Recent Progress on Instanton Partition Functions

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Based on the following papers:

- [arXiv:1205.4741] with Amihay Hanany and Shlomo Razamat
- [arXiv:1111.5624] with Christoph Keller, Jaewon Song and Yuji Tachikawa
- [arXiv:1005.3026] with Sergio Benvenuti and Amihay Hanany

(Please see also

- [arXiv:1205.4722] by Christoph Keller and Jaewon Song

for a closely related work.)

Part I: Introduction

The moduli space of instantons

- Consider instantons in a pure Yang-Mills theory with gauge group G
- **The moduli space of k G-instantons on $\mathbb{C}^2$:**

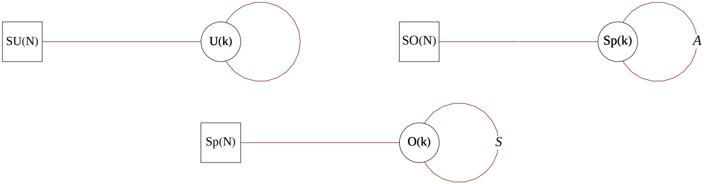
  The space of solutions to the self-dual Yang-Mills equations, modulo gauge transformations, in a giving winding sector k and gauge group G
- For a classical gauge group G, $SU(N)$, $Sp(N)$ or $SO(N)$, such instanton solutions can be constructed using linear algebra!
- Such a simple method of constructions is known as the ADHM construction

  (Atiyah, Drinfeld, Hitchin, Manin '78)

The ADHM construction from string theory

(Douglas, Moore, Witten '94-'96)

- Can be realised on a system of D3-branes and D7-branes (possibly with an O-plane).
- **D3's on top of D7's:** D3-branes $\equiv$ instantons in the w.v. of D7-branes.
- The w.v. theory of the D3-branes has 8 SUSYs (4d $\mathcal{N} = 2$). Can be represented by the ADHM quivers:



- **D3-branes on top of D7-branes** $\longleftrightarrow$ Higgs branch of the ADHM quiver.
- Identified with the moduli space of k $SU(N)$, $SO(N)$ or $Sp(N)$ instantons on $\mathbb{C}^2$.

Comments on the ADHM construction

- The F and D terms give rise to the moment map equations for hyperKähler quotients of the instanton moduli spaces
- For **classical** gauge groups, the moment map equations follow from the Langrangian of the corresponding gauge theory
- For **exceptional** gauge groups, no ADHM construction is known!
  - Although brane constructions are known, they do not admit a perturbative description and hence there is no Lagrangian

Comments on the ADHM construction (continued)

- **For E-type groups:** *One way* is to look at the Higgs branch of theories on M5-branes wrapping Riemann spheres with punctures (Gaiotto '09) and with appropriate boundary conditions.

  (Benini-Benvenuti-Tachikawa '09, Gaiotto-Razamat '12)

  - For 1 instanton, this construction gives rise to theories with E-**type** global sym. proposed by [Minahan-Nemeschansky '96].
  - For F_4 and G_2, there is no known construction of this type so far!

Even though the ADHM construction is not available, it is still possible to compute instanton partitions function exactly and explicitly!

Symmetry of an instanton moduli space

The moduli space of k G instantons on $\mathbb{C}^2$

- is a singular hyperKähler cone
- possesses a symmetry

$$U(2)_{\mathbb{C}^2} \times G$$

where $U(2)_{\mathbb{C}^2}$ is a symmetry of $\mathbb{C}^2$, the overall position of the instantons

Symmetry

- The $U(1)_{\mathbb{C}^2}$ subgroup of $U(2)_{\mathbb{C}^2}$ can be identified with the Cartan of the R symmetry $SU(2)_R$
- The $SU(2)_{\mathbb{C}^2}$ subgroup of $U(2)_{\mathbb{C}^2}$ rotates the two chiral multiplets in the {adjoint, A, S} hypermultiplet of the ADHM quiver for $\{SU(N), SO(N), Sp(N)\}$ instantons

Part II: Hilbert series for instanton moduli spaces

Hilbert series for instanton moduli spaces

- In order to study the instanton moduli space, we compute a partition function that counts holomorphic functions on the space wrt. the global $U(1)_{\mathbb{C}^2}$ charge
- Such a partition function is known as the Hilbert series (HS) of instanton moduli space. It takes the form

$$g(t; x; y_1, \ldots, y_r) = \sum_{k=0}^{\infty} \mathbf{R}^{(k)}_{SU(2)_{\mathbb{C}^2}}(x) \; \mathbf{r}^{(k)}_G(y_1, \ldots, y_r) \; t^k$$

