6j symbols for the modular double, quantum hyperbolic geometry, and supersymmetric gauge theories

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Introduction

- fusion kernel/b-6j symbol;
- relation to 3d hyperbolic geometry;
- application to SuSy theories;

Fusion kernel I

$$\langle V_{\alpha_4}(z_4,\bar z_4)V_{\alpha_3}(z_3,\bar z_3)V_{\alpha_2}(z_2,\bar z_2)V_{\alpha_1}(z_1,\bar z_1)\rangle =$$

$$= \int_{Q/2+i\mathbb{R}} d\alpha_s\, C(\alpha_4,\alpha_3,\alpha_s)C(Q-\alpha_s,\alpha_2,\alpha_1)\mathcal{F}^{(s)}_{\alpha_s}(A|Z)\mathcal{F}^{(s)}_{\alpha_s}(A|\bar Z)$$

$$= \int_{Q/2+i\mathbb{R}} d\alpha_t\, C(\alpha_4,\alpha_t,\alpha_1)C(Q-\alpha_t,\alpha_3,\alpha_2)\mathcal{F}^{(t)}_{\alpha_t}(A|Z)\mathcal{F}^{(t)}_{\alpha_t}(A|\bar Z)$$

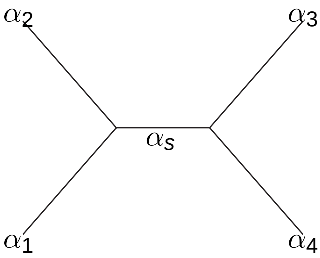
where $A=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$, $Z=(z_1,z_2,z_3,z_4)$, and DOZZ 3-point function is

$$C(\alpha_1,\alpha_2,\alpha_3) = (\pi\mu\gamma(b^2)b^{2-2b^2})^{\frac{1}{b}(Q-\alpha_1-\alpha_2-\alpha_3)}$$

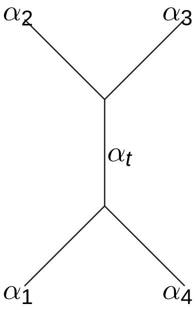
$$\times \frac{\Upsilon_0\Upsilon(2\alpha_1)\Upsilon(2\alpha_2)\Upsilon(2\alpha_3)}{\Upsilon(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon(\alpha_1+\alpha_3-\alpha_2)\Upsilon(\alpha_1+\alpha_2-\alpha_3)\Upsilon(\alpha_2+\alpha_3-\alpha_1)},$$

$\Upsilon(x)=\Gamma_b(x)\Gamma_b(Q-x)$ and $Q=b+b^{-1}$.

Fusion kernel II



$$= \int d\alpha_t F_{\alpha_s \alpha_t}\left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix}\right]$$



Fusion kernel III

$$\mathcal{F}^{(s)}_{\alpha_s}(A|Z) = \int_{Q/2+i\mathbb{R}} d\mu(\alpha_t)\, F_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right] \mathcal{F}^{(t)}_{\alpha_t}(A|Z)\,,$$ Teschner '01

where F-kernel is Ponsot, Teschner '99; Teschner '01

$$F_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right] = \frac{N(\alpha_s,\alpha_2,\alpha_1)N(\alpha_4,\alpha_3,\alpha_s)}{N(\alpha_t,\alpha_3,\alpha_2)N(\alpha_4,\alpha_t,\alpha_1)}\left\{\begin{smallmatrix}\alpha_1 & \alpha_2 & | & \alpha_s\\ \alpha_3 & \alpha_4 & | & \alpha_t\end{smallmatrix}\right\}_{\mathrm{PT}}\,,$$

where

$$N(\alpha_3,\alpha_2,\alpha_1) = \frac{\Gamma_b(2Q-2\alpha_3)\Gamma_b(2\alpha_2)\Gamma_b(2\alpha_1)}{\Gamma_b(2Q-\alpha_1-\alpha_2-\alpha_3)\Gamma_b(Q-\alpha_1-\alpha_2+\alpha_3)} \times \frac{1}{\Gamma_b(\alpha_1+\alpha_3-\alpha_2)\Gamma_b(\alpha_2+\alpha_3-\alpha_1)}.$$

