Semiclassical Framed BPS States

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Based on work with Dieter van den Bleeken and Greg Moore

Motivation

framed BPS state[1]: a BPS state in $\mathcal{N} = 2$ SYM with line operator defects

why study them?

- a simple conceptual approach to KS wall-crossing formula
- compute Wilson-'t Hooft operator vev's exactly

why study them semiclassically?

1. define quantities of interest in terms of differentio-geometric structures $\Rightarrow$ use mathematical tools to obtain interesting physics results—"physical mathematics"
2. translate recent physically motivated conjectures on $\mathcal{N} = 2$ spectrum into interesting new conjectures about Dirac operators on hyperkähler manifolds—"mathematical physics"

[1] Gaiotto, Moore and Neitzke (2010)

Outline

- ✓ **Motivation**
- **Background**
  $\mathcal{N} = 2$ *semiclassical methodology, line operators*
- **Application and Examples: physical mathematics**
- **Protected Spin Characters and their Positivity Conjectures**
- **Examples: mathematical physics**

$\mathcal{N} = 2$ SYM on $\mathbb{R}^{1,3}$, gauge group G, no hypers

UV description

- $(A, \varphi, \psi_A) \in \Omega^1(\mathbb{R}^{1,3}, \mathfrak{g}) \oplus \Omega^0(\mathbb{R}^{1,3}, \mathfrak{g}_\mathbb{C}) \oplus \Omega^0(S^+(\mathbb{R}^{1,3}), \mathfrak{g}_\mathbb{C})$
  $(\mathbf{1},\ \mathbf{1},\ \mathbf{2})$ of $SU(2)_R$
- $S = -\frac{1}{2{g_0}^2} \int \left\{ ||F||^2 + ||D\varphi||^2 - \frac{1}{4}||[\varphi, \bar{\varphi}]||^2 + \cdots \right\}$

IR description à la Seiberg–Witten

- Coulomb branch $\mathcal{B}[u^i]$; e.m. charge lattice Γ, $\langle \gamma_1, \gamma_2 \rangle \in \mathbb{Z}$
- $U(1)^{\mathrm{rnk}\,\mathfrak{g}}$ abelian $\mathcal{N} = 2$ v.m.'s $\supset (a^J, A^J)$
- $a_{D,I} \equiv \frac{\partial \mathcal{F}}{\partial a^I}$, $\quad Z_{\gamma = (\gamma_m, \gamma_e)} = a_{D,I}\gamma_m^I + a^I \gamma_{e,I}$, $\quad M_\gamma^{\mathrm{BPS}} = |Z_\gamma|$
- SW solution: $\left(a^I(u), a_{D,I}(u)\right) \to \mathcal{F}(a)$

Semiclassical regime and UV $\leftrightarrow$ IR connection

Identifying low energy quantities

- let $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$; basis $\{H_{\alpha_I}; E_\alpha\}$, H_{α_I} simple co-roots
- $A = A^I H_{\alpha_I} + \cdots$, $\varphi = a^I H_{\alpha_I} + \cdots$,
- $\Gamma = \Gamma_m \oplus \Gamma_e = \Lambda_{cr} \oplus \Lambda_{wt}$ with charges $\begin{matrix} \gamma_m^I = \frac{1}{2\pi}\int_{S^2_\infty} F^I \\ \gamma_{e,I} = \frac{1}{2\pi}\int_{S^2_\infty} (\mathrm{Im}\,\tau_{IJ}) \star F^J \end{matrix}$

Semiclassical regime

- $\mathcal{F} = \mathcal{F}^{\mathrm{cl}} + \mathcal{F}^{\text{1-lp}} + \mathcal{F}^{\mathrm{np}} \;\Rightarrow\; a_{D,I} = a_{D,I}^{\mathrm{cl}} + \cdots \;\Rightarrow\; Z_\gamma = Z_\gamma^{\mathrm{cl}} + \cdots$
- **s.c. regime**: choice of (g_0, Λ_0) and $\mathcal{R}^{\mathrm{s.c.}} \subset \mathcal{B}$, s.t. $|Z_\gamma^{\mathrm{cl}}|$ dominates $|Z_\gamma^{\text{1-lp}}|, |Z_\gamma^{\mathrm{np}}|$

