

# Defects and adjunctions in Landau-Ginzburg models

based on work with Daniel Murfet and Ingo Runkel

**Nils Carqueville**

LMU München

# Bigger picture

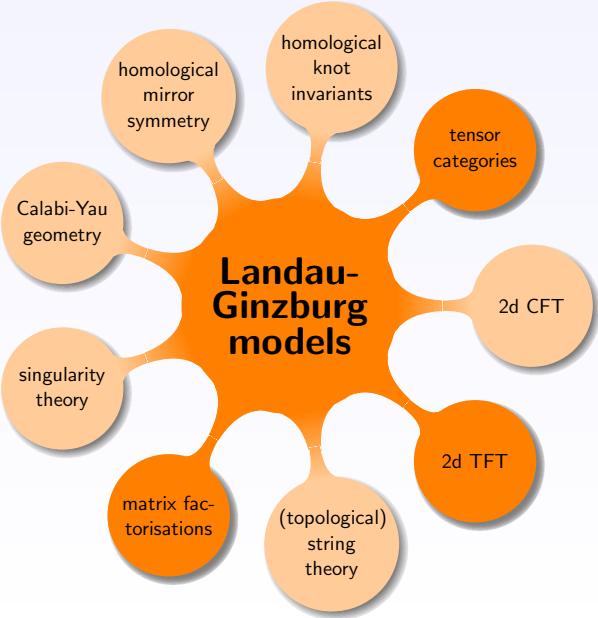
A solid orange circle is centered on the page. Inside the circle, the text "Landau-Ginzburg models" is written in a bold, black, sans-serif font, arranged in three lines.

**Landau-  
Ginzburg  
models**

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**2d TFTs with defects** are naturally described in terms of **bicategories** with extra structure.

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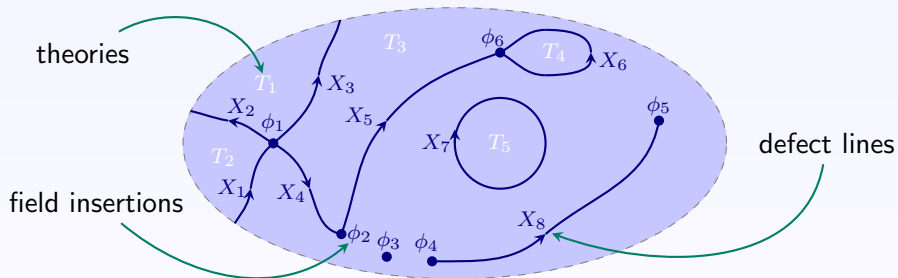
## Applications.

- understand open/closed TFT *universally* from within bicategory
- compute any correlator as a 2-morphism
- new proof of Cardy condition
- ...
- **generalised orbifolds**



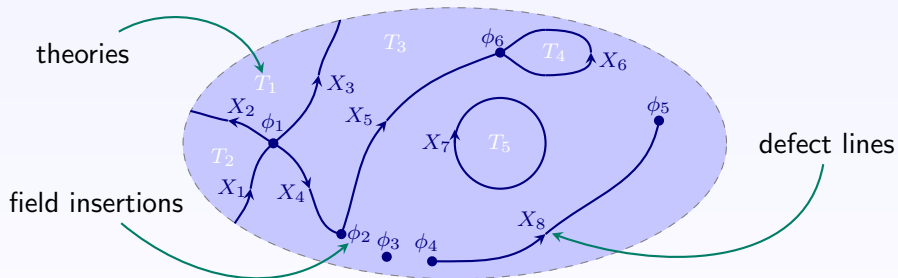
## 2d TFTs with defects

**Worldsheet**, partitioned into domains:



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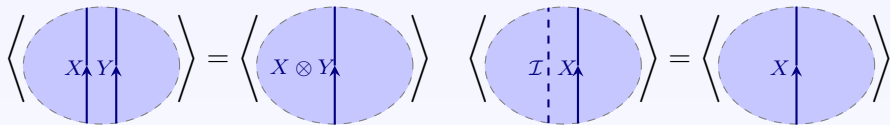


A **TFT** assigns a number  $\langle \dots \rangle$ , the **correlator**, to any worldsheet, depending only on isotopy class of defect lines:

$$\langle \text{Worldsheet with a vertical defect line} \rangle = \langle \text{Worldsheet with a curved defect line} \rangle$$