  - The variable (fugacity) t keeps track of the charge k under $U(1)_{\mathbb{C}^2}$
  - $\mathbf{R}^{(k)}_{SU(2)_{\mathbb{C}^2}}(x)$ is the character of the rep $\mathbf{R}^{(k)}$ of $SU(2)_{\mathbb{C}^2}$
  - $\mathbf{r}^{(k)}_G(y_1, \ldots, y_r)$, with $r = \mathrm{rk}\ G$, is the character of the rep $\mathbf{r}^{(k)}$ of G

Hilbert series for instanton moduli spaces (continued)

$$g(t;x;y_1,\ldots,y_r) = \sum_{k=0}^{\infty} \mathbf{R}^{(k)}_{SU(2)_{\mathbb{C}^2}}(x)\ \mathbf{r}^{(k)}_G(y_1,\ldots,y_r)\ t^k$$

- Interpretation: Holomorphic functions carrying $U(1)_{\mathbb{C}^2}$ charge k transform under the rep $[\mathbf{R}^{(k)}_{SU(2)_{\mathbb{C}^2}};\mathbf{r}^{(k)}_G]$ of $SU(2)_{\mathbb{C}^2} \times G$.
- The number of such functions are $\dim \mathbf{R}^{(k)}_{SU(2)_{\mathbb{C}^2}} \times \dim \mathbf{r}^{(k)}_G$.
- Dimension of the moduli space. Setting $x = y_1 = \ldots = y_r = 1$, we have

$$\begin{aligned} g(t;x=1;\{y_i=1\}) &= \sum_{k=0}^{\infty} \dim \mathbf{R}^{(k)}_{SU(2)_{\mathbb{C}^2}} \times \dim \mathbf{r}^{(k)}_G\ t^k \\ &\sim \frac{1}{(1-t)^{2kh^\vee_G}}\ , \qquad t \to 1\ . \end{aligned}$$

The cplx dim of the moduli space of k G instantons is $2kh^\vee_G$.

Example 1: Hilbert series of $\mathbb{C}^2$

- Holomophic coordinates are z_1, z_2
- $U(2) = U(1) \times SU(2)$. Both z_1 and z_2 carry charge $+1$ under $U(1)$ and transform under the fund. rep. $[1]$ of $SU(2)$. Note the character $[1]_{SU(2)}(x) = x + x^{-1}$
- Assign the fugacities t of $U(1)$ and x of $SU(2)$ to $z_{1,2}$:

$$z_1 \to tx, \qquad z_2 \to tx^{-1}$$

- Any holomorphic function on $\mathbb{C}^2$ takes the form $z_1^{k_1} z_2^{k_2}$, with $k_1, k_2 \geq 0$.
- The Hilbert series is

$$g(t,x) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} t^{k_1+k_2} x^{k_1-k_2} = \boxed{\sum_{n=0}^{\infty} [n]_{SU(2)}(x)\ t^n}$$

- This can also be written as $g(t,x) = \mathrm{PE}\left[t\ [1]_{SU(2)}(x)\right]$, where

$$\mathrm{PE}[f(t_1, \ldots, t_n)] = \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} f(t_1^k, \ldots, t_n^k)\right) .$$

Example 2: Hilbert series of $\mathbb{C}^2/\mathbb{Z}_2$

- Holomophic coordinates are z_1, z_2, with $\mathbb{Z}_2$ action $(z_1, z_2) \mapsto (-z_1, -z_2)$.
- Focus on the $\mathbb{Z}_2$ invariant quantities:
  - 3 generators: $G_1 = z_1^2, \quad G_2 = z_1 z_2, \quad G_3 = z_2^2$
  - 1 relation: $G_1 G_3 - G_2^2 = 0$ (defining equation for $\mathbb{C}^2/\mathbb{Z}_2$)
- The symmetry of $\mathbb{C}^2/\mathbb{Z}_2$ is $U(2) = U(1) \times SU(2)$.
- G_1, G_2, G_3 transform as a triplet $[2]$ under the isometry $SU(2)$ and each carries charge $+2$ under $U(1)$.
- **Hilbert series:** Let t be a fugacity for $U(1)$, and x be an $SU(2)$ fugacity