Semiclassical vanilla BPS states I

recipe for studying BPS spectrum in s.c. regime:

1. construct moduli space of classical BPS field configurations
2. approximate dynamics via motion on moduli space
3. quantize associated c.c. d.o.f.'s $\Rightarrow$ $\mathcal{N} = 4$ SQM
4. s.c. BPS states = BPS states in the SQM

Semiclassical vanilla BPS states II

Step 1: monopole moduli space

- $H^{\mathrm{cl}} \geq -\mathrm{Re}(\zeta^{-1} Z_{\gamma}^{\mathrm{cl}})$

Semiclassical vanilla BPS states II

Step 1: monopole moduli space

- $H^{\rm cl} = -\,{\rm Re}(\zeta^{-1} Z^{\rm cl}_{\gamma})$ when $\begin{cases} \text{with } i\varphi \equiv \zeta(X + iY)\ ,\ A_0 = Y : & \\ \partial_0 A_i = \partial_0 X = \partial_0 Y = 0\ , & \\ \star_3 F = D_{(3)} X\ , & (p) \\ D^2_{(3)} Y + [X,[X,Y]] = 0\ , & (s) \end{cases}$
  - (p) primary, Bogomolny BPS eqn.
  - (s) secondary BPS eqn, $F_{i0} = D_i Y$.
  - maximal bound: $\zeta = {\rm Arg}(-Z^{\rm cl}_{\gamma})$

Semiclassical vanilla BPS states II

Step 1: monopole moduli space

- $$H^{\rm cl} = -\,{\rm Re}(\zeta^{-1} Z_\gamma^{\rm cl}) \ \text{ when } \begin{cases} \text{with } i\varphi \equiv \zeta(X + iY)\ ,\ A_0 = Y : \\ \partial_0 A_i = \partial_0 X = \partial_0 Y = 0\ , \\ \star_3 F = D_{(3)} X\ , & (p) \\ D_{(3)}^2 Y + [X,[X,Y]] = 0\ , & (s) \end{cases}$$
  - (p) primary, Bogomolny BPS eqn. moduli
  - (s) secondary BPS eqn, $F_{i0} = D_i Y$. unique solution, given Y_∞
  - maximal bound: $\zeta = {\rm Arg}(-Z_\gamma^{\rm cl})$

- $$\mathcal{M}(\gamma_m; X_\infty) = \left\{ (A_i, X) \,\middle|\, (p),\ \begin{array}{l} X \to X_\infty \\ F \to \frac{1}{r^2}\gamma_m {\rm vol}_{S^2} \end{array} \right\} \Big/ \mathcal{G} \ \simeq \mathbb{R}^3 \times \frac{\mathbb{R} \times \tilde{\mathcal{M}}}{\mathbb{Z}}$$

- admits $T_{\rm ad} \oplus SO(3) \oplus SU(2)_R$ action:
  - $G_{\rm ad} \supset T_{\rm ad} \leftrightarrow$ global g.t.'s; hyperkähler isometries
  - $SO(3)$ isometry inherited from spatial rotations
  - $SU(2)_R$ action on $T\mathcal{M}$: generators = cmplx structures $J^a_{(A)b}$

Semiclassical vanilla BPS states III

Step 2: motion on moduli space

low energy approx: give time dep. to collective coords $z^a(t), \lambda^a(t)$...

Step 3: Quantize

$\hat{\lambda}^a = \gamma^a, \quad \hat{p}_{z^a} = -iD_a$...

Step 4: s.c. BPS states: $\not{D}_G^{\mathcal{M}} \Psi_{\rm BPS} = 0$

- G a tri-holomorphic vector field on $\mathcal{M}$; originates from Y
- $\boxed{\oplus_{\gamma_e} \mathcal{H}^{BPS}_{u;(\gamma_m,\gamma_e)} \simeq \ker_{``L^2"} \not{D}_{G(Y_\infty)}^{\mathcal{M}(\gamma_m,X_\infty)}}$
- $\mathfrak{t} \oplus \mathfrak{so}(3) \oplus \mathfrak{su}(2)_R$ action lifts to kernel;
  - $\mathfrak{t} \oplus \mathfrak{so}(3)$ acts by Lie derivative
  - $\mathfrak{su}(2)_R$ generators $= \hat{\mathcal{J}}_{(A)} = \frac{1}{8} J_{(A)ab}[\gamma^a, \gamma^b]$

