Quasi-Poisson structures, Courant algebroids and Bianchi identities for non-geometric fluxes

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Contents

Contents

Contents

Contents

Contents

Non-geometric fluxes and gauged supergravity

Bosonic low energy effective field theory

$$S = \int d^n x \sqrt{-g} e^{-\Phi} \left(\mathfrak{R} + (\nabla\Phi)^2 - \tfrac{1}{12} \mathcal{H}_{\mu\nu\lambda} \mathcal{H}^{\mu\nu\lambda} \right)$$

- Scherk-Schwarz reduction, e.g on d-torus gives $\mathcal{O}(d,d)$-invariant action, containing $2d$ vectorfields V^a, B_a.
- Gauging a subgroup of $\mathcal{O}(d,d)$ leads to gauge generators X_a, corresponding to V^a and X^a corresponding to B_a.
- Most general commutation relations for the generators:

$$\begin{aligned} \left[X_a, X_b\right] &= \mathcal{F}_{ab}{}^c X_c + \mathcal{H}_{abc} X^c \\ \left[X_a, X^b\right] &= \mathcal{Q}_a{}^{bc} X_c - \mathcal{F}_{ac}{}^b X^c \\ \left[X^a, X^b\right] &= \mathcal{R}^{abc} X_c + \mathcal{Q}_c{}^{ab} X^c \end{aligned} \quad (1)$$

- where $\mathcal{H}$ is the field strength of the NS-NS B-field and $\mathcal{F}, \mathcal{Q}$ and $\mathcal{R}$ contain geometric and non-geometric fluxes f, Q, R, arising in the conjectured chain of T-dualities:

$$\mathcal{H}_{abc} \rightarrow f_{ab}{}^c \rightarrow Q_a{}^{bc} \rightarrow R^{abc}$$

Representation on TM and Bianchi identities

Representation on TM and Bianchi identities

- Basis of TM: $e_a = e_a{}^\mu \partial_\mu$, dual $e^a = e^a{}_\mu dx^\mu$
- Geometric flux: $\left[e_a \, , e_b\right] = f_{ab}{}^c e_c$
- Bi-vectorfield $\beta = \frac{1}{2}\beta^{ab} e_a \wedge e_b$ defines anchor map:

$$\beta^\# : T^\star M \to TM, \quad \beta^\#(e^a) = \beta^{ab} e_b =: e^a_\#$$

$\to$ easy to compute

$$\left[e_a, e^b_\#\right] = Q_a{}^{bc} e_c - f_{ac}{}^b e^c_\#$$
$$\left[e^a_\#, e^b_\#\right] = R^{abc} e_c + Q_c{}^{ab} e^c_\#$$

where we defined

$$Q_a{}^{bc} := \partial_a \beta^{bc} + 2 f_{am}{}^{[\underline{b}} \beta^{m\underline{c}]}$$
$$R^{abc} := 3\left(\beta^{[\underline{a}m} \partial_m \beta^{\underline{bc}]} + f_{mn}{}^{[\underline{a}} \beta^{\underline{b}m} \beta^{\underline{c}]n}\right)$$

Include $\mathcal{H}$-flux?

Representation on TM and Bianchi identities

Perform a field-redefinition

$$\mathcal{F}_{ab}{}^{c} = f_{ab}{}^{c} - \mathcal{H}_{abm}\beta^{mc}$$
$$\mathcal{Q}_{a}{}^{bc} = Q_{a}{}^{bc} + \mathcal{H}_{amn}\beta^{mb}\beta^{nc}$$
$$\mathcal{R}^{abc} = R^{abc} - \mathcal{H}_{mnp}\beta^{ma}\beta^{nb}\beta^{pc}$$

to arrive at the algebra

$$\begin{aligned} [e_a, e_b] &= \mathcal{F}_{ab}{}^{c} e_c + \mathcal{H}_{abc} e_{\#}^{c} \\ [e_a, e_{\#}^{b}] &= \mathcal{Q}_{a}{}^{bc} e_c - \mathcal{F}_{ac}{}^{b} e_{\#}^{c} \\ [e_{\#}^{a}, e_{\#}^{b}] &= \mathcal{R}^{abc} e_c + \mathcal{Q}_{c}{}^{ab} e_{\#}^{c} \end{aligned} \tag{2}$$

