

Heterotic Vector Bundles, Deformations and Geometric Transitions

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A smooth $E_8 \times E_8$ heterotic model:

- The geometric ingredients include:
 - A Calabi-Yau 3-fold, X
 - A holomorphic vector bundle, V , on X (with structure group $G \subset E_8$)
- Compactifying on X leads to $\mathcal{N} = 1$ SUSY in 4D, while V breaks $E_8 \rightarrow G \times H$, where H is the Low Energy GUT group
 - $G = SU(n)$, $n = 3, 4, 5$ leads to $H = E_6, SO(10), SU(5)$
- Matter and Moduli
 - H -charged matter, $H^1(X, V)$, $H^1(X, V^\vee)$, $H^1(X, \wedge^2 V)$, \dots
 - $X \Rightarrow h^{1,1}(X)$ - Kähler moduli and $h^{2,1}(X)$ - Complex structure moduli
 - $V \Rightarrow h^1(X, \text{End}_0(V))$ Bundle moduli
- A susy vacuum must satisfy the Hermitian Yang-Mills Equations

$$\delta\chi = 0 \Rightarrow \begin{cases} F_{ab} = F_{\bar{a}\bar{b}} = 0 \\ g^{a\bar{b}} F_{a\bar{b}} = 0 \end{cases}$$

- Begin with a holomorphic bundle, $F_{ab} = F_{\bar{a}\bar{b}} = 0$, w.r.t a **fixed complex structure**. What happens as we vary the complex structure? Must a bundle stay holomorphic for any variation $\delta\mathfrak{z}^I v_I \in h^{2,1}(X)$? \Rightarrow No.
- Infinitesimally, we must solve:

$$\delta\mathfrak{z}^I v_I^c{}_{[\bar{a}]|c|\bar{b}} F_{|c|\bar{b}}^{(0)} + 2D_{[\bar{a}}^{(0)} \delta A_{\bar{b}]} = 0$$

- **For the 4d Theory:** Superpotential $W = \int_X \Omega \wedge H$ where $H = dB - \frac{3\alpha'}{\sqrt{2}} (\omega^{3YM} - \omega^{3L})$. $\omega^{3YM} = A \wedge dA - \frac{1}{3} A \wedge A \wedge A$.
- In Minkowski vacuum, F-terms: $F_{C_i} = \frac{\partial W}{\partial C_i} = -\frac{3\alpha'}{\sqrt{2}} \int_X \Omega \wedge \frac{\partial \omega^{3YM}}{\partial C_i} \Rightarrow$

$$\delta(F_{C_i}) = \int_X \epsilon^{\bar{a}\bar{c}\bar{b}} \epsilon^{abc} \Omega_{abc}^{(0)} 2\bar{\omega}_{\bar{c}}^{xi} \text{tr}(T_x T_y) \left(\delta\mathfrak{z}^I v_I^c{}_{[\bar{a}]|c|\bar{b}} F_{|c|\bar{b}}^{(0)y} + 2D_{[\bar{a}}^{(0)} \delta A_{\bar{b}]}^y \right)$$

Physics Idea Use bundle holomorphy to constrain C.S. Moduli of X .

New moduli stabilization tool.

Math Question Given a CY X , how to find/engineer bundles that are holomorphic only at special loci (ideally isolated pts) in CS moduli space?

There are three objects in Deformation Theory that we need:

- $Def(X)$: Deformations of X as a complex manifold. First order defs parameterized by the vector space $H^1(TX) = H^{2,1}(X)$. These are the *complex structure* deformations of X .
- $Def(V)$: The deformation space of V (changes in connection, δA) *for fixed* C.S. moduli. Infinitesimally $H^1(End(V))$. The *bundle moduli* of V .
- $Def(V, X)$: Simultaneous holomorphic deformations of V and X . The tangent space is $H^1(X, \mathcal{Q})$ where

$$0 \rightarrow End(V) \rightarrow \mathcal{Q} \xrightarrow{\pi} TX \rightarrow 0$$