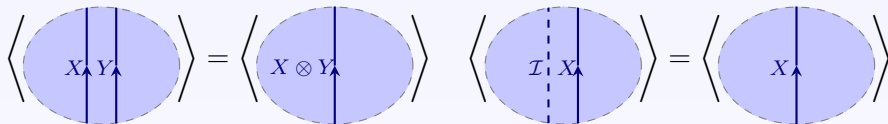
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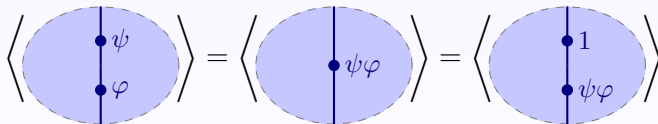


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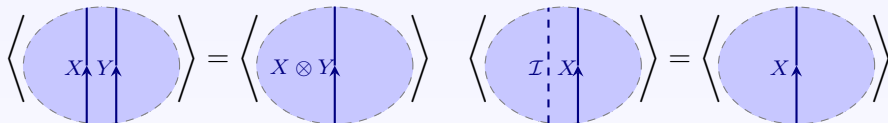


**operator product** of fields, unit = identity field

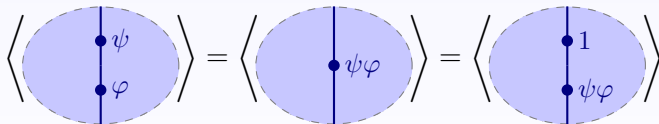


## 2d TFTs with defects

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**Claim.** 2d TFTs with defects give **bicategory**:

- objects (domains) = theories
- 1-morphisms (lines) = defects
- 2-morphisms (points) = fields

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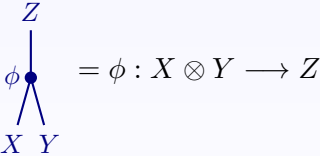
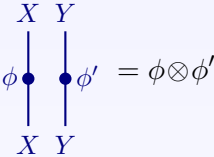
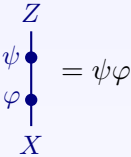
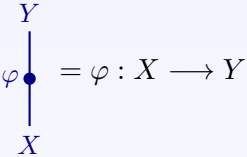
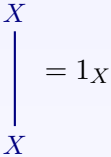
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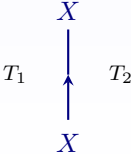
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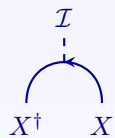
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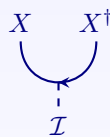
$$T_1 \begin{array}{c} X \\ | \\ \uparrow \\ X \end{array} \quad T_2 \begin{array}{c} X^\dagger \\ | \\ \downarrow \\ X^\dagger \end{array} \quad T_1$$

## Orientation and adjoints



The diagram shows a blue arc with an arrow pointing downwards from its center to a vertical dashed line labeled  $\mathcal{I}$ . The left end of the arc is labeled  $X^\dagger$  and the right end is labeled  $X$ .

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**Definition.** A bicategory **has adjoints** if for each 1-morphism  $X$  there is a 1-morphism  $X^\dagger$  with 2-morphisms  $\text{ev}_X, \text{coev}_X$  such that the above *Zorro moves* hold.

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- fields between  $X$  and  $Y$ : **(BRST) cohomology** of

$$\text{Hom}(X, Y) \ni \psi \longmapsto d_Y \psi - (-1)^{|\psi|} \psi d_X$$

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- operator product: matrix multiplication

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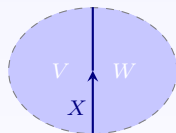
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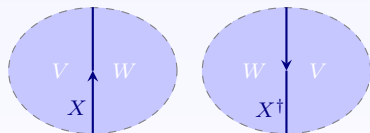


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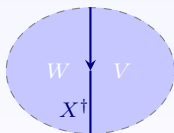
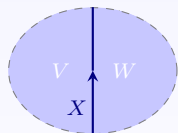


