Nested Donaldson-Thomas invariants and the Elliptic genera in String theory

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Elliptic Genus in String theory:

1. Gaiotto, Strominger, Yin (hep-th/0702012v)
2. Gaiottoe, Yin (hep-th/0702012v1)
3. Frederik Denef, Gregory W. Moore (hep-th/0702146)
4. Hirosi Ooguri, Andrew Strominger, Cumrun Vafa (hep-th/0405146v2)

Study of D4-D2-D0 systems over a threefold.

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Study of D4-D2-D0 systems over a threefold.

X, D4-D2-D0

$(D \subset X, K_D)$ a divisor in X

$(p \text{ and } \mathcal{C}) \subset D \subset X$, Nested subschemes

An integral linear combination of holomorphic curves $\mathcal{C} \subset D \subset X$

Definition

The modified elliptic genus of an M5 Brane is given by the generating series of the invariants of D4-D2-D0 bound states in an M5-Brane background:

$$Z_{X,D}(\tau,\overline{\tau},y^A) = \sum_{\delta\in\Lambda^*/\Lambda} Z_\delta(\tau)\Theta_{\Lambda+\delta}(\tau,\overline{\tau},y^A).$$

$\Lambda \in \mathrm{H}^2(D,\mathbb{Z})$ is the image of $\mathrm{H}^2(X,\mathbb{Z}) \hookrightarrow \mathrm{H}^2(D,\mathbb{Z})$, $\Theta_{\Lambda+\delta} =$ Complicated!, but well known modular forms (Jacobi Theta functions!):

$$\Theta_{\Lambda+\delta}(\tau,\overline{\tau},y^A) = \sum_{q\in\Lambda+\delta+\frac{J}{2}} (-)^{J\cdot q}\exp[-\pi i\tau q^2 + \pi i(\tau-\overline{\tau})\frac{(J\cdot q)^2}{J\cdot J} + 2\pi i y\cdot q].$$

$Z_\delta(\tau)$ are given by holomorphic modular vectors.

$Z_{X,D}(\tau,\overline{\tau},y^A)$ is a modular form!
It is **difficult** to compute $Z_{X,D}(\tau,\overline{\tau},y^A)$ due to singularities of the moduli space of D4-D2-D0 systems.

Main Goal:

Use Algebraic Geometry to compute the corresponding generating series.

A D4-D2-D0 system of (p and $\mathcal{C}$) $\subset D \subset X$ can be characterized with a torsion 2 dimensional sheaf F such that $\mathrm{Ch}(F) = (0, [D], \beta = [\mathcal{C}], n[pt])$.

1. Need the condition that $[D]$ is irreducible.
2. We consider the moduli space of stable sheaves over X with $\mathrm{Ch}(F) = (0, D, \beta, n)$ denote it $\mathcal{M} := \mathcal{M}_X^{(0,D,\beta,n)}$.
3. Assume that $F \otimes K_X \cong F$, then $\mathrm{Ext}^3(F, F) \cong \mathrm{Hom}(F, F) \cong \mathbb{C}$, perfect obstruction theory of dim=0.
4. We compute invariants over $\mathcal{M}$ using Deformation-Obstruction theories and virtual classes: $\int_{[\mathcal{M}]^{vir}} 1$. We call them *Nested DT* invariants (NDT_X).
5. When X :=CY3, these are also given by weighted Euler characteristic and Behrend's functions.
6. Toda studied similar torsion sheaves with $(0, n[D], \beta, n)$ when $n \gg 0$, using wallcrossing and Behrend's function. For us $n = 1$.

Example

Let $j : D \hookrightarrow X$ be a divisor. Fix a linear system of divisors with representative D (i.e $\mid D \mid$) over X. For an element $S \in \mid D \mid$ (a surface) let $\mathcal{C} \subset S$ with $[\mathcal{C}] = \beta$ be a curve lying scheme theoretically on S. The sheaf $F := j_* \mathcal{I}_{\mathcal{C}/S}$ satisfies the property that $\text{Ch}(F) = (0, D, \beta, n)$.