\mathcal{Q} is defined by the projectivized total space of the bundle $\mathbb{P}(V) \rightarrow X$.

$$0 \rightarrow H^1(End(V)) \rightarrow H^1(\mathcal{Q}) \xrightarrow{d\pi} H^1(TX) \xrightarrow{\alpha} H^2(End(V)) \rightarrow \dots$$

Must determine: $H^1(X, \mathcal{Q}) = H^1(X, End(V)) \oplus Ker(\alpha)$ where

$\alpha = [F^{1,1}] \in H^1(End(V) \otimes TX^\vee)$ is the *Atiyah Class*. *These are the moduli of a heterotic theory.*

A simple example

- Consider an $SU(2)$ bundle defined by an extension in $Ext^1(\mathcal{L}^\vee, \mathcal{L})$:

$$0 \rightarrow \mathcal{L} \rightarrow V \rightarrow \mathcal{L}^\vee \rightarrow 0$$

In principle, such a bundle can stabilize arbitrarily many moduli.

- For example, consider $\mathcal{L} = \mathcal{O}(-2, -2, 1, 1)$ on the CY, $X = \left[\begin{array}{c|c} \mathbb{P}^1 & 2 \\ \mathbb{P}^1 & 2 \\ \mathbb{P}^1 & 2 \\ \mathbb{P}^1 & 2 \end{array} \right]^{4,68}$
- Why this one?** Here $Ext^1(\mathcal{L}^\vee, \mathcal{L}) = H^1(X, \mathcal{O}(-4, -4, 2, 2)) = 0$ generically. Hence cannot define the bundle for general complex structure!
- Happily, cohomology can “jump” at higher co-dimensional loci in Complex Structure moduli space.
- This is an easy example of “structural” C.S. dependence in V .

- Explicitly: Let $\mathcal{A} = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Given, $p \in H^0(\mathcal{A}, \mathcal{O}(2, 2, 2, 2))$, the Koszul sequence for X gives us

$$0 \rightarrow \mathcal{O}(-2, -2, -2, -2) \otimes \mathcal{L}_{\mathcal{A}}^{\otimes 2} \xrightarrow{p} \mathcal{L}_{\mathcal{A}}^{\otimes 2} \rightarrow \mathcal{L}_X^{\otimes 2} \rightarrow 0$$

$$H^1(X, \mathcal{L}^{\otimes 2}) = \ker(p), \quad p : H^2(\mathcal{A}, \mathcal{O}(-2, -2, -2, -2) \otimes \mathcal{L}_{\mathcal{A}}) \xrightarrow{p} H^2(\mathcal{A}, \mathcal{L}^{\otimes 2})$$

$$H^2(X, \mathcal{L}^{\otimes 2}) = \text{coker}(p)$$

Question: How can we vary $p = p_0 + \delta p$ so that $\ker(p) \neq 0$?

In field theory:

- E_7 Singlets: $C_+ \in H^1(\mathcal{L}^{\otimes 2})$, $C_- \in H^1(\mathcal{L}^{\vee \otimes 2})$. (“Jump” together)
- Superpotential: $W = \lambda_{ia}(\mathfrak{z}) C_+^i C_-^a$

- Choose Vacuum: $\langle C_+ \rangle \neq 0$ and $\langle C_- \rangle = 0$.

- Non-trivial F-term: $\frac{\partial W}{\partial C_-^a} = \lambda_{ia}(\mathfrak{z}) \langle C_+^i \rangle = 0$

- In fluctuation $\delta \left(\frac{\partial W}{\partial C_-^b} \right) = \frac{\partial \lambda_{ib}}{\partial \mathfrak{z}_+^I} \langle C_+^i \rangle \delta \mathfrak{z}_+^I$

$\frac{\partial \lambda_{ib}}{\partial \mathfrak{z}_+^I}$ vanishes along locus with $\lambda = 0$. \perp to locus, $\delta \mathfrak{z}_+^I$ gets a mass. All agree with Atiyah Computation.

Global questions...