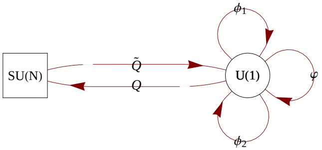
$$g(t,x) = (1-t^4)\,\mathrm{PE}\left[[2]_{SU(2)}(x) t^2\right]$$

$$= \boxed{\sum_{n=0}^{\infty} [2n]_{SU(2)}(x)\; t^{2n}}$$

- Observe that the symmetry $U(2)$ is manifest in this expression.

Example 3: One $SU(N)$ instanton on $\mathbb{C}^2$

- Translate the ADHM quiver from $\mathcal{N}=2$ language to $\mathcal{N}=1$ language
- In $\mathcal{N}=1$ notation, the quiver looks like



  - Superpotential $W = \widetilde{Q}^i \varphi Q_i \quad \longrightarrow \quad F$ terms: $\widetilde{Q}^i Q_i = 0$
  - Global symmetry: $SU(2)_{\mathbb{C}^2} \times U(1)_{\mathbb{C}^2} \times SU(N)$
  - ϕ_1, ϕ_2 transform as a doublet under $SU(2)_{\mathbb{C}^2}$

$$g_{1,SU(N)}(t; x, y_1, \ldots, y_{n-1}) = \overbrace{\frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{2\pi i z}}^{\text{D-terms and modding out by } U(1)} \times \overbrace{(1-t^2)}^{\text{F-terms}} \times$$

$$\mathrm{PE}\left[\underbrace{[1]_{SU(2)_{\mathbb{C}^2}}(x)\, t}_{\phi_1,\phi_2} + \underbrace{[1,0,\ldots,0]_{SU(N)}(\boldsymbol{y})\, t z^{-1}}_{\widetilde{Q}} + \underbrace{[0,\ldots,0,1]_{SU(N)}(\boldsymbol{y})\, t z}_{Q} \right]$$

Example 3: One $SU(N)$ instanton on $\mathbb{C}^2$ (continued)

- The result of the integration gives the HS:

$$g_{1,SU(N)}(t;x,\boldsymbol{y}) = \underbrace{\frac{1}{(1-tx)\left(1-tx^{-1}\right)}}_{\substack{\text{HS for } \mathbb{C}^2 \text{ assoc. with} \\ \text{position of the instanton}}} \widetilde{g}_{1,SU(N)}(t,\boldsymbol{y})$$

$$\widetilde{g}_{1,SU(N)}(t,\boldsymbol{y}) = \sum_{n=0}^{\infty} [n,0,\ldots,0,n]_{SU(N)}(\boldsymbol{y})\; t^{2n}$$

$$= \boxed{\sum_{n=0}^{\infty} n(\text{highest weight of } \mathbf{Adj})\; t^{2n}}$$

- $\widetilde{g}_{1,SU(N)}(t,\boldsymbol{y})$ is said to be the HS of the reduced instanton moduli space

  (i.e. excluding the $\mathbb{C}^2$ component corresponding to the position of the instanton)

- **An example of holomorphic function at t^2:** This can be written as $M^i{}_j = \widetilde{Q}^i_a Q^a_j$, with a constraint $M^i{}_i = 0$ due to the F term. Hence $M^i{}_j$ transform under the adjoint rep $[1,0,\ldots,0,1]$ of $SU(N)$.

Example 4: One G instanton on $\mathbb{C}^2$ (with any simple group G)

- Repeat this computation for $G = SO(N),\ Sp(N)$ using the ADHM quivers. The HS of the reduced instanton moduli space take the same form as before:

$$\widetilde{g}_{1,G}(t;\boldsymbol{y}) = \sum_{n=0}^{\infty} n(\text{highest weight of } \mathbf{Adj})\ t^{2n}$$

- **Claim:** This holds for any simple group G, i.e. the $ABCDEFG$ type groups!

  (Benvenuti, Hanany, NM '08)

- The symmetry G is manifest in this expression.
- This claim can be mathematically proven. The proof relies on a special property of the moduli space of one instanton:
  - It is the orbit of the highest weight vector in the Lie algebra of $G_{\mathbb{C}}$ (Kronheimer '90).
  - The space of holomorphic functions on such a space is known (e.g. Vinberg-Popov '72 and Garfinkle '73); from which the HS can be deduced.