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Relationship to Joyce-Song pairs theory:

$$0 \to \mathcal{O}_X(-S) \xrightarrow{s} \mathcal{I}_{\mathcal{C}/X} \to j_*\mathcal{I}_{\mathcal{C}/S} \to 0$$

The tuple $(s, \mathcal{I}_{\mathcal{C}/X})$ is a stable Joyce-Song pair. However if S is not very ample then $(s, \mathcal{I}_{\mathcal{C}/X})$ and $j_*\mathcal{I}_{\mathcal{C}/S}$ do not have the same deformation theory!

As objects in derived category

$$I^\bullet := [\mathcal{O}_X(-S) \xrightarrow{s} \mathcal{I}_{\mathcal{C}/X}] \cong j_*\mathcal{I}_{\mathcal{C}/S}.$$

Therefore if $I^\bullet$ and $(s, \mathcal{I}_{\mathcal{C}/X})$ deform in the same way then a theory for $j_*\mathcal{I}_{\mathcal{C}/S}$ can be computed via Joyce-Song stable pairs theory.

Lemma

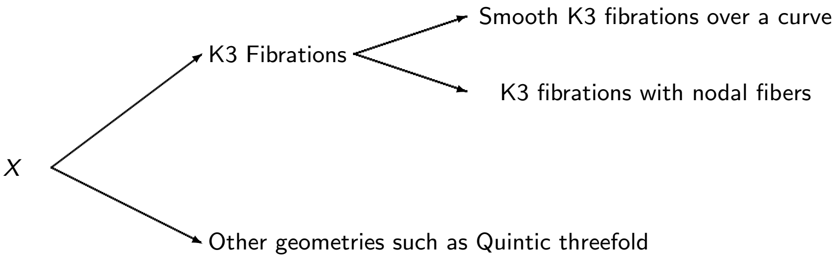
Let the curve $\mathcal{C}$ satisfy the property that $\mathrm{H}^i(\mathcal{I}_{\mathcal{C}/X}(S)) = 0$ for all $i > 0$. Then every deformation $I^\bullet$ over B of $I_0^\bullet$ is quasi-isomorphic to a complex

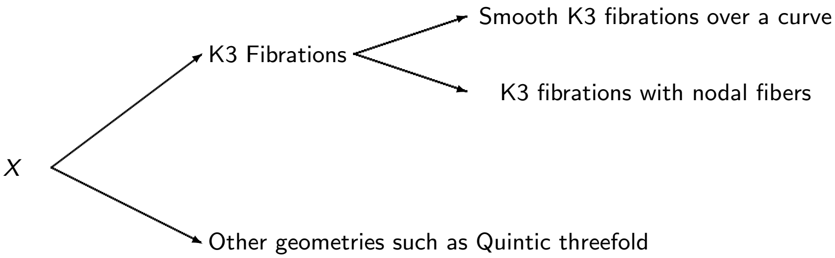
$$\mathcal{O}_X(-S) \boxtimes \mathcal{O}_B \xrightarrow{\phi} \tilde{\mathcal{I}}_{\mathcal{C}/X}$$

where $\tilde{\mathcal{I}}_{\mathcal{C}/X}$ is a B-flat deformation of $(\tilde{\mathcal{I}}_{\mathcal{C}/X})_0$ with section ϕ.

$$\begin{array}{ccc} \mathcal{O}_X(-S) & \xrightarrow{s} & \mathcal{I}_{\mathcal{C}/X} \\ \big\downarrow & & \\ \mathcal{I}_{\mathcal{C}/X} & & \end{array}$$

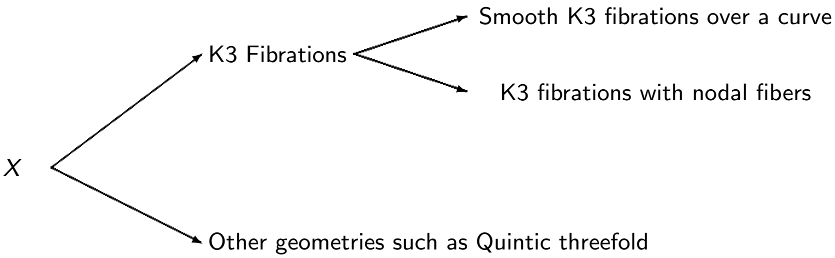
In that case the NDT invariants can be related to MNOP invariants. Other wise a new theory is required for 2 dimensional torsion sheaves.