- Everything we have discussed so far involves fluctuation around a point in C.S. moduli space. **Big limitation:** You have to know where to start.
- Hard to find isolated solutions (wanted for moduli stabilization) this way.
- **New Idea:** Represent CS Loci (vacuum solutions to the F-terms) as an **algebraic variety**. From Koszul Sequence:
- $H^1(X, \mathcal{L}^{\otimes 2}) = \ker(p)$ forms an ideal in the combined moduli space

$$p(\text{complex structure})s(\text{bundle moduli}) = 0 \text{ in cohomology}$$

- Using computational algebraic geometry (i.e. Groebner Basis methods)

Form the Ideal \Rightarrow Primary Decompose \Rightarrow Elimination to remove s

- Now we can scan over all possible starting points!

In order to describe the ideal on one slide, quotient by freely acting $\mathbb{Z}_2 \times \mathbb{Z}_4$ symmetry. Then we have $p(c)s(b) = 0 \in H^2(\mathcal{A}, \mathcal{L}^{\otimes 2})$, where

$$\begin{aligned}
 p = & \quad c_1 x_{1,0} x_{1,1} x_{2,0} x_{2,1} x_{3,0} x_{3,1} x_{4,0} x_{4,1} + \\
 & \quad c_9 \left(x_{1,0}^2 x_{3,0} x_{3,1} x_{4,0} x_{4,1} x_{2,0}^2 + x_{1,1}^2 x_{3,0} x_{3,1} x_{4,0} x_{4,1} x_{2,0}^2 + x_{1,0}^2 x_{2,1}^2 x_{3,0} x_{3,1} x_{4,0} x_{4,1} + x_{1,1}^2 x_{2,1}^2 x_{3,0} x_{3,1} x_{4,0} x_{4,1} \right) + \\
 & \quad c_3 \left(x_{1,1}^2 x_{2,0} x_{2,1} x_{4,0} x_{4,1} x_{3,0}^2 + x_{1,0} x_{1,1} x_{2,1}^2 x_{3,1} x_{4,0} x_{3,0} + x_{1,0} x_{1,1} x_{2,0}^2 x_{3,1} x_{4,1}^2 x_{3,0} + x_{1,0}^2 x_{2,0} x_{2,1} x_{3,1}^2 x_{4,0} x_{4,1} \right) + \\
 & \quad c_4 \left(x_{1,0} x_{1,1} x_{2,0}^2 x_{4,0} x_{4,1} x_{3,0}^2 + x_{1,1}^2 x_{2,0} x_{2,1} x_{3,1} x_{4,0} x_{3,0} + x_{1,0}^2 x_{2,0} x_{2,1} x_{3,1} x_{4,1}^2 x_{3,0} + x_{1,0} x_{1,1} x_{2,1}^2 x_{3,1}^2 x_{4,0} x_{4,1} \right) + \\
 & \quad c_5 \left(x_{1,0} x_{1,1} x_{2,1}^2 x_{4,0} x_{4,1} x_{3,0}^2 + x_{1,0}^2 x_{2,0} x_{2,1} x_{3,1} x_{4,0} x_{3,0} + x_{1,1}^2 x_{2,0} x_{2,1} x_{3,1} x_{4,1}^2 x_{3,0} + x_{1,0} x_{1,1} x_{2,0}^2 x_{3,1}^2 x_{4,0} x_{4,1} \right) + \\
 & \quad c_6 \left(x_{1,0}^2 x_{2,0} x_{2,1} x_{4,0} x_{4,1} x_{3,0}^2 + x_{1,0} x_{1,1} x_{2,0}^2 x_{3,1} x_{4,0} x_{3,0} + x_{1,0} x_{1,1} x_{2,1}^2 x_{3,1} x_{4,1}^2 x_{3,0} + x_{1,1}^2 x_{2,0} x_{2,1} x_{3,1}^2 x_{4,0} x_{4,1} \right) + \\
 & \quad c_7 \left(x_{1,1}^2 x_{2,1}^2 x_{3,0} x_{4,0}^2 + x_{1,0}^2 x_{2,1}^2 x_{3,1} x_{4,0}^2 + x_{1,1}^2 x_{2,0}^2 x_{3,0} x_{4,1}^2 + x_{1,0}^2 x_{2,0}^2 x_{3,1} x_{4,1}^2 \right) + \\
 & \quad c_8 \left(x_{1,0}^2 x_{2,1}^2 x_{3,0} x_{4,0}^2 + x_{1,0}^2 x_{2,0}^2 x_{3,1} x_{4,0}^2 + x_{1,1}^2 x_{2,1}^2 x_{3,0} x_{4,1}^2 + x_{1,1}^2 x_{2,0}^2 x_{3,1} x_{4,1}^2 \right) + \\
 & \quad c_2 \left(x_{1,0} x_{1,1} x_{2,0} x_{2,1} x_{3,0}^2 x_{4,0}^2 + x_{1,0} x_{1,1} x_{2,0} x_{2,1} x_{3,1}^2 x_{4,0}^2 + x_{1,0} x_{1,1} x_{2,0} x_{2,1} x_{3,0}^2 x_{4,1}^2 + x_{1,0} x_{1,1} x_{2,0} x_{2,1} x_{3,1}^2 x_{4,1}^2 \right) + \\
 & \quad c_{10} \left(x_{1,1}^2 x_{2,0}^2 x_{3,0} x_{4,0}^2 + x_{1,1}^2 x_{2,1}^2 x_{3,1} x_{4,0}^2 + x_{1,0}^2 x_{2,0}^2 x_{3,0} x_{4,1}^2 + x_{1,0}^2 x_{2,1}^2 x_{3,1} x_{4,1}^2 \right) + \\
 & \quad c_{11} \left(x_{1,0}^2 x_{2,0}^2 x_{3,0} x_{4,0}^2 + x_{1,1}^2 x_{2,0}^2 x_{3,1} x_{4,0}^2 + x_{1,0}^2 x_{2,1}^2 x_{3,0} x_{4,1}^2 + x_{1,1}^2 x_{2,1}^2 x_{3,1} x_{4,1}^2 \right)
 \end{aligned}$$