To get Bianchi-type identities, define the Jacobiator for $X, Y, Z \in \Gamma(TM)$ by

$$\mathcal{J}(X, Y, Z) := [[X, Y], Z] + \text{cycl.}$$

$\rightarrow$ evaluate it on all different combinations of $e_a, e_{\#}^{a}$

Bianchi identities

Blumenhagen,A.D.,Plauschinn,Rennecke, arXiv:1205.1522

Example:
$\mathcal{J}(e^a_\#, e^b_\#, e^c_\#)$ leads to

$$0 = \Big(\beta^{[\underline{c}m}\,\partial_m \mathcal{R}^{\underline{ab}]d} - 2\mathcal{R}^{[\underline{ab}m}\,\mathcal{Q}_m{}^{\underline{cd}]}\Big) + \Big(\beta^{[\underline{c}m}\,\partial_m \mathcal{Q}_n{}^{\underline{ab}]} + \mathcal{R}^{[\underline{ab}m}\,\mathcal{F}_{mn}{}^{\underline{c}]} + \mathcal{Q}_m{}^{[\underline{ab}}\,\mathcal{Q}_n{}^{\underline{c}]m}\Big)\beta^{nd} \quad (3)$$

Observation:
For constant fluxes in toroidal compactifications, a special case of (3) was already obtained earlier (Shelton, Taylor, Wecht, hep-th/0508133 and Ihl, Robbins, Wrase, arXiv:0705.3410)

$$\begin{aligned} 0 &= \mathcal{R}^{[\underline{ab}m}\,\mathcal{F}_{mn}{}^{\underline{c}]} + \mathcal{Q}_m{}^{[\underline{ab}}\,\mathcal{Q}_n{}^{\underline{c}]m}\,, \\ 0 &= \mathcal{R}^{[\underline{ab}m}\,\mathcal{Q}_m{}^{\underline{cd}]} \end{aligned} \quad (4)$$

Representation on $TM \bigoplus T^{\star}M$

Lie algebroids

Definition

A vector bundle $E \to M$ is called a Lie algebroid, if it has the following additional structure:

- *A bracket $[,]_E : E \times E \to E$*
- *A bundle homomorphism (called anchor) $\rho : E \to TM$*
- *Leibnitz rule: For sections s_1, s_2 of E and $f \in \mathcal{C}^\infty(M)$:*

$$[s_1, fs_2]_E = f[s_1, s_2]_E + \rho(s_1)(f)\, s_2$$

$\to$ immediately: $\Gamma(\wedge^\bullet E^\star)$ is graded differential, i.e.

$$(d_E\,\omega)(s_0, \ldots, s_k) = \sum_{i=0}^{k} (-1)^i \rho(s_i)\,(\omega(s_0, \ldots, \hat{s}_i, \ldots, s_k)) + \sum_{i<j} (-1)^{i+j} \omega\,([s_i, s_j]_E, s_0, \ldots, \hat{s}_i, \ldots, \hat{s}_j, \ldots, s_K)$$

Standard examples

- $(TM, [,], \rho = \mathrm{id})$
  $d_{TM}\omega = d\omega$, (Physics: $\mathcal{H} = dB$)
- If (M, β) is a (quasi-)Poisson manifold, we have $(T^\star M, [,]_{KS(\beta)}, \rho = \beta^\#)$, where

$$[\alpha, \omega]_{KS(\beta)} = \mathcal{L}_{\beta^\#(\alpha)}\omega - \mathcal{L}_{\beta^\#(\omega)}\alpha - d\left(\beta(\alpha, \omega)\right)$$

Differential on TM? $\rightarrow$ need the *Shouten-Nijenhuis* bracket:

$$[f, g]_{SN} = 0, \quad [X, f]_{SN} = X(f), \quad [X, Y]_{SN} = [X, Y]$$

and extended to $\Gamma(\wedge^\bullet TM)$ by $(X \in \Gamma(\wedge^k TM), Y \in \Gamma(\wedge^l TM))$

$$[X, Y \wedge Z]_{SN} = [X, Y]_{SN} \wedge Z + (-1)^{(k-1)l} Y \wedge [X, Z]_{SN}$$
$$[X, Y]_{SN} = -(-1)^{(k-1)(l-1)}[Y, X]_{SN}$$

$\rightarrow$ differential on TM is given by

$$d_{T^\star M}X = d_\beta X := [\beta, X]_{SN}$$

Interlude: R-flux and quasi-Poisson-geometry?

Physics:

$$R = \frac{1}{2}\left(\beta^{[\underline{a}m}\partial_m\beta^{\underline{bc}]} + f_{mn}{}^{[\underline{a}}\beta^{\underline{b}m}\beta^{\underline{c}n]}\right)\ e_a \wedge e_b \wedge e_c$$
$$= [\beta, \beta]_{SN} = d_\beta \beta$$

$\rightarrow$ if β defines a quasi-Poisson-structure

$$\{f, g\} = \beta^{ij}(\partial_i f)(\partial_j g)$$
$$\{\{f, g\}, h\} + \mathrm{cycl.} = R^{ijk}(\partial_i f)(\partial_j g)(\partial_k h)$$