Special functions I

The function $\Gamma_b(x)$ is closely related to the double Barnes function

$$\log\Gamma_b(x) \;=\; \int_0^\infty \frac{dt}{t}\left(\frac{e^{-xt}-e^{-Qt/2}}{(1-e^{-bt})(1-e^{-t/b})}-\frac{(Q-2x)^2}{8e^t}-\frac{Q-2x}{t}\right) \quad .$$

Important properties of $\Gamma_b(x)$ are

functional equation $\quad \Gamma_b(x+b)=\sqrt{2\pi}\,b^{bx-\frac{1}{2}}\Gamma^{-1}(bx)\Gamma_b(x).$

analyticity $\quad \Gamma_b(x)$ is meromorphic,

poles $\quad x=-nb-mb^{-1}, n, m\in\mathbb{Z}^{\geq 0}.$

Special functions II

Double Sine functions

$$S_b(x) = \frac{\Gamma_b(x)}{\Gamma_b(Q-x)},$$

satisfies

self–duality	$S_b(x) = S_{b^{-1}}(x)\,,$
functional equation	$S_b(x+b^{\pm 1}) = 2\,\sin(\pi b^{\pm 1}x)\,S_b(x)\,,$
reflection property	$S_b(x)\,S_b(Q-x) = 1\,.$

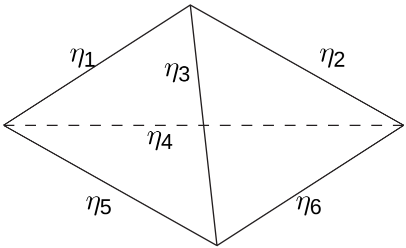
Fusion kernel IV

$$\left\{\begin{smallmatrix}\alpha_1 & \alpha_2 \\ \alpha_3 & \bar{\alpha}_4\end{smallmatrix}\middle|\begin{smallmatrix}\alpha_s \\ \alpha_t\end{smallmatrix}\right\}_{\mathrm{PT}} = \frac{S_b(\alpha_2+\alpha_s-\alpha_1)S_b(\alpha_t+\alpha_1-\alpha_4)}{S_b(\alpha_2+\alpha_t-\alpha_3)S_b(\alpha_s+\alpha_3-\alpha_4)}$$
$$\times \int_{\mathcal{C}} du\; S_b(-\alpha_2 \pm (\alpha_1 - Q/2) + u) S_b(-\alpha_4 \pm (\alpha_3 - Q/2) + u)$$
$$\times S_b(\alpha_2 + \alpha_4 \pm (\alpha_t - Q/2) - u) S_b(Q \pm (\alpha_s - Q/2) - u)\,.$$

Here, $S_b(\alpha \pm u) := S_b(\alpha+u)S_b(\alpha-u)$. A contour $\mathcal{C}$ approaches $Q + \mathrm{i}\mathbb{R}$ near infinity, and passes the real axis in $(Q/2, Q)$.

Volume of non-ideal tetrahedron I

A non-ideal tetrahedron defined by 6 dihedral angles $\eta_i, i = 1, \ldots, 6$



Volume of non-ideal tetrahedron II

The volume Murakami Yano '05

$$\mathrm{Vol}(\underline{A}) = \frac{1}{2}\,\mathrm{Im}\big[U(u_+,\underline{A}) + \Delta(\underline{A})\big] = -\frac{1}{2}\,\mathrm{Im}\big[U(u_-,\underline{A}) + \Delta(\underline{A})\big]\,,$$

where $A_k = e^{\mathrm{i}\eta_k}$ and $u_\pm$ are the two roots

$$\frac{dU(u,\underline{A})}{du} = -\frac{2\pi \mathrm{i}}{u}\,.$$

Here

$$\begin{aligned} U(u,\underline{A}) = {} & \mathrm{Li}_2(u) + \mathrm{Li}_2(A_{st13}u) + \mathrm{Li}_2(A_{st24}u) + \mathrm{Li}_2(A_{1234}u) \\ & - \mathrm{Li}_2(-A_{12s}u) - \mathrm{Li}_2(-A_{s34}u) - \mathrm{Li}_2(-A_{4t1}u) - \mathrm{Li}_2(-A_{32t}u)\,, \end{aligned}$$

where $A_{ijk} := A_iA_jA_k$, $A_{ijkl} := A_iA_jA_kA_l$.