Smooth K3 fibrations over a curve
K3 Fibrations
K3 fibrations with nodal fibers
X
Other geometries such as Quintic threefold



Fix a smooth K3 fibration $X \to \mathcal{Z}$ and an ample polarization L over X. We are following the work of Pandharipande-Maulik (Gromov-Witten theory and Noether-lefschetz theory):

$$\mathrm{GW}(X) = \mathrm{GW}(X/\mathcal{Z}) \cdot \mathrm{NL}(X \to \mathcal{Z})$$



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Our Goal: $\mathrm{DT}(X) = \mathrm{DT}(X/\mathcal{Z}) \cdot \mathrm{NL}(X \to \mathcal{Z})$

Let $\mathcal{M}$ and $\mathcal{M}/\mathcal{Z}$ denote the absolute and relative moduli spaces. Let $i : S \hookrightarrow X$ be a smooth surface. Consider a stable torsion free sheaf G over S and let $F \cong i_* G$. The points in $\mathcal{M}$ parameterize F and points in $\mathcal{M}/\mathcal{Z}$ parameterize G.

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2. Need to relate them!

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1. Need deformation-obstruction theories for $\mathcal{M}$ and $\mathcal{M}/\mathcal{Z}$.
2. Need to relate them!

Theorem

There exists a map in the derived category

$$R\pi_{\mathcal{M}*}(R\mathscr{H}om_{X\times\mathcal{M}}(\mathbb{F},\mathbb{F}) \otimes \omega_{\pi_{\mathcal{M}}}[2])^{\vee} \to \mathbb{L}^{\bullet}_{\mathcal{M}}$$

out of which one can obtain a perfect relative deformation obstruction theory of amplitude $[-1,0]$ for $\mathcal{M}$, i.e

$$E^{\bullet} := \left(\tau^{\geq 1} R\pi_{\mathcal{M}*}(R\mathscr{H}om_{X\times\mathcal{M}}(\mathbb{F},\mathbb{F})\right)^{\vee}[-1] \to \mathbb{L}^{\bullet}_{\mathcal{M}}$$

is a perfect deformation-obstruction theory of amplitude $[-1,0]$ for $\mathcal{M}$.

Theorem

Let $\tilde{i} : X \times_{\mathcal{Z}} \mathcal{M} \hookrightarrow X \times \mathcal{M}$ *be the natural inclusion. Denote by* $\pi'_{\mathcal{M}}$ *the composition map* $\pi'_{\mathcal{M}} : \pi_{\mathcal{M}} \circ \tilde{i}$ *(note that* $\pi'_{\mathcal{M}}$ *is relative dimension 2). There exists a map in the derived category*

$$R\pi'_{\mathcal{M}*}\left(R\mathscr{H}om_{X\times_{\mathcal{Z}}\mathcal{M}}(\mathbb{G},\mathbb{G})\otimes\omega_{\pi'_{\mathcal{M}}}[1]\right)^{\vee}\to\mathbb{L}^{\bullet}_{\mathcal{M}/\mathcal{Z}}$$

out of which one can obtain a perfect deformation obstruction theory of amplitude $[-1,0]$ *for* $\mathcal{M}/\mathcal{Z}$*, i.e*

$$F^{\bullet}:=\left(\tau^{\geq 1}R\pi'_{\mathcal{M}*}(R\mathscr{H}om_{X\times_{\mathcal{Z}}\mathcal{M}}(\mathbb{G},\mathbb{G})\right)^{\vee}[-1]\to\mathbb{L}^{\bullet}_{\mathcal{M}/\mathcal{Z}}$$

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is a perfect deformation-obstruction theory of amplitude $[-1,0]$ for $\mathcal{M}/\mathcal{Z}$.