$$\begin{aligned}
 s = & \quad \frac{b_1}{x_{1,0}^2 x_{1,1}^2 x_{2,0}^2 x_{2,1}^2} + \frac{b_2}{x_{1,0}^3 x_{1,1} x_{2,0}^2 x_{2,1}} + \frac{b_2}{x_{1,0} x_{1,1}^3 x_{2,0}^2 x_{2,1}} + \frac{b_3}{x_{1,0}^4 x_{2,0}^2 x_{2,1}^2} + \frac{b_3}{x_{1,1}^4 x_{2,0}^2 x_{2,1}^2} + \frac{b_2}{x_{1,0}^3 x_{1,1} x_{2,0}^2 x_{2,1}^3} + \\
 & \quad \frac{b_2}{x_{1,0} x_{1,1}^3 x_{2,0}^2 x_{2,1}^3} + \frac{b_3}{x_{1,0}^2 x_{1,1}^2 x_{2,0}^4} + \frac{b_4}{x_{1,0}^4 x_{2,0}^2} + \frac{b_4}{x_{1,1}^4 x_{2,0}^2} + \frac{b_3}{x_{1,0}^2 x_{1,1}^2 x_{2,1}^4} + \frac{b_4}{x_{1,0}^4 x_{2,1}^4} + \frac{b_4}{x_{1,1}^4 x_{2,1}^4}
 \end{aligned}$$

$x_{a,i}$ -homogeneous coordinates on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

Disconnected Loci

- Let's perform the analysis for the example... $\mathcal{L} = \mathcal{O}(-4, -4, 2, 2)$ on $X^{4,68}/\mathbb{Z}_2 \times \mathbb{Z}_4$.
- Primary Decomposing and Eliminating gives **27 distinct loci in Complex Structure Moduli Space**
- Branches to the solution space range in dimension from 7 to zero (i.e. from 3 to all of the complex structure moduli fixed by V).
- Having found isolated point-like solutions, we might think we can declare victory...
- But for the given values of C.S., we still have to check transversality of the CY: $p = 0 = dp$
- By Bertini's Theorem, a generic CICY is smooth, but once we are fixed to very special points in CS, singularities are a real concern...