New integral representation

A new integral representation Teschner, GV '12

$$\left\{ \begin{smallmatrix} \alpha_1 & \alpha_2 & \alpha_s \\ \alpha_3 & \alpha_4 & \alpha_t \end{smallmatrix} \right\}_b = \Delta(\alpha_s,\alpha_2,\alpha_1)\Delta(\alpha_4,\alpha_3,\alpha_s)\Delta(\alpha_t,\alpha_3,\alpha_2)\Delta(\alpha_4,\alpha_t,\alpha_1)$$
$$\times \int_{\mathcal{C}} du\, S_b(u-\alpha_{12s})S_b(u-\alpha_{s34})S_b(u-\alpha_{23t})S_b(u-\alpha_{1t4})$$
$$\times S_b(\alpha_{1234}-u)S_b(\alpha_{st13}-u)S_b(\alpha_{st24}-u)S_b(2Q-u)\,,$$

where

- $\alpha_{ijk} = \alpha_i + \alpha_j + \alpha_k$, $\alpha_{ijkl} = \alpha_i + \alpha_j + \alpha_k + \alpha_l$.
-

$$\Delta(\alpha_3,\alpha_2,\alpha_1) = \left(\frac{S_b(\alpha_1+\alpha_2+\alpha_s-Q)}{S_b(\alpha_1+\alpha_2-\alpha_s)S_b(\alpha_1+\alpha_s-\alpha_2)S_b(\alpha_2+\alpha_s-\alpha_1)} \right)^{\frac{1}{2}}.$$

- a contour $\mathcal{C}$ which approaches $2Q + i\mathbb{R}$ near infinity, and passes the real axis in the interval $(3Q/2, 2Q)$.

Semiclassical limit

Let us reparameterize variables

$$e^{-2\pi \mathrm{i} b\alpha_k + \pi \mathrm{i}} \equiv A_k\,, \qquad k \in \{1,2,3,4,s,t\}\,.$$

Introducing $v := 2\pi b(u - Q/2)$ we get an integral of the form

$$I = D(\underline{\alpha}) \int_{\mathcal{C}-Q/2} \frac{dv}{2\pi b}\, \mathcal{J}(a,b;v)$$

whose integrand $\mathcal{J}(a,b;v)$ has quasi-classical asymptotics

$$\mathcal{J}(a,b;v) \;=\; \exp\left(\frac{1}{2\pi b^2}\, U(e^{\mathrm{i}v},\underline{A})\right)\left(1+\mathcal{O}(b^2)\right),$$

since $S_b(x)$ for $b \to 0$ is given as

$$S_b\left(\frac{\nu}{2\pi b}\right) \;=\; e^{-\frac{\mathrm{i}}{2\pi b^2}\left(\frac{1}{4}\nu^2 - \frac{\pi}{2}\nu + \frac{1}{6}\pi^2\right)} \exp\left(-\frac{1}{2\pi \mathrm{i} b^2}\mathrm{Li}_2(e^{\mathrm{i}\nu})\right)\left(1+\mathcal{O}(b^2)\right).$$