In order to relate the two deformation-obstruction theories we study the exact triangle in the derived category of $\mathcal{M}$

Lemma

Let $\tilde{i} : X \times_{\mathcal{Z}} \mathcal{M} \hookrightarrow X \times \mathcal{M}$ *be the inclusion as above and* $\mathbb{G}$ *be the universal sheaf over* $X \times_{\mathcal{Z}} \mathcal{M}$. *By definition* $\mathbb{F} \cong \tilde{i}_* \mathbb{G}$. *Then there exists an exact triangle*

$$R\pi_{\mathcal{M}*}\left(\tilde{i}_* R\mathscr{H}om_{X \times_{\mathcal{Z}} \mathcal{M}}(\mathbb{G}, \mathbb{G})\right) \to R\pi_{\mathcal{M}*}\left(R\mathscr{H}om_{X \times \mathcal{M}}(\mathbb{F}, \mathbb{F})\right)$$
$$\to R\pi_{\mathcal{M}*}\left(\tilde{i}_* R\mathscr{H}om_{X \times_{\mathcal{Z}} \mathcal{M}}(\mathbb{G}, \mathbb{G} \otimes \mathcal{O}_{X \times_{\mathcal{Z}} \mathcal{M}}(X \times_{\mathcal{Z}} \mathcal{M}))\right)[-1] \quad (1)$$

In the level of obstruction bundles we have:

$$\mathrm{Ext}^2_S(G, G) \to \mathrm{Ext}^2_X(F, F) \to \mathrm{Ext}^1_S(G, G \otimes \mathcal{O}_S(S))$$

Theorem

Denote by $\mathsf{E}^\bullet$ *the perfect obstruction theory over* $\mathcal{M}$ *and by* $\mathsf{F}^\bullet$ *the perfect absolute obstruction theory over* $\mathcal{M}/\mathcal{Z}$. *Let* $[\mathcal{M}, \mathsf{E}^\bullet]^{vir}$ *and* $[\mathcal{M}, \mathsf{F}^\bullet]^{vir}$ *denote the induced virtual fundamental classes of* $\mathcal{M}$. *Then the following identity holds true over* $\mathcal{A}_*(\mathcal{M})$:

$$[\mathcal{M}, \mathsf{E}^\bullet]^{vir} = [\mathcal{M}, \mathsf{F}^\bullet]^{vir} \cap c_{top}(\mathcal{E}xt^1_{\pi'_\mathcal{M}}(\mathbb{G}, \mathbb{G}(\mathbb{S}))).$$

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Vanishing Issue:

$$\mathrm{h}^1(F^{\bullet\vee}) = \mathrm{Ext}^2_{X_z}(\tilde{\mathbb{G}}_m, \tilde{\mathbb{G}}_m \otimes \omega_{X_z}) \cong \mathbb{C}$$

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Need to compute the reduced deformation obstruction theory over the surfaces!

$$(\mathsf{F}^\bullet)^\vee \to \mathrm{h}^1(\mathsf{F}^\bullet)^\vee[-1] \xrightarrow{tr} R^2_{\pi'_{\mathcal{M}*}} \mathcal{O}_{X \times_{\mathcal{Z}} \mathcal{M}}[-1] \cong \mathcal{K}^\vee[-1]$$

The fiber of $\mathcal{K}^\vee$ over a point $z \in \mathcal{Z}$ is given by

$$\mathrm{H}^0(X_z, \omega_X|_{X_z} \otimes \mathcal{N}_{X_z/X}) \cong \mathrm{H}^0(X_z, K_{X_z})$$

where $\mathcal{N}_{X_z/X}$ and K_{X_z} denote the normal bundle and the canonical bundle (respectively) of the fiber of $X \to \mathcal{Z}$ over a point $z \in \mathcal{Z}$.

Given the dual map in the derived category

$$\Psi : \mathcal{K}[1] \to \left(h^1(\mathrm{F}^\bullet)^\vee[-1]\right)^\vee \to \mathrm{F}^\bullet$$

The reduced obstruction theory is obtained by coning off the map Ψ:

Definition

Let $\mathrm{D}^\bullet := \mathrm{Cone}(\Psi)[-1]$. It can be shown that $\mathrm{D}^\bullet$ is an absolute obstruction theory over $\mathcal{M}$ (with rather a different obstruction bundle) .