Singularities in the CY

Dimension in CS	7	5	4	3	2	1	0
Dimension of Singularities in X	0	0	smooth	0	0	0	2

- For this example, of the 27 branches to the solution space, all but one force the CY to be singular. The 4-dimensional locus given by

$$c_{10} - c_{11} = 0, c_8 - c_{11} = 0, c_7 - c_{11} = 0, c_4 + c_5 = 0, c_3 + c_6 = 0, c_2 c_9 - c_1 c_{11} = 0$$

leads to a smooth CY.

- Can we do anything with the singular solutions?
- Locally, for $\dim(\text{Sing}) \leq 1$ we can imagine resolving (i.e. blowing-up) the singularities.
- Unfortunately, for the case at hand, all isolated point-like solutions are too badly singular. However, we can consider one of the larger loci...

Blowing up and “Splitting” CICYs

- Consider the 5-dimensional locus with pt-like singularities in CY
- Locally, we can resolve these singular pts. But have to worry about global issues: **CY condition? What happens to the bundle? Symmetries?**
- Happily, there are some resolutions of singular CYs that we have good control over: **Conifold Transitions.**
- CY defining poly takes the form $p = f_1 f_3 - f_2 f_4 = 0$. Topologically a cone over $S^3 \times S^2$. Can be resolved by introducing new \mathbb{P}^1 direction

$$\begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = 0$$

- This “resolved” manifold can be described as a new CICY, called a “split” of the initial singular one.
- Can explicitly track divisors, extension bundles, and symmetries through this geometric transition.

- Consider the following split of the Tetraquadric:

$$x = \left[\begin{array}{c|c} \mathbb{P}^1 & 2 \\ \mathbb{P}^1 & 2 \\ \mathbb{P}^1 & 2 \\ \mathbb{P}^1 & 2 \end{array} \right] \Rightarrow [p_{(2,2,2,2)} \rightarrow f^1_{(2,0,2,0)} f^3_{(0,2,0,2)} - f^2_{(2,0,2,0)} f^4_{(0,2,0,2)} = 0] \Rightarrow X_{split} = \left[\begin{array}{c|cc} \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^1 & 2 & 0 \\ \mathbb{P}^1 & 0 & 2 \\ \mathbb{P}^1 & 2 & 0 \\ \mathbb{P}^1 & 0 & 2 \end{array} \right]$$

- Quotienting both sides by $\mathbb{Z}_2 \times \mathbb{Z}_4$, this “split” locus intersects the dim 5 CS locus above, leads to 8 singular pts on the CY. Resolving such pts, $X \rightarrow X_{split}$.
- To check, can repeat the analysis independently with $\mathcal{L} = \mathcal{O}(0, -2, -2, 1, 1)$ on X_{split} .
- This time, 14 branches ranging from dimension 3 to 0. The resolution X gives a 2 dimensional locus in the CS space of a smooth X_{split} .
- All CICYs connected by such transitions. Reid’s Fantasy?
- Not yet dynamical transitions, but provides interesting web of “stabilizing” bundles on CYs...

Conclusions

- The presence of a **holomorphic** vector bundle constrains C.S. moduli \Rightarrow can be used as a hidden sector mechanism for moduli stabilization
- The C.S. can be stabilized at the perturbative level without moving away from a CY manifold
 - Avoids problems of naive KKL'T scenarios in heterotic
 - Allows us to keep geometric model-building toolkit
- Stabilized values fully determined for use in physical couplings
- The **full moduli-dependent vacuum space can be determined** using computational algebraic geometry
- Some singular solutions can be resolved \Rightarrow Gives insight into connections between possible base manifold CYs.

The End