    (See also Gaiotto, Neitzke, Tachikawa '08)

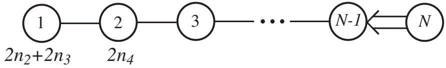
Example 5: Two $Sp(N)$ instantons on $\mathbb{C}^2$

The Hilbert series can be computed from the ADHM quiver and can be written in terms of $U(2) \times Sp(N)$ character expansion as

$$\widetilde{g}_{2,Sp(N)}(t, x, y_1, \ldots, y_N) = f(0; 0, \ldots, 0) + f(0; 0, 1, 0, \ldots, 0)t^4 + [f(1; 2, 0, 0, \ldots, 0) + f(1; 2, 1, 0, \ldots, 0)]\, t^5 \ ,$$

where the function f is defined as

$$f(a; b_1, b_2, \ldots, b_N) = \frac{1}{1-t^4} \sum_{m_2=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} t^{2m_2+2n_2+3n_3+4n_4} \times [2m_2 + n_3 + a; 2n_2 + 2n_3 + b_1, 2n_4 + b_2, b_3, \ldots, b_N] \ .$$



- Observe the **lattice** spanned by certain highest weight vectors associated with $SU(2) \times SU(N)$ reps.

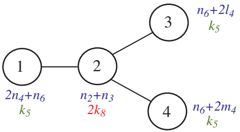
Example 6: Two $SO(8)$ instantons on $\mathbb{C}^2$

The Hilbert series can be computed from the ADHM quiver and can be written in terms of $U(2) \times SO(8)$ character expansion as

$$\widetilde{g}_{2,SO(8)}(t, x, y_1, \ldots, y_4) = \sum_{k_8=0}^{\infty} \Big\{ f(0; 0, 2k_8, 0, 0) t^{8k_8} + f(1; 0, 2k_8 + 1, 0, 0) t^{8k_8+5} + f(1; 1, 2k_8, 1, 1) t^{8k_8+7}$$
$$+ f(0; 1, 2k_8 + 1, 1, 1) t^{8k_8+10} \Big\} + \sum_{k_5=0}^{\infty} \Big\{ f(k_5 + 1; k_5 + 1, 0, k_5 + 1, k_5 + 1) t^{5k_5+5}$$
$$+ f(k_5 + 2; k_5 + 2, 0, k_5 + 2, k_5 + 2) t^{5k_5+12} \Big\} .$$

where the function f is defined as

$$f(a; b_1, \ldots, b_r) = \frac{1}{1 - t^4} \sum_{m_2=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} \sum_{m_4=0}^{\infty} \sum_{l_4=0}^{\infty} \sum_{n_6=0}^{\infty} t^{2m_2+2n_2+3n_3+4n_4+4m_4+4l_4+6n_6} \times$$
$$[2m_2 + n_3 + a; 2n_4 + n_6 + b_1, n_2 + n_3 + b_2, n_6 + 2l_4 + b_3, 2m_4 + n_6 + b_4] ,$$



Lattice structures

- Detailed study of the lattice structures allows for general and explicit expressions for the Hilbert series for 1 and 2 instantons in any simple group.
  - **Universal lattice:** denoted by n's, m's and l's.
  - **Non-universal lattices:** denoted by k's.
  - **Shifts:** denoted by a and b's.

The HS for two instantons in any simple group on $\mathbb{C}^2$ can be found at [arXiv:1205.4741].

Part III: Hilbert series as instanton partition functions

Hilbert series and Nekrasov's partition functions

- **HS:** $g(t; x; y_1, \ldots, y_r) = \sum_{m=0}^{\infty} \mathbf{R}^{(m)}_{SU(2)_{\mathbb{C}^2}}(x)\ \mathbf{r}^{(m)}_G(y_1, \ldots, y_r)\ t^m$.
- **Nekrasov's partition function:** Nekrasov's partition function for k G instantons can be obtained from the HS (Bruzzo-Fucito-Morales-Tanzini '02, Nakajima-Yoshioka '03):

$$Z_k(\epsilon_1, \epsilon_2, \boldsymbol{a}) = \lim_{\beta \to 0} \beta^{2kh^\vee_G}\ g(e^{-\frac{1}{2}\beta(\epsilon_1+\epsilon_2)}; e^{-\frac{1}{2}\beta(\epsilon_1-\epsilon_2)}; e^{-\beta a_1}, \ldots, e^{-\beta a_r})\ .$$

