$(2,0)$ theory, cigars, and AGT

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String-Math 2012, Bonn

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Relates two kinds of theories:

$$\boxed{\text{4d } \mathcal{N} = 2 \text{ gauge theory} \iff \text{2d Liouville/Toda CFT}}$$

Mappings between various objects. In particular,

$$\text{partition function} = \text{correlator}.$$

The CFT symmetry seen in 4d:

$$\mathcal{W} \circlearrowright H^{\bullet}(\mathcal{M}),$$

where $\mathcal{M}$ is the instanton moduli space.

6d derivation

Start with the $(2,0)$ theory of $\mathfrak{g} = A, D, E$ on $M_4 \times C_2$.

Twist the theory to preserve some supersymmetries.

The twisted theory has a supercharge Q with $Q^2 = 0$ (up to a “gauge transformation”).

It is topological along M and holomorphic along C.

Roughly speaking, we have performed the topological twist $\mathcal{N} = 2$ SUSY along M and the holomorphic twist of $\mathcal{N} = (0,2)$ SUSY along C.

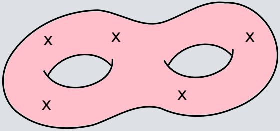
Q-cohomology $\supset$ chiral (or vertex) algebra on C.

The latter is similar to the chiral algebras of 2d $(0,2)$ theories, related to chiral differential operators, geometric Langlands, etc.

6d derivation

Q-invariant quantities are protected.

Place Q-invariant codim-2 defects at points on C:



Then we get

$$\begin{tikzcd}
& \langle\cdots\rangle_{\text{6d}} \arrow[dl, "C \to 0"'] \arrow[dr, "M \to 0"] & \\
Z = \langle 1 \rangle_{\text{4d}} & & \langle\cdots\rangle_{\text{2d}}
\end{tikzcd}$$

6d derivation

Suppose $\mathcal{W} \subset H_Q(\text{6d})$. Then $\mathcal{W} \circlearrowright H_Q(\text{6d})$ naturally.

The action will descend to 4d:

$$\begin{tikzcd}
 & \mathcal{W} \subset H_Q(\text{6d}) \arrow[dl, "C \to 0"'] \arrow[dr, "M \to 0"] & \\
\mathcal{W} \circlearrowright H_Q(\text{4d}) & & \mathcal{W} \subset H_Q(\text{2d})
\end{tikzcd}$$

So we expect $\mathcal{W} \circlearrowright H_T^\bullet(\mathcal{M}) = H_Q(\text{4d})$.

This has been proven in some cases. [Maulik–Okounkov, Schiffmann–Vasserot]

6d derivation

We can derive the AGT correspondence if

$$\begin{tikzcd}
 & \text{$(2,0)$ theory on } \mathbb{R}^4 \times C \arrow[dl, "C \to 0"'] \arrow[dr, "\text{“}\mathbb{R}^4 \to 0\text{”}"] & \\
\mathcal{N} = 2 \text{ theory on } \mathbb{R}^4 & & \text{Toda theory on } C
\end{tikzcd}$$

The $C \to 0$ part is understood. [Gaiotto, Gaiotto–Moore–Neitzke]

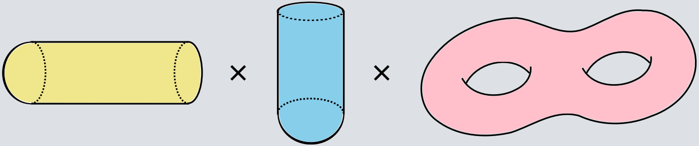
We want to understand the “$\mathbb{R}^4 \to 0$” part.

This “compactification” is done by the “Ω-deformation,” which confines the dynamics near $0 \in \mathbb{R}^4$.

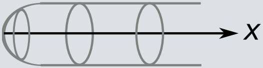
Down to 4d

We don't know much about the 6d theory, so let's compactify to lower dimensions.

Bend $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ into the product of cigars $D_1 \times D_2$:



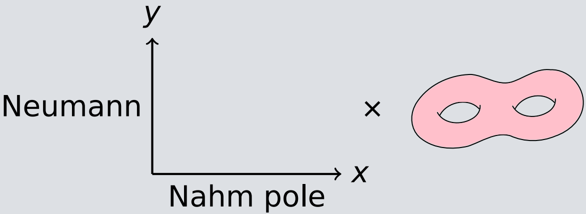
And make the radii of the cigars very small. Each reduces to a half-line $\mathbb{R}_{\geq 0} = [0, \infty]$:



We get $\mathcal{N} = 4$ super Yang–Mills on $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times C$.