Similar structure was found in closed-string cft-computations (Blumenhagen,A.D.,Lüst,Plauschinn,Rennecke, arXiv:1106.0316)

$$\langle \mathcal{X}^a(z_1,\bar{z}_1)\mathcal{X}^b(z_2,\bar{z}_2)\mathcal{X}^c(z_3,\bar{z}_3)\rangle = \frac{\alpha'^2}{12}\theta^{abc}\left[\mathcal{L}\left(\frac{z_{12}}{z_{13}}\right) - \mathcal{L}\left(\frac{\bar{z}_{12}}{\bar{z}_{13}}\right)\right]$$

computing 3-(n-)point tachyon correlators suggested a 3-product

$$f \triangle g \triangle h := f\ g\ h + \theta^{ijk}(\partial_i f)(\partial_j g)(\partial_k h) + \mathcal{O}(\theta^2)$$

Inclusion of $\mathcal{H}$: Quasi-Lie algebroids

- $(TM, [,], \mathrm{id})$

$$[X, Y] \to [X, Y]^{\mathcal{H}} := [X, Y] + \beta^{\#}(\iota_X \iota_Y \mathcal{H})$$
$$d \to d^{\mathcal{H}}$$

evaluation on basis

$$[e_a, e_b]^{\mathcal{H}} = f_{ab}{}^{n} e_n - \mathcal{H}_{abm}\beta^{mp} e_p$$
$$= \mathcal{F}_{ab}{}^{c} e_c$$

- $(T^{\star}M, [,]_{KS(\beta)}, \beta^{\#})$

$$[\alpha, \omega]_{KS(\beta)} \to [\alpha, \omega]^{\mathcal{H}}_{KS(\beta)} := [\alpha, \omega]_{KS(\beta)} + \iota_{\beta^{\#}(\alpha)} \iota_{\beta^{\#}(\omega)} \mathcal{H}$$
$$d_\beta \to d^{\mathcal{H}}_\beta$$

evaluation on basis

$$[e^a, e^b]^{\mathcal{H}}_{KS(\beta)} = (\partial_p \beta^{ab} + 2 f_{pm}{}^{[\underline{a}} \beta^{m\underline{b}]} + \beta^{am}\beta^{bn}\mathcal{H}_{mnp}) e^p$$
$$= \mathcal{Q}_p{}^{ab} e^p$$

Courant algebroids

Definition

A vector bundle $E \to M$ together with a non-degenerate symmetric bilinear form $\langle , \rangle$, a skew-symmetric bracket $[,]$ on $\Gamma(E)$ and a bundle map $\rho : E \to TM$ is called a Courant algebroid, if the following properties hold ($s_i, e \in \Gamma(E),\ f, g \in \mathcal{C}^\infty(M)$)

- $\mathcal{J}(s_1, s_2, s_3) = \mathcal{D}F(s_1, s_2, s_3)$,
- $\rho([s_1, s_2]) = [\rho(s_1), \rho(s_2)]$,
- $[s_1, fs_2] = f\,[s_1, s_2] + (\rho(s_1)f)s_2 - \frac{1}{2}\langle s_1, s_2\rangle \mathcal{D}f$
- $\rho \circ \mathcal{D} = 0$, *i.e.* $\langle \mathcal{D}f, \mathcal{D}g\rangle = 0$
- $\rho(e)\langle s_1, s_2\rangle = \langle [e, s_1] + \frac{1}{2}\mathcal{D}\langle e, s_1\rangle, s_2\rangle + \langle s_1, [e, s_2] + \frac{1}{2}\mathcal{D}\langle e, s_2\rangle\rangle$

where we also defined F by $F(s_1, s_2, s_3) = \frac{1}{6}\langle [s_1, s_2], s_3\rangle + cycl.$ and the derivative operator $\mathcal{D} : \mathcal{C}^\infty(M) \to \Gamma(E)$ is defined by

$$\langle \mathcal{D}f, s\rangle = \rho(s)f .$$

Courant algebroid structure on $TM \oplus T^\star M$

Result in Mathematics (Roytenberg, arXiv:math/9910078): Two quasi-Lie algebroids $E, E^\star$ together with compatibility conditions give rise to an associated Courant algebroid structure on $E \oplus E^\star$.

