

AGT and the topological string

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AGT vs. the topological string

- AGT is a relation between CFT and $\mathcal{N} = 2$ gauge theory in an Ω background, (in its original form) exact in the ϵ -parameters, but in an instanton expansion in the gauge coupling.
- Topological string techniques (the holomorphic anomaly) yield results that are exact in the gauge coupling, but in an ϵ expansion.

What can we learn by comparing the CFT and the holomorphic anomaly approach?

Work in progress with Jan Troost.

Outline

- 1 Z_{top} and how to compute it via the holomorphic anomaly
- 2 Review of AGT
- 3 AGT and topological string

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The protagonist

$$Z_{top} = \exp \left[\sum_{n,g} F^{(n,g)} g_s^{2g-2} s^n \right]$$

Topological string

- $F^{(0,g)}(t)$ counts maps: $\Sigma_g \rightarrow X$.
- $F^{(n,g)}(t)$, $n > 0$ does not have a topological string interpretation yet.
- $g_s^2 = \epsilon_1 \epsilon_2$, $s = \epsilon_1 + \epsilon_2$.

The protagonist

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Gauge theory

- $F^{(0,0)}(a)$ is the prepotential: governs special geometry of $\mathcal{N} = 2$ theory, e.g. $\tau_{IR} = \frac{d^2 F^{(0,0)}}{da^2}$.
- $F^{(0,g)}(a)$, $g > 0$ is graviphoton-curvature coupling when theory is coupled to a gravitational background:

$$F^{(0,g)} R_+^2 F_{gp}^{2g-2}$$

- $F^{(n,g)}(a)$, $n > 0$: field theory interpretation not yet settled.

Instanton calculus

Localization via rotations in spacetime with ϵ_1, ϵ_2 equivariant parameters computes Z_{inst} ($Z_{top} = Z_{pert}Z_{inst}$),

$$\begin{aligned}
 Z_{inst}(a, \epsilon_1, \epsilon_2) &= \sum_{Y_1, Y_2} q^{|Y_1|+|Y_2|} \\
 &\times \prod_{(i,j) \in Y_1} \frac{\prod_{k=1}^{N_f} (a + \epsilon_1(i-1) + \epsilon_2(j-1) + m_k)}{E_{i,j}^{Y_1, Y_1}(0)(\epsilon - E_{i,j}^{Y_1, Y_1}(0))E_{i,j}^{Y_1, Y_2}(2a)(\epsilon - E_{i,j}^{Y_1, Y_2}(2a))} \\
 &\times \prod_{(i,j) \in Y_2} \frac{\prod_{k=1}^{N_f} (-a + \epsilon_1(i-1) + \epsilon_2(j-1) + m_k)}{E_{i,j}^{Y_2, Y_2}(0)(\epsilon - E_{i,j}^{Y_2, Y_2}(0))E_{i,j}^{Y_2, Y_1}(-2a)(\epsilon - E_{i,j}^{Y_2, Y_1}(-2a))},
 \end{aligned}$$

where

$$E_{i,j}^{Y_1, Y_2}(a) \equiv a + \epsilon_1(Y_{1,j}^T - i + 1) - \epsilon_2(Y_{2,i} - j).$$

The holomorphic anomaly recursion

$F^{(0,1)}, F^{(1,0)}$ determined via geometry

$$F^{(0,1)} = -\frac{1}{2} \log(G_{u\bar{u}} |\Delta|^{\frac{1}{3}}), \quad F^{(1,0)} = \frac{1}{24} \log(\Delta).$$

$F^{(n,g)}$ for $n + g > 1$ determined by recursion (supplemented by boundary condition at conifold points),

$$\bar{\partial}_i F^{(n,g)} = \frac{1}{2} \bar{C}_i^{jk} (D_j D_k F^{(n,g-1)} + \sum'_{m,h} D_j F^{(m,h)} D_k F^{(n-m,g-h)}).$$

UV vs. IR

Seiberg-Witten curve for massless $N_f = 4$,

$$y^2 = (x - u e_1(\tau))(x - u e_2(\tau))(x - u e_3(\tau))$$

\Updownarrow

$$y^2 = x(x - u)(x - u q_{inst}), \quad q_{inst} = \frac{e_3 - e_2}{e_1 - e_2}(\tau) = \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)}$$

Here, q_{inst} is the instanton expansion parameter (UV-parameter), τ is the effective coupling (IR-parameter).