Theorem

The following identity holds true over $\mathcal{A}_*(\mathcal{M})$:

$$\int_{[\mathcal{M},\mathrm{E}^\bullet]^{vir}} 1 = \int_{[\mathcal{M},\mathrm{D}^\bullet]^{vir}} c_{top}(\mathcal{E}xt^1_{\pi'_{\mathcal{M}}}(\mathbb{G},\mathbb{G}(\mathbb{S}))) \cdot c_1(\mathcal{K}^\vee)$$
$$= \int_{[\mathcal{M}(K3)]^{red}} c_{top}(T_{\mathcal{M}(K3)}) \cdot \int_{B(m,h,\gamma)} c_1(\epsilon^*\mathcal{K}^\vee) \quad (2)$$

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$$= \int_{[\mathcal{M}(K3)]^{red}} c_{top}(T_{\mathcal{M}(K3)}) \cdot \int_{B(m,h,\gamma)} c_1(\epsilon^*\mathcal{K}^\vee)$$
$$= (\textit{Reduced NDT invariants}) \cdot (\textit{Noether-Lefschetz numbers}) \qquad (3)$$

Given $\mathrm{Ch}(G) = (0, D, \beta, n)$, note that if $G \in \mathcal{M}$ is supported on the smooth surface $i : S \hookrightarrow X$ then $G \cong M \otimes \mathcal{I}$ for some $M \in Pic(S)$ and ideal sheaf of n points on S, denoted by $\mathcal{I}$. Let $i_* \beta = \gamma$ and $\beta^2 = 2h - 2$. Then

$$\mathrm{NDT}(X, P) = \sum_{h \in \mathbb{Z}} \sum_{\substack{\gamma \in H_2(X) \\ 2L \cdot \gamma = \frac{\partial P_{2L}(m)}{\partial m} |_{m=0}}} \chi(\mathrm{Hilb}^{h+1-P_{2L}(0)}(S)) \cdot \mathrm{NL}^{\pi}_{h,\gamma}$$
$$+ \delta_{0, \frac{\partial}{\partial m} P_L(0)} \chi(Hilb^{2-P_L(0)}(S)) \cdot \int_{\mathcal{C}} c_1(\mathcal{K}^{\vee}) \tag{4}$$

where

$$P_L = P_L^F(m) = \frac{L^2}{2} m^2 + (L \cdot \beta) m + \frac{\beta^2}{2} - n + 2$$

is the Hilbert polynomial of F with respect to polarization L.

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$NDT(X) = $ (Goettsche's Invariants) $\cdot$ (Noether-Lefschetz numbers).

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$$NDT(X) = (\text{Goettsche's Invariants}) \cdot (\text{Noether-Lefschetz numbers}).$$

1. Goettsche invariants $\rightarrow$ Modular

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1. Goettsche invariants $\rightarrow$ Modular
2. NL numbers $\sim$ Fourier coefficients of Modular forms [Borcherds]

Given $\mathrm{Ch}(G) = (0, D, \beta, n)$, note that if $G \in \mathcal{M}$ is supported on the smooth surface $i : S \hookrightarrow X$ then $G \cong M \otimes \mathcal{I}$ for some $M \in Pic(S)$ and ideal sheaf of n points on S, denoted by $\mathcal{I}$. Let $i_*\beta = \gamma$ and $\beta^2 = 2h - 2$. Then

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Modularity of NDT invariants!

We obtain the generating series for the NDT invariants

$$Z_{\mathrm{NDT}}(X,P)(q) = 2 \cdot q^{P(0)} \cdot \frac{\Theta(q)}{\prod_{h\geq 1}(1-q^h)^{24}}$$

Where Θ is a vector valued modular form.

Example

In case where $\tilde{X} \to \tilde{\mathcal{C}}$ is obtained by a pencil of quartics over $\mathbb{P}^1$, Θ is given by a vector valued modular form of degree 21 and level 8:

$$\Theta(q) = -1 + 108q + 320q^{\frac{9}{8}} + 5016q^{\frac{3}{2}} + \cdots$$

Nodal Fibrations

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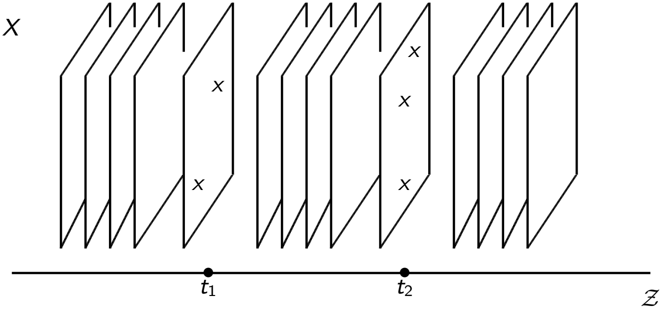