$\rightarrow$ Take the following Courant algebroid structure on $TM \oplus T^\star M$:

- For sections $X + \xi, Y + \eta \in \Gamma(TM \oplus T^\star M)$, $\langle X + \xi, Y + \eta \rangle_\pm = \xi(Y) \pm \eta(X)$
- Anchor $\alpha(X + \xi) = X + \beta^\#(\xi)$,
- Derivative operator $\mathcal{D} = d + d_\beta$,
- Bracket $[\![,]\!]$ on $\Gamma(TM \oplus T^\star M)$:

$$
\begin{aligned}
[\![X, Y]\!] &= [X, Y]^{\mathcal{H}} + \iota_Y \iota_X \mathcal{H} \, , \\
[\![X, \xi]\!] &= [\iota_X, d^{\mathcal{H}}]_+ \xi - [\iota_\xi, d_\beta^{\mathcal{H}}]_+ X + \tfrac{1}{2}\left(d^{\mathcal{H}} - d_\beta^{\mathcal{H}}\right) \langle X, \xi \rangle_- \, , \\
[\![\xi, X]\!] &= [\iota_\xi, d_\beta^{\mathcal{H}}]_+ X - [\iota_X, d^{\mathcal{H}}]_+ \xi + \tfrac{1}{2}\left(d^{\mathcal{H}} - d_\beta^{\mathcal{H}}\right) \langle \xi, X \rangle_- \, , \\
[\![\xi, \nu]\!] &= [\xi, \nu]_K^{\mathcal{H}} + \iota_\nu \iota_\xi \mathcal{R} \, ,
\end{aligned} \tag{5}
$$

Result I

Blumenhagen,A.D.,Plauschinn,Rennecke, arXiv:1205.1522

This is indeed a Courant algebroid.

One step of the calculation for the 1^{st} axiom

Special case: $f=0, \mathcal{H}=0$. *The Jacobiator of* e^a, e^b, e^c *is given by*

$$\mathcal{J}(e^a,e^b,e^c) = -3\Big(\beta^{[\underline{c}m}\,\partial_m R^{\underline{ab}]d} - 2R^{[\underline{ab}m}\,\mathcal{Q}_m{}^{\underline{cd}]}\Big)e_d - 3\Big(\beta^{[\underline{c}m}\,\partial_m \mathcal{Q}_d{}^{\underline{ab}]} + \mathcal{Q}_m{}^{[\underline{ab}}\,\mathcal{Q}_d{}^{\underline{c}]m}\Big)e^d + \tfrac{3}{2}\,\mathcal{D}\mathcal{R}^{abc}$$

Taking the $[e^a_\#,[e^b_\#,e^c_\#]]$*-Bianchi identity, we get*

$$= -3\Big(\beta^{[\underline{c}m}\,\partial_m \mathcal{Q}_d{}^{\underline{ab}]} + \mathcal{Q}_m{}^{[\underline{ab}}\,\mathcal{Q}_d{}^{\underline{c}]m}\Big)\Big(-\beta^{dn}e_n + e^d\Big) - \tfrac{1}{2}\mathcal{D}(R^{abc})$$

Finally, taking the $[e_a,[e^b_\#,e^c_\#]]$*-Bianchi identity, leads to*

$$= \;\partial_d R^{abc}\Big(-\beta^{dn}e_n + e^d\Big) = \;(d_\beta + d)(R^{abc})$$

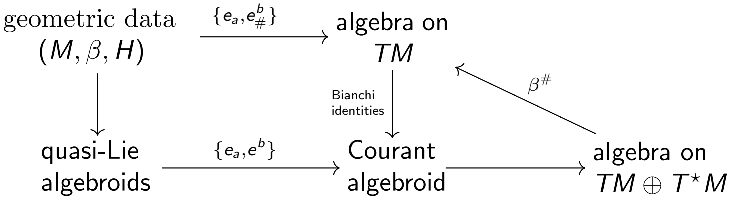
Result II

Blumenhagen,A.D.,Plauschinn,Rennecke, arXiv:1205.1522

With the bracket $[\![,]\!]$, we again get the right commutation relations, but now realized on $\Gamma(TM \oplus T^\star M)$:

$$[\![e_a, e_b]\!] = \mathcal{F}_{ab}{}^c\, e_c + \mathcal{H}_{abc}\, e^c \ ,$$
$$[\![e_a, e^b]\!] = \mathcal{Q}_a{}^{bc}\, e_c - \mathcal{F}_{ac}{}^b\, e^c \ ,$$
$$[\![e^a, e^b]\!] = \mathcal{Q}_c{}^{ab}\, e^c + \mathcal{R}^{abc}\, e_c \ ,$$

Conclusion


geometric data
(M, β, H)
$\{e_a, e^b_\#\}$
algebra on
TM
Bianchi
identities
$\beta^\#$
quasi-Lie
algebroids
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Courant
algebroid
algebra on
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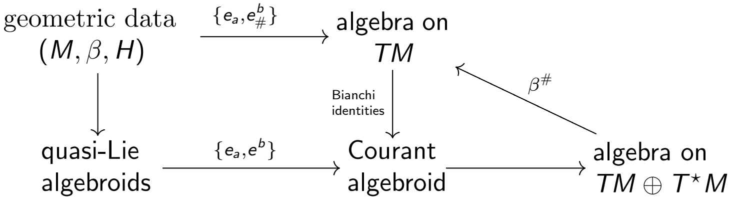
Conclusion



Thank you!