Down to 4d

The boundaries are endowed with half-BPS boundary conditions. (cf. Witten's approach to Khovanov homology)



On cigars, the effect of the Ω-deformation is localized near the tips [Nekrasov–Witten].

Turning it on induces Q-invariant boundary couplings.

Down to 4d

The twisted theory has a Q-invariant $\mathfrak{g}_{\mathbb{C}}$-connection

$$\mathcal{A} = (A_x + iY_x)dx + (A_y + iX_y)dy + X_z dz + A_{\bar{z}} d\bar{z},$$

where Xs and Ys are scalars in the untwisted theory.

Here we have the topological-holomorphic twist of Kapustin.

A natural candidate for the boundary coupling at $x = 0$ (Neumann) is a Chern-Simons term

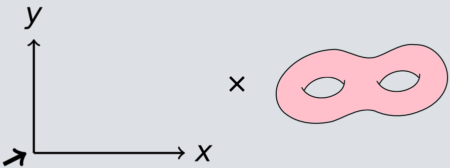
$$\frac{k}{4\pi i} \int_{x=0} \mathrm{Tr}\left(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right).$$

It satisfies criteria derived from the quasi-topological invariance of the 6d theory. Other possibilities don't seem very interesting.

The other boundary coupling is the S-dual.

The 2d theory

But the boundary has a boundary!



The usual statement is that Chern-Simons theory on M_3 gives a WZW model on ∂M_3.

This is because the gauge degrees of freedom turn into dynamical ones on the boundary.

The affine currents are

$$J = k\mathcal{A}_z = kX_z.$$

The 2d theory

Here we have something different.

At $y = 0$, it has the Nahm pole

$$J = \frac{kt_+}{y} + \text{less singular},$$

with t_+ the raising operator for a principal $\mathfrak{sl}_2 \subset \mathfrak{g}_\mathbb{C}$.

After a singular gauge transformation, this reads

$$J = kt_+ + \sum_{t_a \in \mathfrak{g}_0 \oplus \mathfrak{g}_-} J^a t_a,$$

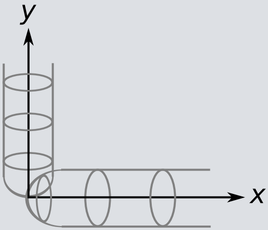
which implements quantum Drinfeld–Sokolov reduction

$$\widehat{\mathfrak{g}} \longrightarrow \mathcal{W}(\mathfrak{g}).$$

The 2d theory

The 2d theory is (the holomorphic half of) Toda theory.

As expected, it lives at the origin of $\mathbb{R}^4$:



For $\mathfrak{g} = A_{N-1}$, the W-algebra is $\mathcal{W}_N$.

Virasoro $\subset \mathcal{W}_N$ and $c \sim \# N^3$ for $N \gg 1$.

c exhibits the N^3 scaling behavior of the entropy of N M5-branes, reflecting the 6d origin of the 2d degrees of freedom.

Seiberg–Witten curve

Semiclassically (level $k \gg 1$),

$$\det(x - J) = x^N - \sum W_i x^{N-i}.$$

The SW curve of the $\mathcal{N} = 2$ theory is given by

$$\langle \det(x - X_z) \cdots \rangle = \left(x^N + \sum u_i(z) x^{N-i} \right) \langle \cdots \rangle = 0$$

in the undeformed limit.

Since $J = kX_z$, we have

$$\boxed{\langle W_i \cdots \rangle \sim u_i \langle \cdots \rangle}$$

[Alday–Gaiotto–Tachikawa]

Concluding remarks

We argued that the $(2,0)$ theory “compactified on the Ω-background” is Toda theory.

Our argument

- works for general $\epsilon_1, \epsilon_2 \in \mathbb{C}$.
- can incorporate a surface operator, by placing a codim-2 defect at the tip of D_1. This changes the $\mathfrak{sl}_2$ embedding and the resulting $\mathcal{W}$, explaining the conjecture of Braverman et al. and Wyllard.
- can deal with the non-ADE case, by including outer-automorphism twists on C. [Tachikawa]

Concluding remarks

Future work:

- Study the case $M = \mathbb{C}^2/\mathbb{Z}_k$. One should find para-Toda theory. [Belavin–Feigin, Nishioka–Tachikawa]
- Connect to Nekrasov–Shatashvili. The connection is more or less clear; $\mathcal{W}_\infty(\mathfrak{g})$ quantizes the Hitchin hamiltonians, while the SUSY configurations of $\mathcal{N} = 4$ SYM form the Hitchin moduli space.
- Connect to geometric Langlands.
- Carry out the compactification. We can lift the Ω-deformation to M-theory. [Hellerman–Orlando–Reffert]