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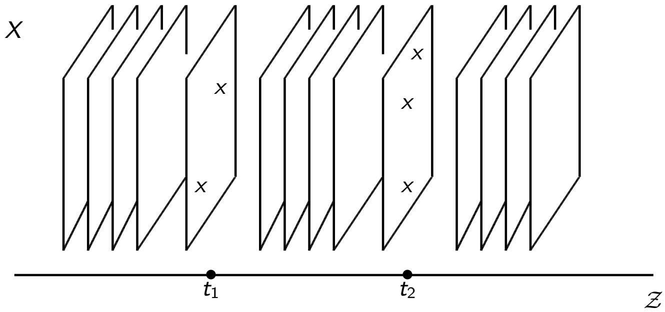
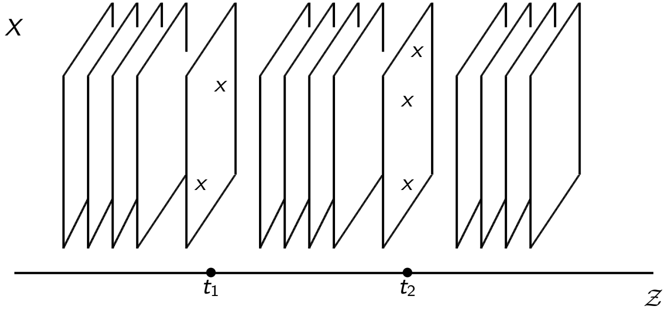


Main idea: To compute the *NDT* invariants of X via relating X to smooth fibrations.

Nodal Fibrations



Main idea: **Conifold Transition!**

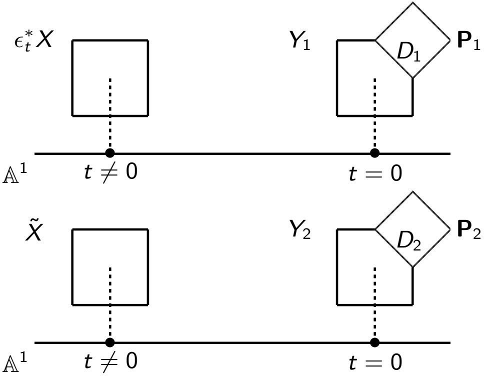
Let $\epsilon_0 : \mathscr{Z}_0 \xrightarrow{2:1} \mathscr{Z}$ be the double cover of $\mathscr{Z}$ ramified at t_i. Then $\epsilon^* X$ has conifold singularities at t_i. We resolve the singularities and obtain $\tilde{X} \longrightarrow \epsilon_0^* X$.

Let $\epsilon_t : \mathscr{Z}_t \xrightarrow{2:1} \mathscr{Z}$ be the double cover of $\mathscr{Z}$ at $s_i(t)$ such that $s_i(t) \longrightarrow t_i$ as $t \longrightarrow 0$. Then $\epsilon_t^* X$ is smooth as $t \neq 0$ and $\epsilon_t^* X \cong \epsilon_0^* X$ at $t = 0$.

Relate $\epsilon_t^* X$ to $\tilde{X}$ via degenerations

$\mathscr{BL}_{\{s_1(0),\cdots s_r(0)\}}(\epsilon_t^* X)$:

$\mathscr{BL}_{\{e_1,\cdots e_r\}\times 0}(\tilde{X} \times \mathbb{A}^1)$:



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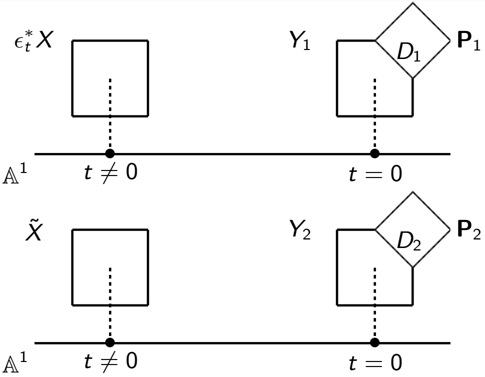

1. $\mathbf{P}_1$: A double cover of $\mathbb{P}^3$ branched over a smooth quartic surface $(\cong \mathbb{P}^1 \times \mathbb{P}^1)$
2. $D_1 \cong \mathbb{P}^1 \times \mathbb{P}^1 \cong D_2$
3. $\mathbf{P}_2 \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1})$
4. $Y_1 \cong Y_2 \cong \mathscr{BL}_{t_1,\cdots t_r}(\epsilon_0^* X)$

It turns out that $\mathbf{P}_1$ and $\mathbf{P}_2$ do not contribute to our invariants!

