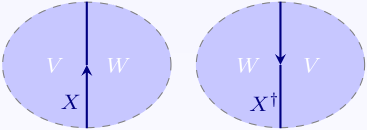


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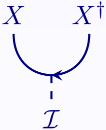


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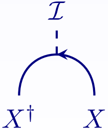


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*Proof*: homological perturbation, associative Atiyah classes

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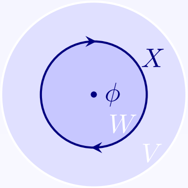

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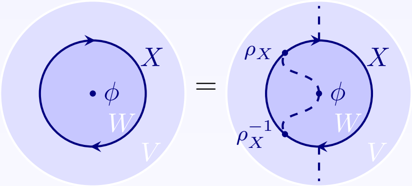


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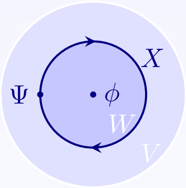


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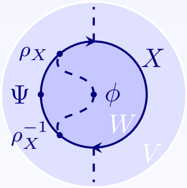


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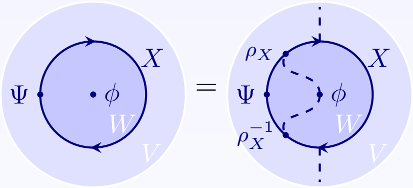


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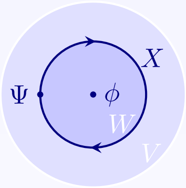


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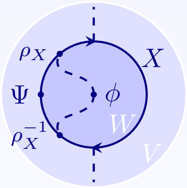


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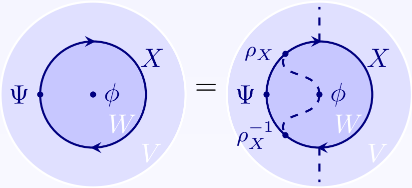


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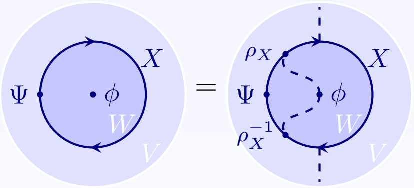


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Polishchuck/Vaintrob 2010, Carqueville/Murfet 2012

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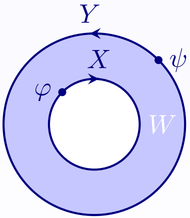


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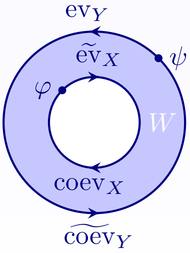


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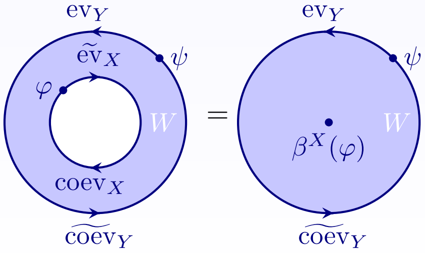


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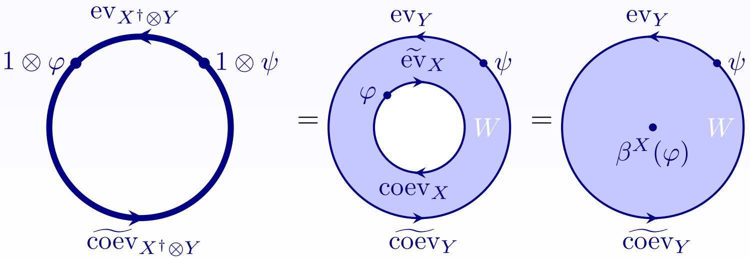


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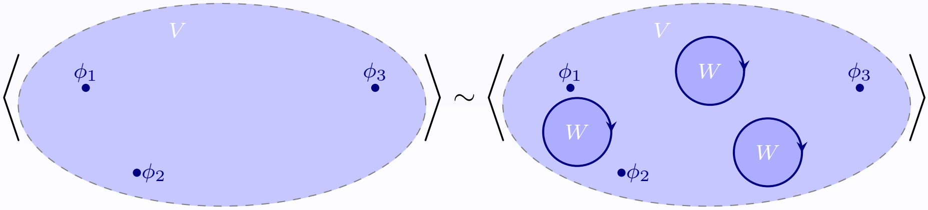


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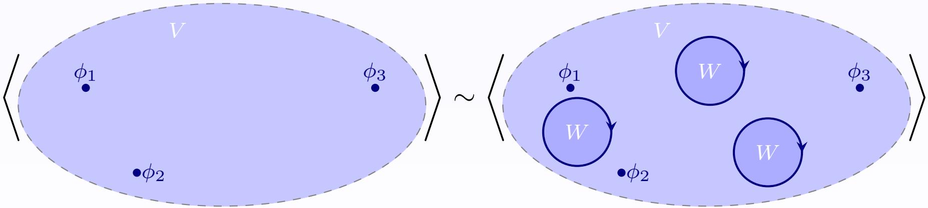


This **orbifold completion** works in any pivotal bicategory, not only $\mathcal{LG}$.

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- "ordinary" orbifolds: for discrete symmetry group $G$ of $W$ we have

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$$X = \begin{pmatrix} 0 & \frac{x^d-u^{2d}}{x-u^2} - y^2 \\ x-u^2 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & z+uy \\ z-uy & 0 \end{pmatrix}$$

is defect between $W = u^{2d}$ and $V = x^d - xy^2 + z^2$, has invertible quantum dimensions

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**Task.** Classify all defects with invertible quantum dimensions (and find new equivalences this way)!

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- boundary-bulk maps (actually: all open/closed TFT data)
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Also allows to find new structure: **generalised orbifolds**