- **One G instanton:** Nekrasov's partition function is

$$Z_{k=1}(\epsilon_1, \epsilon_2, \boldsymbol{a}) = -\frac{1}{\epsilon_1\epsilon_2} \sum_{\boldsymbol{\gamma} \in \Delta_l} \frac{1}{(\epsilon_1 + \epsilon_2 + \boldsymbol{\gamma} \cdot \boldsymbol{a})(\boldsymbol{\gamma} \cdot \boldsymbol{a}) \prod_{\boldsymbol{\gamma}^\vee \cdot \boldsymbol{\alpha}=1,\ \boldsymbol{\alpha} \in \Delta} (\boldsymbol{\alpha} \cdot \boldsymbol{a})}\ ,$$

where Δ and Δ_l are the sets of the roots and the long roots, and $\boldsymbol{\gamma}^\vee = \frac{2\boldsymbol{\gamma}}{\boldsymbol{\gamma} \cdot \boldsymbol{\gamma}}$.

(Keller, NM, Song, Tachikawa '11; thanks to A. Bondal and S. Carnahan)

- **AGT relation:** This is equal to the norm of a certain coherent state of the W-algebra. For non-simply laced G, the coherent state is in the twisted sector of a simply-laced W-algebra.

(Keller, NM, Song, Tachikawa '11)

Hilbert series and superconformal indices

- The superconformal index (SCI) is a partition function of a SCFT on $S^3 \times S^1$ with periodic BCs for fermions around S^1.
  (e.g. Römelsberger '05,'07, Kinney-Maldacena-Minwalla-Raju '05, Dolan-Osborn '09,
  Spiridonov-Vatanov '08-'09, Gadde, Pomoni, Rastelli, Razamat, Yan '10-onwards)
- For a 4d $\mathcal{N} = 2$ SCFT, the SCI can be thought as a trace over the states of the theory on S^3. It gets contribution from all states annihilated by one of the supercharges. (Any choice of supercharges yields the same result.)
- One can assign to such states certain combinations of global charges that commute with this supercharge. There are fugacities associated with those global charges.
- Some of such fugacities can be set to zero and the SCI simplifies tremendously. A special case of our interests is known as the Hall-Littlewood (HL) index.
  (Gadde, Rastelli, Razamat, Yan '11)

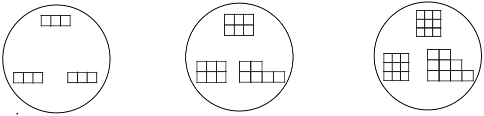
Hilbert series and Hall-Littlewood indices

- For a theory with Lagrangian, the HL index gets contributions only from one of the complex scalars in the h-plet and one of the fermions in the $\mathcal{N} = 2$ v-plet.
- For a 4d $\mathcal{N} = 2$ gauge theory arise from M5-branes wrapping a Riemann sphere (i.e. genus 0) with punctures, it is conjectured that the HL index is equal to the HS.

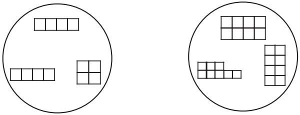
  (Gadde, Rastelli, Razamat, Yan '11)
- $E_{6,7,8}$ instantons can be realised in this way! For F_4 and G_2, there is no known construction of this type.

Instantons in E-type groups from M5 branes on Riemann spheres

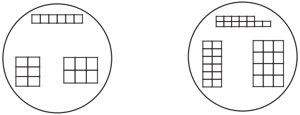
- One, two and three E_6 instantons



- One and two E_7 instantons



- One and two E_8 instantons



- The HL indices for these theories can be computed **exactly** in terms of HL polynomials (Gaiotto-Razamat '12). But the $E_{6,7,8}$ symmetry are not manifest.