Explicit results for massless $N_f = 4$

$$F^{(2,0)} = \frac{E_2(\tau)}{2^5 3 a^2}, \quad F^{(1,1)} = -\frac{E_2(\tau)}{2^3 3 a^2}, \quad F^{(0,2)} = \frac{E_2(\tau)}{2^5 a^2},$$

$$F^{(3,0)} = -\frac{1}{2^8 3^2 5 a^4} (5E_2^2 + 13E_4)(\tau),$$

$$F^{(2,1)} = \frac{1}{2^6 3^2 5 a^4} (10E_2^2 + 17E_4)(\tau),$$

$$F^{(1,2)} = -\frac{1}{2^8 3^2 5 a^4} (95E_2^2 + 94E_4)(\tau),$$

$$F^{(0,3)} = \frac{1}{2^7 3 a^4} (2E_2^2 + E_4)(\tau).$$

Explicit results for massless $N_f = 4$

$$F^{(2,0)} = \frac{E_2(\tau)}{2^5 3 a^2}, \quad F^{(1,1)} = -\frac{E_2(\tau)}{2^3 3 a^2}, \quad F^{(0,2)} = \frac{E_2(\tau)}{2^5 a^2},$$

General structure

$$F^{(n,g)} = \frac{1}{a^{2(g+n)-2}} P_{2(g+n)-2}(E_2, E_4, E_6)(\tau)$$

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Degenerate weights

- Conformal primary: $L_0|\Delta\rangle = \Delta|\Delta\rangle$, $L_k|\Delta\rangle = 0$ for $k > 0$
- Descendant of level n : $L_{-i_1} \cdots L_{-i_k}|\Delta\rangle$, $\sum_m i_m = n$
- Null vector at level n : a descendant that is also a primary

A descendant χ at level mn of the weight

$$\Delta_{mn} = \frac{c-1}{24} + \frac{1}{4}(m\alpha_+(c) + n\alpha_-(c))^2$$

is a null vector: $L_0|\chi\rangle = (\Delta_{mn} + mn)|\chi\rangle$, $L_k|\chi\rangle = 0$ for $k > 0$.

Δ_{mn} are called degenerate weights.

Conformal blocks I

ϕ_1, ϕ_2 primaries:

$$\phi_1(z)\phi_2(0) = \sum_{\Delta} z^{\Delta-\Delta_1-\Delta_2} \Psi_{\Delta} C_{12}^{\Delta}$$

with

$$\Psi_{\Delta} = \phi_{\Delta}(0) + z\beta_{12}^{\Delta, \{1\}} L_{-1}\phi_{\Delta}(0) + \bar{z}\beta_{12}^{\bar{\Delta}, \{1\}} \bar{L}_{-1}\phi_{\Delta}(0) + \dots$$

The coefficients $\beta_{12}^{\Delta, \{i_1, \dots, i_n\}}$ are determined by conformal symmetry alone, and are the essential ingredients in conformal blocks.

Conformal blocks II

To define the **4-point conformal block**, map

$$(z_1, z_2, z_3, z_4) \rightarrow (0, x, 1, \infty),$$

$$\langle \phi_1(z_1) \phi_2(z_2) \phi_3(z_3) \phi_4(z_4) \rangle = \prod z_{ij}^{\gamma_{ij}} \bar{z}_{ij}^{\tilde{\gamma}_{ij}} \langle \Delta_4 | \phi_3(1) \phi_2(x) | \Delta_1 \rangle$$