b-6j symbol, identities

b-6j symbol satisfies

$$\int_{Q/2+i\mathbb{R}^+} d\mu(\delta_1) \left\{ \begin{matrix} \alpha_1 & \alpha_2 & \beta_1 \\ \alpha_3 & \beta_2 & \delta_1 \end{matrix} \right\}_b \left\{ \begin{matrix} \alpha_1 & \delta_1 & \beta_2 \\ \alpha_4 & \alpha_5 & \gamma_2 \end{matrix} \right\}_b \left\{ \begin{matrix} \alpha_2 & \alpha_3 & \delta_1 \\ \alpha_4 & \gamma_2 & \gamma_1 \end{matrix} \right\}_b$$

$$= \left\{ \begin{matrix} \beta_1 & \alpha_3 & \beta_2 \\ \alpha_4 & \alpha_5 & \gamma_1 \end{matrix} \right\}_b \left\{ \begin{matrix} \alpha_1 & \alpha_2 & \beta_1 \\ \gamma_1 & \alpha_5 & \gamma_2 \end{matrix} \right\}_b ,$$

$$\int_{Q/2+i\mathbb{R}^+} d\mu(\alpha_s) \left\{ \begin{matrix} \alpha_1 & \alpha_2 & \alpha_s \\ \alpha_3 & \alpha_4 & \alpha_t \end{matrix} \right\}_b^* \left\{ \begin{matrix} \alpha_1 & \alpha_2 & \alpha_s \\ \alpha_3 & \alpha_4 & \alpha_t' \end{matrix} \right\}_b = (M(\alpha_t))^{-1} \delta(\alpha_t - \alpha_t') ,$$

which together with the semiclassical limit $b \to 0$ suggests Turaev-Viro type state-sum models.

3d gauge theories on duality walls I

Recently a nice generalization of AGT duality was proposed Drukker, Gaiotto, Gomis '10: one may consider two four-dimensional theories from class $\mathcal{S}$ (4d $\mathcal{N} = 2$ SYM theories) on the upper- and lower semispheres of S^4, respectively, coupled to a three-dimensional theory on the defect S^3 separating the two semi-spheres.

$$\int_{(Q/2+i\mathbb{R})^2} d\alpha_s d\alpha_t \, (\mathcal{G}^{(s)}_{\alpha_s}(A|Z))^* \, G_{\alpha_s\alpha_t} \left[\ldots \right] \mathcal{G}^{(t)}_{\alpha_t}(A|Z') \, .$$

PF for 3d $\mathcal{N}=2$ SuSy theories

PF on S^3 Kapustin, Willett, Yaakov '10, Jafferis '10; Hama, Hosomichi, Lee '10 and PF on S_b^3 Hama, Hosomichi, Lee '11

$$Z(\underline{f}) = \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \prod_{j=1}^{\mathrm{rank}\, G} du_j \, J(\underline{u}) Z^{vec}(\underline{u}) \prod_I Z_{\Phi_I}^{chir}(\underline{f}, \underline{u}).$$

Here f_k are mass parameters of matter while u_j-variables – Weyl weights for the Cartan subalgebra of the gauge group G. For Chern-Simons theories one has $J(\underline{u}) = e^{-\pi \mathrm{i} k \sum_{j=1}^{\mathrm{rank}\, G} u_j^2}$, where k is the level of CS-term, and for SYM theories one has $J(\underline{u}) = e^{2\pi \mathrm{i} \lambda \sum_{j=1}^{\mathrm{rank}\, G} u_j}$, where λ is the Fayet-Illiopoulos term.

$$Z_{SU(2)}^{vec}(\underline{u}) = \frac{1}{S_b(\pm 2u)}.$$

3d gauge theories on duality walls, $N = 2^*$ I

Explicitly the above idea was checked only for $4d$ $\mathcal{N} = 2^*$ SYM theory where on the domain wall the so-called $T[SU(2)]$ theory lives. Explicit check of DGG idea Hosomichi, Lee, Park '10. $3d - 3d$ relation Terashima, Yamazaki '11; Dimofte, Gukov '11; Dimofte, Gaiotto, Gukov '11.
Here, $\mathcal{G}^{(s)}_{\alpha_s}(A|Z)$, $\mathcal{G}^{(t)}_{\alpha_t}(A|Z)$ are one-point CBs on a torus and $G_{\alpha_s\alpha_t}$ is the so-called S-kernel transformation Teschner '03

$$S = \frac{2^{3/2}}{S_b(m)} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{S_b(Q/4 - \mu + m/2 \pm z)}{S_b(3Q/4 - \mu - m/2 \pm z)} e^{4\pi \mathrm{i}\xi z} dz$$