Line operators of type ζ, L_ζ

- defect at $\mathbf{x}' = 0 \in \mathbb{R}^3$, preserving
  - $\mathfrak{so}(3) \oplus \mathfrak{su}(2)_R$ bosonic symmetry
  - SUSY's $\mathcal{R}^A_\alpha = \zeta^{1/2} Q^A_\alpha + \zeta^{-1/2} (\sigma^0)_{\alpha\dot{\beta}} \bar{Q}^{\dot{\beta}A}$, where $|\zeta| = 1$
- a modification of the theory in the UV; $\mathcal{H} \to \mathcal{H}_L$

Line operator charges, $L_\zeta(P, Q)$

- $(P, Q) \in (\Lambda_{wt}(G)^* \times \Lambda_{wt}(G))/\mathcal{W}$
- $L_\zeta(0, Q)$: Wilson operator in irrep $R(Q)$
- $L_\zeta(P, 0)$: 't Hooft operator, defined by boundary conditions

$$F \to \frac{P}{2} \mathrm{vol}_{S^2} + \mathrm{reg}, \quad i\zeta^{-1}\varphi = \frac{P}{2r} + \mathrm{reg}, \quad r = |\mathbf{x} - \mathbf{x}'| \qquad (b)$$

(Semiclassical) Framed BPS states

- a state $\in \mathcal{H}_L$, preserving $\mathcal{R}^A_\alpha$'s $\Rightarrow$ saturates bound $E \geq -\mathrm{Re}(\zeta^{-1} Z)$
- space of such states: $\mathcal{H}^{\mathrm{BPS}}_{L;u} = \oplus_{\gamma \in \Gamma_L} \mathcal{H}^{\mathrm{BPS}}_{L;u;\gamma}$

Semiclassical description:

- classical BPS equations same as before, with $i\zeta^{-1}\varphi = X + iY$
- $\overline{\mathcal{M}}(P; \gamma_m, X_\infty) = \left\{ (A, X) \;\middle|\; (p)\ ,\ (b)\ ,\ \begin{array}{l} X \to X_\infty \\ F \to \frac{1}{r^2}\gamma_m \mathrm{vol}_{S^2} \end{array} \right\} \Big/ \mathcal{G}$
- admits $T_{\mathrm{ad}} \oplus SO(3) \oplus SU(2)_R$ action
- c.c. expansion of low energy action, quantization proceeds identically
- framed BPS states: $\oplus_{\gamma_e} \mathcal{H}^{\mathrm{BPS}}_{L;u;(\gamma_m,\gamma_e)} \simeq \ker_{L^2} \not{D}^{\overline{\mathcal{M}}(P;\gamma_m;X_\infty)}_{G(Y_\infty)}$

Application: A vanishing theorem

Vanilla case: There is a special locus on the Coulomb branch where we know the spectrum exactly. On this locus if γ_m not a simple co-root, implying that $\dim \tilde{\mathcal{M}} > 0$, then the kernel vanishes.

Argument:

- special locus $\{Y_\infty = 0\} \bigcap \mathcal{R}^{\mathrm{s.c.}}$
- $Y_\infty = 0 \;\Rightarrow\; G = 0$, so $\not{D}^{\mathcal{M}}_{G} \to \not{D}^{\mathcal{M}}$
- $\not{D}^{\tilde{\mathcal{M}}} \tilde{\Psi} = 0 \;\Rightarrow\; \left(\not{D}^{\tilde{\mathcal{M}}}\right)^2 \tilde{\Psi} = 0 \xRightarrow{\tilde{\mathcal{M}}\ \mathrm{h.k.}} \Delta \tilde{\Psi} = 0$
- $0 = \int_{\tilde{\mathcal{M}}} \bar{\tilde{\Psi}} \Delta \tilde{\Psi} = -\int_{\tilde{\mathcal{M}}} ||\nabla \tilde{\Psi}||^2 \;\Rightarrow\; \nabla \tilde{\Psi} = 0 \;\Rightarrow\; \tilde{\Psi} = 0$