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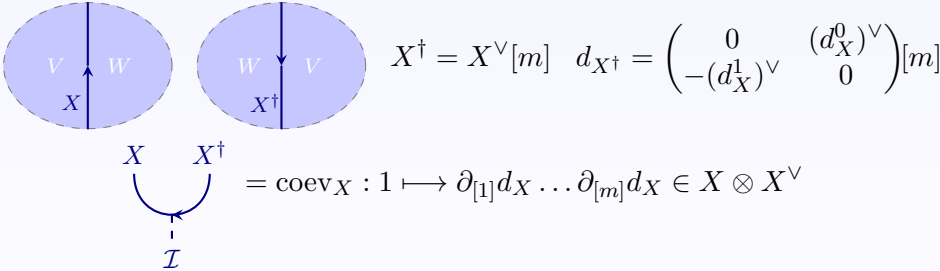
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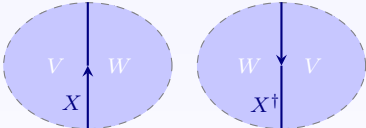


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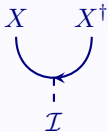
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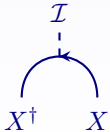
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$$= \text{coev}_X : 1 \mapsto \partial_{[1]} d_X \dots \partial_{[m]} d_X \in X \otimes X^\vee$$



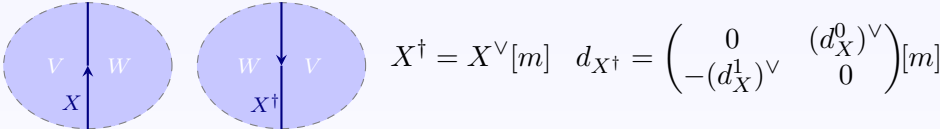
$$= \text{ev}_X = \text{Res} \left[ \frac{\text{str} \left( (-) \circ \partial_{z_1} d_X \dots \partial_{z_m} d_X \right) dz}{\partial_{z_1} V \dots \partial_{z_m} V} \right] + \mathcal{O}(\theta)$$

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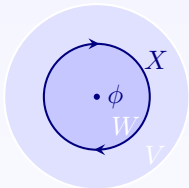
$$= \text{coev}_X : 1 \mapsto \partial_{[1]} d_X \dots \partial_{[m]} d_X \in X \otimes X^\vee$$

$$= \text{ev}_X = \text{Res} \left[ \frac{\text{str} \left( (-) \circ \partial_{z_1} d_X \dots \partial_{z_m} d_X \right) dz}{\partial_{z_1} V \dots \partial_{z_m} V} \right] + \mathcal{O}(\theta)$$

*Proof:* homological perturbation, associative Atiyah classes

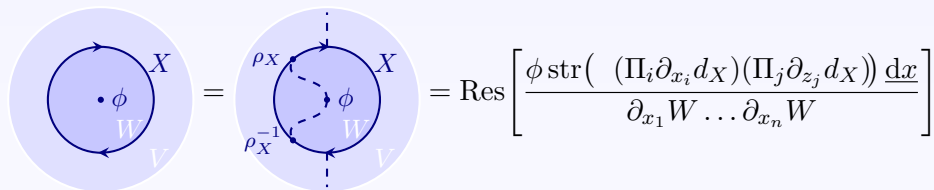
# Applications

**Defect action on bulk fields** for defect  $X$  between  $W(x)$  and  $V(z)$ :



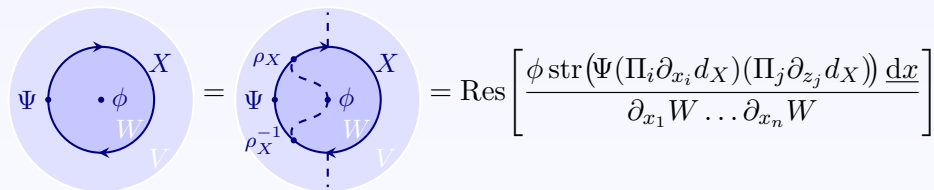
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**Defect action on bulk fields** for defect  $X$  between  $W(x)$  and  $V(z)$ :


$$= \text{Res} \left[ \frac{\phi \text{str} \left( (\Pi_i \partial_{x_i} d_X) (\Pi_j \partial_{z_j} d_X) \right) \underline{dx}}{\partial_{x_1} W \dots \partial_{x_n} W} \right]$$

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The diagram illustrates the defect action on bulk fields. It consists of three parts: a diagram on the left, an equals sign, a diagram in the middle, and another equals sign followed by a residue integral on the right.

The left diagram shows a large light blue circle containing a smaller dark blue circle. The inner circle has a counter-clockwise arrow and is labeled  $X$  at the top. A dot in the center is labeled  $\phi$ . A dot on the left boundary of the inner circle is labeled  $\Psi$ . The region between the inner and outer circles is divided into two sectors: the upper-right sector is labeled  $W$  and the lower-left sector is labeled  $V$ .