and substitute OPE

$$\begin{aligned} G_{43}^{21} &= \langle \Delta_4 | \phi_3(1) \phi_2(x) | \Delta_1 \rangle \\ &= \sum_{\Delta} C_{43}^{\Delta} C_{21}^{\Delta} \langle \Psi_{\Delta}^{43} | \Psi_{\Delta}^{21} \rangle x^{\Delta - \Delta_1 - \Delta_2} \bar{x}^{\Delta - \Delta_1 - \Delta_2} \\ &= \sum_{\Delta} C_{43}^{\Delta} C_{21}^{\Delta} |x^{\Delta - \Delta_1 - \Delta_2 + \sum k_i}|^2 \\ &\quad \left| \sum_{\{k\}, \{k'\}} \beta_{43}^{\Delta, \{k\}} \beta_{21}^{\Delta, \{k'\}} \langle \Delta | L_{k_1} \cdots L_{k_n} L_{-k'_1} \cdots L_{-k'_m} | \Delta \rangle \right|^2 \end{aligned}$$

Conformal blocks III

⇒ We have split the four point function

$$G_{43}^{21} = \sum_{\Delta} C_{43}^{\Delta} C_{21}^{\Delta} |F_{\Delta}(c, \Delta_i|x)|^2$$

into a CFT specific contribution $C_{43}^{\Delta} C_{21}^{\Delta}$ and a generic contribution

$$F_{\Delta}(c, \Delta_i|x) = \sum_{\{k\}, \{k'\}} \beta_{43}^{\Delta, \{k\}} \beta_{21}^{\Delta, \{k'\}} \langle \Delta | L_{k_1} \cdots L_{k_n} L_{-k'_1} \cdots L_{-k'_m} | \Delta \rangle \\ \times x^{\Delta - \Delta_1 - \Delta_2 + \sum k_i} .$$

The AGT dictionary

AGT

The four-point conformal block is equal to the instanton part of the partition function of $N_f = 4$ Seiberg-Witten theory:

$$F_{\Delta}(c, \Delta_i | x) = Z_{inst}(\epsilon_1, \epsilon_2, a, m_i, q_{inst})(1 - x)^{-\nu}$$

With $Q = b + \frac{1}{b}$ such that $c = 1 + 6Q^2$,

$$b = \sqrt{\frac{\epsilon_1}{\epsilon_2}}, \quad \Delta = \frac{Q^2}{4} - \frac{a^2}{\epsilon_1 \epsilon_2}, \quad \Delta_i = \frac{Q^2}{4} - \frac{m_{1/4} \pm m_{2/3}}{\epsilon_1 \epsilon_2},$$

$$x = q_{inst}.$$

How to compute conformal blocks

The equations determining $\beta_{12}^{\Delta, \{k\}}$ lead to poles when Δ is a degenerate weight.

Given c , Δ is degenerate at

$$\Delta_{mn} = \frac{c-1}{24} + \frac{1}{4}(m\alpha_+(c) + n\alpha_-(c))^2.$$

To render a fixed Δ degenerate, c must be chosen as

$$c_{mn} = 13 - 6(\alpha_+^2(\Delta) + \alpha_-^2(\Delta)).$$

Recursion relations for $F_{\Delta}(c, \Delta_i|x)$ exist which make either the poles in c or those in Δ manifest.

The large c -recursion for conformal blocks

$$F_{\Delta}(c, \Delta_i|x) = f_{\Delta}(\Delta_i|x) + \sum_{m,n} \frac{R'_{mn}(\Delta, \Delta_i)}{c - c_{mn}(\Delta)} F_{\Delta+mn}(c_{mn}, \Delta_i|x)$$

where

$$\begin{aligned} f_{\Delta}(\Delta_i|x) &= \lim_{c \rightarrow \infty} F_{\Delta}(c, \Delta_i|x) \\ &= x^{\Delta - \Delta_1 - \Delta_2} {}_2F_1(\Delta + \Delta_2 - \Delta_1, \Delta + \Delta_3 - \Delta_4, 2\Delta, x) \\ &= x^{\Delta - \Delta_1 - \Delta_2} (1 + \mathcal{O}(x)) \end{aligned}$$

\Rightarrow yields recursion relation in x .

AGT use this recursion to compare to gauge theory results calculated via Nekrasov formula, which yields expansion in q_{inst} .