Similarly in the framed case when $\dim \underline{\overline{\mathcal{M}}} > 0$

Example[2] $G = SU(3)$, $\gamma_m = (1,1)$

relative moduli space is Taub-NUT

- $ds^2_{\tilde{\mathcal{M}}} = \mu\ ds^2_{\mathrm{TN}}(\ell)$ and $G|_{\tilde{\mathcal{M}}} = \mu^{-1}\ell^{-2}P\frac{\partial}{\partial\psi}$
- where $\mu = \frac{2\pi}{g_0{}^2}\frac{x_1 x_2}{x_1+x_2}$, $\ell = \frac{\pi}{g_0{}^2\mu}$, $P = \frac{\pi}{g_0{}^2}\left(\frac{y_1}{x_1} - \frac{y_2}{x_2}\right)$
- with $x_I = \langle \alpha_I, X_\infty \rangle$ and $y_I = \langle \alpha_I, Y_\infty \rangle$

zero modes of $\not{D}_G$

- $\tilde{\Psi} = (\lambda^A, 0)^T$, with $\lambda^A = \varphi_1 o^A + \varphi_2 \iota^A$

  $\varphi_1 = \frac{r^{j+\frac{1}{2}}}{r+\ell} e^{-(P-j-\frac{1}{2})r/\ell} D^j_{-j,m}(\psi,\theta,\phi) e^{-i\psi/2}$, $\varphi_2 = 0$, $(P > \frac{1}{2})$
- $j \in \{0, \frac{1}{2}, 1, \ldots, [|P| - \frac{1}{2}]\}$, and $m \in \{-j, -j+1, \ldots, j\}$

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[2] Pope (1978); Lee, Weinberg and Yi (1997); Gauntlett, Kim, Park and Yi (1999)

Example $G = SU(3)$, $\gamma_m = (1,1)$

Checks

- walls consistent with $\mathrm{Im}\left(Z_{\gamma_1}\bar{Z}_{\gamma_2}\right) = 0$ in s.c. regime, where $\gamma_1 = (1,0;n_1,0)$, $\gamma_2 = (0,1;0,n_2)$ and we identify $j = |n_1 - n_2|$
- for j large, and near wall ($0 < P - j - \frac{1}{2} \ll 1$), wavefunction is sharply peaked at $r = 2R_{\mathrm{bnd}}$, where R_{bnd} consistent[3] with Denef's bound state radius[4]:

$$R_{\mathrm{bnd}} = \frac{1}{2}|\langle\gamma_1,\gamma_2\rangle|\frac{|Z_{\gamma_1} + Z_{\gamma_2}|}{\mathrm{Im}\left(Z_{\gamma_1}\bar{Z}_{\gamma_2}\right)}$$

[3] Up to a pesky factor of 2 we haven't yet tracked down.
[4] Denef (2002), Denef and Moore (2007)

Protected Spin Characters[5]

vanilla protected spin character

- Wigner $\Rightarrow$ $\begin{array}{lll} \text{long rep:} & \rho_{hh}\otimes\rho_{hh}\otimes\mathfrak{h}, & E > |Z_{\gamma}(u)| \text{ generically} \\ \textbf{short rep}: & \rho_{hh}\otimes\mathfrak{h}, & E = |Z_{\gamma}(u)| \end{array}$
    - $\rho_{hh} = (0,\frac{1}{2})\oplus(\frac{1}{2},0)$ "half-hypermultiplet," and
    - $\mathfrak{h}$ an arbitrary $\mathfrak{so}(3)\oplus\mathfrak{su}(2)_R$ rep.
- protected spin character counts **rigid** BPS states:

  $\Omega(\gamma, u; y) := \mathrm{Tr}_{\tilde{\mathcal{H}}^{\mathrm{BPS}}_{u,\gamma}}(-y)^{2I_3}y^{2J_3}$, where $\mathcal{H}^{\mathrm{BPS}}_{u,\gamma} = \rho_{hh}\otimes\tilde{\mathcal{H}}^{\mathrm{BPS}}_{u,\gamma}$

framed protected spin character

- $\begin{array}{lll} \text{long rep:} & \rho_{hh}\otimes\mathfrak{h}_L, & E > -\mathrm{Re}(\zeta^{-1}Z_{\gamma}(u)) \text{ generically} \\ \text{short rep:} & \mathfrak{h}_L, & E = -\mathrm{Re}(\zeta^{-1}Z_{\gamma}(u)) \end{array}$
- $\underline{\overline{\Omega}}(L_{\zeta}, \gamma, u; y) := \mathrm{Tr}_{\mathcal{H}^{\mathrm{BPS}}_{L,u,\gamma}}(-y)^{2I_3}y^{2J_3}$