The middle diagram is identical to the left one but includes a vertical dashed line passing through the center. This line intersects the inner circle at two points, labeled  $\rho_X$  at the top and  $\rho_X^{-1}$  at the bottom. A dashed arrow points from  $\rho_X$  to  $\rho_X^{-1}$  along the inner circle.

The right part of the equation is a residue integral:

$$= \text{Res} \left[ \frac{\phi \text{str}(\Psi(\Pi_i \partial_{x_i} d_X)(\Pi_j \partial_{z_j} d_X)) \underline{dx}}{\partial_{x_1} W \dots \partial_{x_n} W} \right]$$

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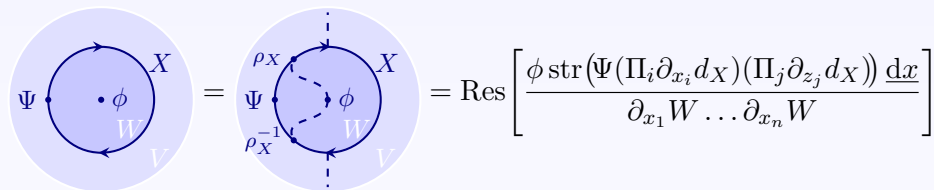
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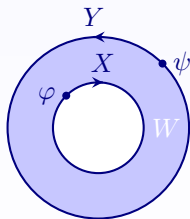
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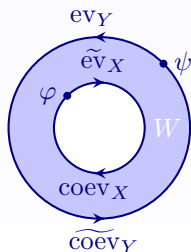
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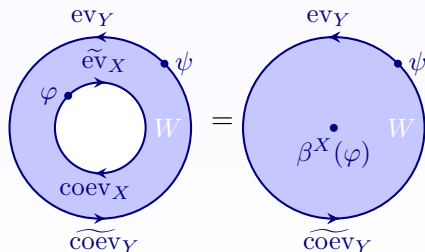
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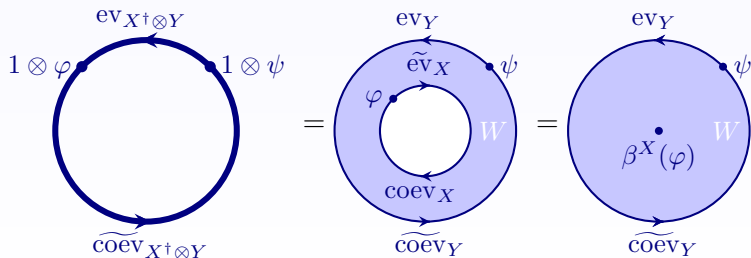
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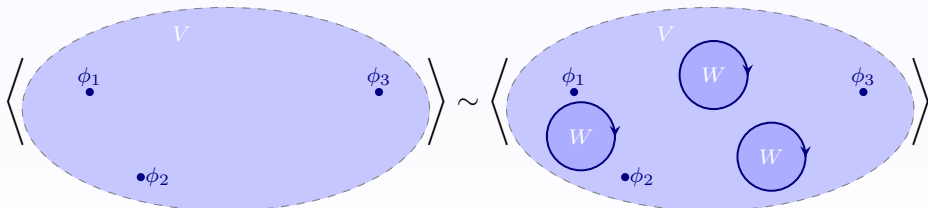
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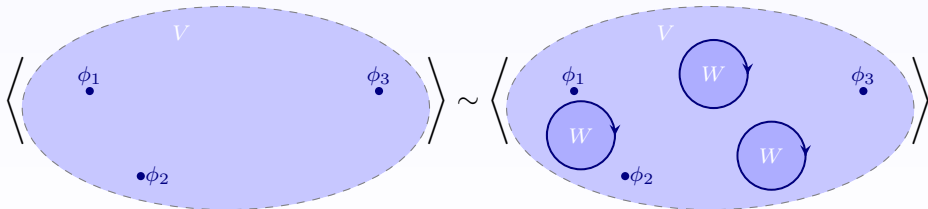


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This **orbifold completion** works in any pivotal bicategory, not only  $\mathcal{LG}$ .

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- “ordinary” orbifolds: for discrete symmetry group  $G$  of  $W$  we have

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**Task.** Classify all defects with invertible quantum dimensions (and find new equivalences this way)!

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Also allows to find new structure: **generalised orbifolds**