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$$x = q_{inst}.$$

The large Δ -recursion for conformal blocks

$$F_{\Delta}(c, \Delta_i | x) = (16y)^{\Delta - \frac{c-1}{24}} x^{\frac{c-1}{24} - \Delta_1 - \Delta_2} (1-x)^{\frac{c-1}{24} - \Delta_1 - \Delta_3} \theta_3^{\frac{c-1}{2} - 4 \sum_{i=1}^4 \Delta_i}(y) H_{\Delta}(c, \Delta_i | y)$$

with

$$H_{\Delta}(c, \Delta_i | y) = 1 + \sum_{m,n} (16y)^{mn} \frac{R_{mn}(c, \Delta_i)}{\Delta - \Delta_{mn}(c)} H_{\Delta_{mn} + mn}(c, \Delta_i | y)$$

where

$$x = \frac{\theta_2^4(\nu)}{\theta_3^4(\nu)}, \quad y = \exp \pi i \nu.$$

Comparison in massless limit

Simplification in massless limit: only massive parameter aside from $\epsilon_{1,2}$ is a .

$\Rightarrow a$ keeps track of joint order in g_s and s : $\frac{\sum_{n+g=k} \# g_s^{2g-2} s^n}{a^{2k-2}}$.

$$F_{\Delta}(c, \Delta_i | x) = (16y)^{\frac{a^2}{8s}} x^{-\frac{s^2}{4g_s}} (1-x)^{-\frac{s^2}{4g_s}} \theta_3(y)^{3\frac{s^2}{8s}} H_{\Delta}(c, \Delta_i | y)$$

with

$$H_{\Delta}(c, \Delta_i | y) = 1 + \sum_{m,n} (16y)^{mn} \frac{R_{mn}(c, \Delta_i)}{\Delta - \Delta_{mn}(c)} H_{\Delta_{mn}+mn}(c, \Delta_i | y)$$

$$\Rightarrow H_{\Delta} = \exp \sum_{n+g>1} F^{(n,g)} g_s^{2g-2} s^n.$$

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$$\Rightarrow H_{\Delta} = \exp \sum_{n+g>1} F^{(n,g)} g_s^{2g-2} s^n.$$

From topological string to CFT

We can obtain all order results in q for H_Δ .

E.g. $g + n = 2$: isolate order a^2 contribution:

$$H_\Delta(c, \Delta_i | y) = 1 + \sum_{m,n} \frac{(16y)^{mn} R_{mn}(c, \Delta_i)}{-\frac{a^2}{g_s^2} + \alpha_{mn}} H_{\Delta_{mn} + mn}(c, \Delta_i | y)$$

$$\begin{aligned} \Rightarrow & -\frac{g_s^2}{a^2} \sum_{m,n} (16y)^{mn} R_{mn}(c, \Delta_i) H_{\Delta_{mn} + mn}(c, \Delta_i | y) \\ & = \left[\frac{s^2}{g_s^2} F^{(2,0)} + s F^{(1,1)} + g_s^2 F^{(0,2)} \right]' \\ & = \left(\frac{s^2}{g_s^2} - 4s + 3g_s^2 \right) \frac{[E_2(\nu)]'}{96a^2} \end{aligned}$$

From CFT to topological string I

Likewise, we can obtain all-order results in g_s and s for topological string.

E.g., calculate first non-vanishing order in y :

$$\begin{aligned} H_{\Delta}(c, \Delta_i) &= 1 + (16y)^2 \left(\frac{R_{12}}{\Delta - \Delta_{1,2}} + \frac{R_{21}}{\Delta - \Delta_{2,1}} \right) + \dots \\ &= 1 + \left[\sum_{n+g>1} F^{(n,g)} g_s^{2g-2} s^n \right]_1 e^{2\pi i \tau} + \dots \end{aligned}$$

From CFT to topological string II

$$\frac{R_{12}}{\Delta - \Delta_{1,2}} + \frac{R_{21}}{\Delta - \Delta_{2,1}} = \frac{\epsilon_2^5}{2^9 \epsilon_1 (\epsilon_1^2 - \epsilon_2^2) a^2} \frac{1}{\left(1 - \frac{(\epsilon_1 + 2\epsilon_2)^2}{4a^2}\right)} + (\epsilon_1 \leftrightarrow \epsilon_2).$$

\Rightarrow at each order in y , the powers $\frac{1}{a^{2(g+n)-2}}$ in $F^{(n,g)}$ can be resummed via a geometric series.