[5] GMN 2010

Positivity Conjectures

- presence of $\mathfrak{su}(2)_R$ *essential* for rigidity of $\Omega, \underline{\overline{\Omega}}$
- interesting question: What $\mathfrak{su}(2)_R$ reps appear in $\Omega, \underline{\overline{\Omega}}$?
- surprising empirical answer: only the trivial one
  note: $\Rightarrow \ \Omega(y) = \mathrm{Tr}_{\mathcal{H}^{\mathrm{BPS}}}\, y^{2J_3} \ \Rightarrow \ \Omega(1) = \dim \mathcal{H}^{\mathrm{BPS}}$

Positivity Conjectures:

PC1: short reps in $\tilde{\mathcal{H}}^{\mathrm{BPS}}_{u,}, \mathcal{H}^{\mathrm{BPS}}_{L,u}$ always have $I_3 = 0$

PC2: I_3 eigenvalues all (half-)integral
note: still $\Rightarrow \Omega(y=1) = \dim \mathcal{H}^{BPS}$

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Positivity Conjectures:

PC1: short reps in $\tilde{\mathcal{H}}^{\mathrm{BPS}}_{u,}, \mathcal{H}^{\mathrm{BPS}}_{L,u}$ always have $I_3 = 0$ "no exotics"

$\Downarrow$

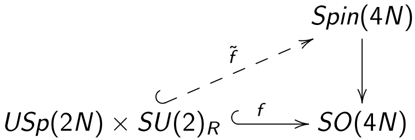
PC2: I_3 eigenvalues all (half-)integral "(half-)integral R-spin"

note: still $\Rightarrow \Omega(y=1) = \dim \mathcal{H}^{BPS}$

PC's and the Kernel of $\not{D}_G$ I

$SU(2)_R$ as the commutant of holonomy

- recall $SU(2)_R$ acts on $T\mathcal{M}$ with cmplx structures as generators
- lifts to action on $S_{\mathrm{D}}(\mathcal{M})$ via $\mathcal{J}_{(A)} = \frac{1}{8} J_{(A)ab}[\hat{\lambda}^a, \hat{\lambda}^b]$
- $\mathrm{Hol}(\nabla_{\mathcal{M}}) \simeq USp(2N) \hookrightarrow SO(4N)$ preserves cmplx structures



- features:
  1. f canonically defined from $T_X\mathcal{M} \simeq \mathbb{H}^N$, $USp(2N) \simeq U(N, \mathbb{H})$
  2. $\tilde{f}(1, -1) = \omega$, volume form in $Spin(4N)$
  3. $\tilde{f}^*(\rho_{\mathrm{Dirac}}) = \oplus_{k=1}^{N+1} R_k \otimes S_k$, where S_k the k-dim $SU(2)$ rep

PC's and the Kernel of $\not{D}_G$ II

Consequences for the Dirac operator

- $$\begin{aligned} Spin(4N) &\rightarrow USp(2N) \times SU(2)_R \\ \ker \not{D}_G &\rightarrow \bigoplus_{k=1}^{N+1} n_k R_k \otimes S_k \end{aligned}$$
- **PC1** $\Rightarrow n_k = 0, \forall k > 1$; $\ker \not{D}_G$ is irrep of $USp(2N)$
- **PC2** $\Rightarrow \ker \not{D}_G$ is chiral

our previous example

- consistent with **PC1**

Conclusions

- We reviewed the semiclassical construction of the space of one-particle BPS states in $\mathcal{N} = 2$ theories as the kernel of certain twisted Dirac operators on the moduli space of classical (singular) monopole solutions.

- We described the action of $\mathfrak{so}(3) \oplus \mathfrak{su}(2)_R$ on the kernels.

- We translated the Positivity Conjectures of GMN into statements about the kernels. In particular, the "no exotics" and "(half-)integral R-spins only" conjectures imply that the kernels are chiral.

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Thanks!