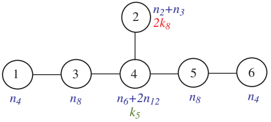
Example 7: Two E_6 instantons

- The HS is equal to the HL index and can be rewritten in terms of $U(2) \times E_6$ character expansion as

$$\widetilde{g}_{2,E_6}(t,x,\boldsymbol{y}) = \sum_{k_8=0}^{\infty} \Big\{ f(0;0,2k_8,0,0,0,0)t^{8k_8} + f(1;0,2k_8+1,0,0,0,1)t^{8k_8+5} + f(1;0,2k_8,0,1,0,0)t^{8k_8+7} + f(0;0,2k_8+1,0,1,0,0)t^{8k_8+10} \Big\} + \sum_{k_5=0}^{\infty} \Big\{ f(k_5+1;0,0,0,k_5+1,0,0)t^{5k_5+5} + f(k_5+2;0,0,0,k_5+2,0,0)t^{5k_5+12} \Big\} ,$$

where the function f is defined as

$$f(a;b_1,\ldots,b_6) = \frac{1}{1-t^4} \sum_{m_2=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_6=0}^{\infty}\sum_{n_8=0}^{\infty}\sum_{n_{12}=0}^{\infty} t^{2m_2+2n_2+3n_3+4n_4+6n_6+8n_8+12n_{12}} [2m_2+n_3+a; n_4+b_1, n_2+n_3+b_2, n_8+b_3, n_6+2n_{12}+b_4, n_8+b_5, n_4+b_6] .$$



- This form of HS provides a way to generalise this to E_7 and E_8.

Example 8: Two E_7 instantons

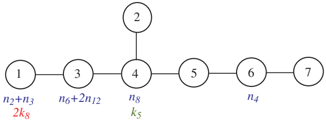
The Hilbert series can be rewritten in terms of $U(2) \times E_7$ character expansion as

$$
\begin{aligned}
&\widetilde{g}_{2,E_7}(t, x, y_1, \ldots, y_7) \\
&= \sum_{k_8=0}^{\infty} \Big\{ f(0; 2k_8, 0, 0, 0, 0, 0, 0) t^{8k_8} + f(1; 2k_8+1, 0, 0, 0, 0, 0, 0) t^{8k_8+5} \\
&\qquad + f(1; 2k_8, 0, 1, 0, 0, 0, 0) t^{8k_8+7} + f(0; 2k_8+1, 0, 1, 0, 0, 0) t^{8k_8+10} \Big\} + \\
&\sum_{k_5=0}^{\infty} \Big\{ f(k_5+1; 0, 0, k_5+1, 0, 0, 0, 0) t^{5k_5+5} + f(k_5+2; 0, 0, k_5+2, 0, 0, 0, 0) t^{5k_5+12} \Big\} ,
\end{aligned}
$$

where the function f is defined as

$$
f(a; b_1, \ldots, b_7) = \frac{1}{1-t^4} \sum_{m_2=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} \sum_{n_6=0}^{\infty} \sum_{n_8=0}^{\infty} \sum_{n_{12}=0}^{\infty} t^{2m_2+2n_2+3n_3+4n_4+6n_6+8n_8+12n_{12}} \times
$$

$$
[2n_2 + n_3 + a; n_2 + n_3 + b_1, b_2, n_6 + 2n_{12} + b_3, n_8 + b_4, b_5, n_4 + b_6, b_7] .
$$



Example 9: Two E_8 instantons

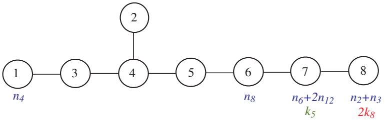
The Hilbert series can be rewritten in terms of $U(2) \times E_8$ character expansion as

$$\begin{aligned}
&\widetilde{g}_{2,E_8}(t, x, y_1, \ldots, y_8) \\
&= \sum_{k_8=0}^{\infty} \Big\{ f(0; 0,0,0,0,0,0,0,2k_8) t^{8k_8} + f(1; 0,0,0,0,0,0,0,2k_8+1) t^{8k_8+5} \\
&\quad + f(1; 0,0,0,0,0,0,1,2k_8+1) t^{8k_8+7} + f(0; 0,0,0,0,0,0,1,2k_8+1) t^{8k_8+10} \Big\} + \\
&\sum_{k_5=0}^{\infty} \Big\{ f(k_5+1; 0,0,0,0,0,0,k_5+1,0) t^{5k_5+5} + f(k_5+2; 0,0,0,0,0,0,k_5+2,0) t^{5k_5+12} \Big\} ,
\end{aligned}$$

where the function f is defined as

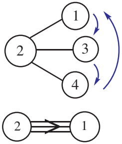
$$f(a; b_1, \ldots, b_8) = \frac{1}{1-t^4} \sum_{m_2=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} \sum_{n_6=0}^{\infty} \sum_{n_8=0}^{\infty} \sum_{n_{12}=0}^{\infty} t^{2m_2+2n_2+3n_3+4n_4+6n_6+8n_8+12n_{12}} \times$$

$$[2n_2+n_3+a; n_4+b_1, b_2, b_3, b_4, b_5, n_8+b_6, n_6+2n_{12}+b_7, n_2+n_3+b_8] .$$