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Obtain a conifold transition identity

$$Z_{NDT}(\mathcal{M}_{\epsilon_t^* X}) = Z_{NDT}(\mathcal{M}_{Y_1/D_1}) = Z_{NDT}(\mathcal{M}_{Y_2/D_2}) = Z_{NDT}(\mathcal{M}_{\tilde{X}})$$

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Application: When the fibration $X \to \mathcal{Z}$ is CY3

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Example

1. $\chi(\mathrm{Hilb}^{[n]}(\text{Nodal Surface}), \nu)$:

$$\chi(\mathrm{Hilb}^{[n]}(X_{\epsilon(p_i)}), \nu_{\mathcal{M}_X^{\beta}}|_{\mathrm{Hilb}^{[n]}(X_{\epsilon(p_i)})}) = \frac{1}{r} \cdot \Big(\frac{1}{2} \cdot \chi(\mathrm{Hilb}^{[n]}(\tilde{X}/\tilde{c})) \cdot \deg(\mathcal{K}_{\tilde{\mathcal{C}}}^{\vee}) - \chi(\mathcal{C} - \{p_1, \cdots, p_r\}) \cdot \chi(\mathrm{Hilb}^{[n]}(\epsilon^* X_q)) \Big). \quad (10)$$

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1. $\chi(\mathrm{Hilb}^{[n]}(\text{Nodal Surface}), \nu)$:

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2. $\chi(\mathcal{M}/\mathcal{Z}, \nu)$ for $\gamma \neq 0$, for a nodal fiber:

$$\chi(\mathcal{M}^{\overline{\beta}}_{X_{\epsilon(p_i)}}, \nu_{\mathcal{M}_X^{\overline{\beta}}}|_{\mathcal{M}^{\overline{\beta}}_{X_{\epsilon(p_i)}}}) = \frac{1}{r} \cdot \Bigg(\frac{1}{2} \sum_{h \in \mathbb{Z}} \sum_{\substack{\gamma \in H_2(\tilde{X}) \\ 2L \cdot \gamma = \frac{\partial P_{2L}(m)}{\partial m}|_{m=0}}} \chi(\mathrm{Hilb}^{h+1-P_{2L}(0)}(\tilde{X}/\tilde{c})) \cdot \mathrm{NL}^{\pi_{\tilde{\mathcal{C}}}}_{h,\gamma} - \chi(\mathcal{C} - \{p_1, \cdots, p_r\}) \cdot \chi(\mathrm{Hilb}^{[n]}(\epsilon^* X_q)) \Bigg). \tag{12}$$

Work in Progress: **NDT invariants of Quintic Threeofold**. Following work of Pandharipande-Maulik (Topological view of GW theory)

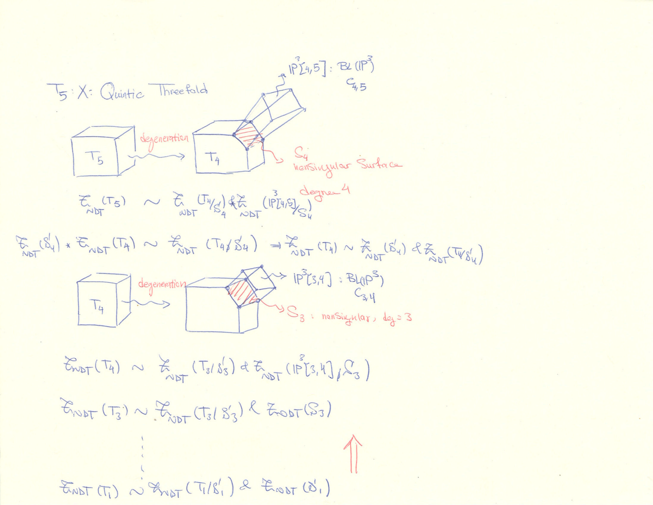

Figure: Degeneration of Quintic

Thank you