Since the $F^{(n,g)}$ are polynomials in the Eisenstein series E_2, E_4, E_6 of fixed weight $2(n+g) - 2$, the *exact* result to all orders in y can be determined from the knowledge of a finite number of expansion coefficients.

Physical significance of poles in a ? They are cancelled by the perturbative contribution to Z_{top} as given by the DOZZ formula.

From CFT to topological string II

$$\frac{R_{12}}{\Delta - \Delta_{1,2}} + \frac{R_{21}}{\Delta - \Delta_{2,1}} = \frac{\epsilon_2^5}{2^9 \epsilon_1 (\epsilon_1^2 - \epsilon_2^2) a^2} \frac{1}{\left(1 - \frac{(\epsilon_1 + 2\epsilon_2)^2}{4a^2}\right)} + (\epsilon_1 \leftrightarrow \epsilon_2).$$

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Physical significance of poles in a ? They are cancelled by the perturbative contribution to Z_{top} as given by the DOZZ formula.

DOZZ and the perturbative contribution to Z_{top} I

AGT

Perturbative contribution to Z_{top} is obtained from 3-point function of Liouville theory, as determined by Dorn-Otto-Zamolodchikov-Zamolodchikov (DOZZ).

$$C(\alpha_1, \alpha_2, \alpha_3) \sim \frac{\prod_i \Upsilon(2\alpha_i)}{\Upsilon(\sum \alpha_i + Q) \prod_i \Upsilon(\alpha_i + \alpha_{i+1} - \alpha_{i+2})}$$

in terms of $\alpha : \Delta = \alpha(Q - \alpha)$, where

$$\Upsilon(a) = \frac{1}{\Gamma_b(a)\Gamma_b(Q-a)}, \quad \Gamma_b(a) \sim \prod_{m,n \geq 0} \frac{1}{mb + \frac{n}{b} + a}$$

DOZZ and the perturbative contribution to Z_{top} II

$$G_{43}^{21} = \sum_{\Delta} C_{43}^{\Delta} C_{21}^{\Delta} |F_{\Delta}(c, \Delta_i | x)|^2$$

3-point function contributions to 4-point function

$$C(Q - \alpha_4, \alpha_3, \alpha) C(Q - \alpha, \alpha_2, \alpha_1)$$

distributed among F_{top} and \bar{F}_{top} such that

$$G_{43}^{21} = \int \frac{d\alpha}{2\pi} |Z_{top}|^2 .$$

DOZZ and the perturbative contribution to Z_{top} III

$$\Gamma_b(a) \sim \prod_{m,n \geq 0} \frac{1}{mb + \frac{n}{b} + a}$$

Poles at $a = -mb - \frac{n}{b} \Rightarrow C(\alpha_1, \alpha_2, \alpha_3)$ vanishes at degenerate weights, canceling poles in conformal block.

g_s , s expansion of DOZZ via integral representation of $\log \Gamma_b$. Closed formula for $\epsilon_1 = -\epsilon_2$, i.e. $s = 0$:

$$[\log \Gamma_b(a)]_g = \frac{B_{2g}}{2g(2g-2)} \frac{1}{a^{2g-2}} \quad \text{for } g > 1,$$

but analytic properties obscured.

DOZZ and the perturbative contribution to Z_{top} III

$$\Gamma_b(a) \sim \prod_{m,n \geq 0} \frac{1}{mb + \frac{n}{b} + a}$$

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Lessons so far, and where to go from here

- Lessons from topological string: modularity in conformal blocks
- Lessons from CFT: analytic structure of all genus result

Next:

- Modularity and crossing symmetry, topological string suggests symmetry *before* the sum over exchanged momenta is performed.
- Holomorphic anomaly in CFT?