Example 10: Two G_2 instantons

- The Dynkin diagram G_2 can be obtained by folding the Dynkin diagram of $SO(8)$ via a $\mathbb{Z}_3$ outer-automorphism.



- The Hilbert series can be rewritten in terms of $U(2) \times G_2$ character expansion as

$$\widetilde{g}_{2,G_2}(t, x, y_1, y_2)$$

$$= \sum_{k_8=0}^{\infty} \left\{ f(0; 0, 2k_8) t^{8k_8} + f(1; 0, 2k_8+1) t^{8k_8+5} + f(1; 3, 2k_8) t^{8k_8+7} + f(0; 3, 2k_8+1) t^{8k_8+10} \right\} +$$

$$\sum_{k_5=0}^{\infty} \left\{ f(k_5+1; 3k_5+3, 0) t^{5k_5+5} + f(k_5+2; 3k_5+6, 0) t^{5k_5+12} \right\},$$

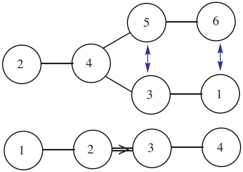
where

$$f(a; b_1, b_2) = \frac{1}{1-t^4} \sum_{m_2=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} \sum_{n_6=0}^{\infty} t^{2m_2+2n_2+3n_3+4n_4+6n_6} \times$$

$$[2m_2 + n_3 + a; 2n_4 + 3n_6 + b_1, n_2 + n_3 + b_2]$$

Example 11: Two F_4 instantons

- The Dynkin diagram F_4 can be obtained by folding the Dynkin diagram of E_6 via a $\mathbb{Z}_2$ outer-automorphism.



- The Hilbert series can be rewritten in terms of $U(2) \times F_4$ character expansion as

$$\widetilde{g}_{2,F_4}(t,x,\boldsymbol{y}) = \sum_{k_8=0}^{\infty} \Big\{ f(0;2k_8,0,0,0)t^{8k_8} + f(1;2k_8+1,0,0,0)t^{8k_8+5} + f(1;2k_8,1,0,0)t^{8k_8+7} + f(0;2k_8+1,1,0,0)t^{8k_8+10} \Big\} + \sum_{k_5=0}^{\infty} \Big\{ f(k_5+1;0,k_5+1,0,0)t^{5k_5+5} + f(k_5+2;0,k_5+2,0,0)t^{5k_5+12} \Big\} .$$

where

$$f(a;b_1,\ldots,b_4) = \frac{1}{1-t^4} \sum_{m_2=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} \sum_{n_6=0}^{\infty} \sum_{n_8=0}^{\infty} \sum_{n_{12}=0}^{\infty} t^{2m_2+2n_2+3n_3+4n_4+6n_6+8n_8+12n_{12}}$$

$$[2m_2+n_3+a; n_2+n_3+b_1, n_6+2n_{12}+b_2, 2n_8+b_3, 2n_4+b_4]$$

Conclusions

- Hilbert series are computed explicitly for one and two instantons in any simple group, regardless of the existence of ADHM constructions.
- Great advantages of writing a Hilbert series in terms of a character expansion:
  - The symmetry $U(2) \times G$ of the instanton moduli space is manifest.
  - The generalisation for higher rank groups or other groups can be done quite straightforwardly.
- For the groups of E-type, recent superconformal index results & character expansions allow for an explicit computation of the HS.
- For G_2 or F_4, discrete symmetries are enough to evaluate the HS exactly, even though neither ADHM construction nor SCI is known for these cases.
- In general, the HS for multi-instantons can also be computed from the blow-up formula due to [Nakajima-Yoshioka '03]. (see, *e.g.* Keller-Song '12.) It'd be interesting to obtain closed forms, in which the symmetry is manifest, from such a formula.