3d gauge theories on duality walls, $N = 2^*$ II

There is an alternative interpretation in terms of 3*d* $\mathcal{N} = 2$ CS theory with $SU(2)$ gauge group and 4 quarks Spiridonov, GV '11; Teschner, GV '12

$$\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{S_b(Q/4 - \mu + m/2 \pm z)}{S_b(3Q/4 - \mu - m/2 \pm z)} e^{4\pi \mathrm{i} \xi z} dz = \frac{1}{2} e^{2\pi \mathrm{i}(\xi^2 - (\frac{Q}{4} + \frac{m}{2})^2 + \mu^2)}$$
$$\times S_b(Q/2 - m \pm 2\xi) \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{S_b(\frac{Q}{4} + \frac{m}{2} \pm \mu \pm \xi \pm y)}{S_b(\pm 2y)} e^{-2\pi \mathrm{i} y^2} dy \,.$$

3d gauge theories on duality walls, $N_f = 4$ I

Now we consider $4d$ $\mathcal{N} = 2$ SYM with $SU(2)$ gauge group and $N_f = 4$ hypermultiplets.

$$\int_{(Q/2+\mathrm{i}\mathbb{R})^2} d\alpha_s d\alpha_t \, (\mathcal{G}^{(s)}_{\alpha_s}(A|Z))^* \, G_{\alpha_s \alpha_t} \left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix} \right] \mathcal{G}^{(t)}_{\alpha_t}(A|Z') \, .$$

$\mathcal{G}^{(s)}_{\alpha_s}(A|Z)$, $\mathcal{G}^{(t)}_{\alpha_t}(A|Z)$ are four-point CBs on a sphere and $G_{\alpha_s \alpha_t}$ is the F-kernel.

3d gauge theories on duality walls, $N_f = 4$ II

The defined F-kernel/b-6j symbol is equal to Teschner, GV '12

$$\mathcal{A}_1 I\left(\begin{array}{ccc} \frac{Q-\alpha_t-\alpha_1-\alpha_4}{2}+\alpha_s & \frac{3Q-\alpha_t-\alpha_1-\alpha_4}{2}-\alpha_s & \frac{Q+\alpha_1-\alpha_4+\alpha_t}{2}-\alpha_3 \\ \frac{-Q-\alpha_1+\alpha_4+\alpha_t}{2}+\alpha_2 & \frac{Q-\alpha_1+\alpha_4+\alpha_t}{2}-\alpha_2 & \frac{-Q+\alpha_1-\alpha_4+\alpha_t}{2}+\alpha_3 \end{array}\right),$$

where

$$I\left(\begin{array}{ccc} \mu_1 & \mu_2 & \mu_3 \\ \mu_4 & \mu_5 & \mu_6 \end{array}\right) = \frac{1}{2}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\prod_{i=1}^{6} S_b(\mu_i \pm \alpha)}{S_b(\pm 2\alpha)} d\alpha.$$

which is PF for $3d$ $\mathcal{N}=2$ SYM theory with $SU(2)$ gauge group and 6 quarks.

Conclusion

- we found a new integral representation for b-6j symbol which
  1. defines a natural normalization from Liouville and Teichmüller theory (quantization of the Fenchel-Nielsen coordinates Teschner's talk);
  2. in a semiclassical limit reproduces the volume of a non-ideal tetrahedron;
- $3d$ $\mathcal{N} = 2$ non-abelian theory arising on a domain wall of $4d$ $\mathcal{N} = 2$ SYM theory with $SU(2)$ gauge group and $N_f = 4$ was identified;
- we construct $3d - 3d$ starting from non-ideal tetrahedrons.