

GENERALIZED GEOMETRY OF TYPE B_n

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- generalized geometry on M^n
- $T \oplus T^*$
- inner product $(X + \xi, X + \xi) = i_X \xi$
- $SO(n, n)$ -structure

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- “extended” generalized geometry on M^n
- $T \oplus 1 \oplus T^*$
- inner product $(X + \lambda + \xi, X + \lambda + \xi) = i_X \xi + \lambda^2$
- $SO(n + 1, n)$ -structure – type B_n

C M Hull, *Generalised geometry for M-theory*, JHEP **07** (2007)

D Baraglia, *Leibniz algebroids, twistings and exceptional generalized geometry*, arXiv: 1101.0856

P Bouwknegt, *Courant algebroids and generalizations of geometry*, String-Math 2011

Roberto Rubio

Mario Garcia Fernandez

- $\mathfrak{so}(n+1, n) = \Lambda^2 T + T + \text{End } T + T^* + \Lambda^2 T^*$

- $A \in T^*, B \in \Lambda^2 T^*$

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- $X + \lambda + \xi \mapsto X + (\lambda + i_X A) + (\xi - 2\lambda A - i_X AA)$
- $X + \lambda + \xi \mapsto X + \lambda + \xi + i_X B$

ACTION ON VECTORS

- $\mathfrak{so}(n+1, n) = \Lambda^2 T + T + \text{End } T + T^* + \Lambda^2 T^*$
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- $X + \lambda + \xi \mapsto X + (\lambda + i_X A) + (\xi - 2\lambda A - i_X AA)$
- $X + \lambda + \xi \mapsto X + \lambda + \xi + i_X B$
- $(A, B).(A', B') = (A + A', \underline{B + B' - 2A \wedge A'})$

SPINORS

- exterior algebra $\Lambda^* T^*$
- $\tau(\varphi) = (-1)^{\deg \varphi} \varphi$
- Clifford multiplication $(X + \lambda + \xi) \cdot \varphi = i_X \varphi + \lambda \tau \varphi + \xi \wedge \varphi$
- $\exp B(\varphi) = e^{-B} \wedge \varphi$
- $\exp A(\varphi) = e^{-A\tau} \varphi = \varphi - A \wedge \tau \varphi$

AUTOMORPHISMS

- generalized diffeomorphisms $\text{GDiff}(M) = \text{Diff}(M) \ltimes \Omega_{cl}^{2+1}$
- $\Omega_{cl}^{2+1} = \{(A, B) \in \Omega^1 \times \Omega^2 : dA = dB = 0\}$
- $(A, B).(A', B') = (A + A', B + B' - 2A \wedge A')$
- $0 \rightarrow \Omega_{cl}^2 \rightarrow \Omega_{cl}^{2+1} \rightarrow \Omega_{cl}^1 \rightarrow 0$

central extension

- generalized vector field $u = X + \lambda + \xi$
- $u \mapsto X - d\lambda - d\xi \in \mathfrak{g}\mathfrak{diff}(M)$
- Lie derivative on $T + 1 + T^* =$ Dorfman “bracket”
- Lie derivative on spinors $d(u \cdot \varphi) + u \cdot d\varphi$

- skew-symmetrize action on $T + 1 + T^*$, get a Courant bracket
- $[u, fv] = f[u, v] + (Xf)v - (u, v)df$
- $[[u, v], w] + \dots = \frac{1}{3}d([u, v], w) + \dots$

Z Chen, M Stienon and P Xu, *On regular Courant algebroids*,
arXiv: 0909.0319

WHAT'S NEW?

- TWISTING AND COHOMOLOGY
- G_2^2 -STRUCTURES
- METRICS, CONNECTIONS AND CURVATURE

TWISTING

- D_n generalized geometry, $T + T^*$
- identify $T + T^*|_U$ with $T + T^*|_V$ over $U \cap V$
- $X + \xi \mapsto X + \xi + i_X B_{UV}$
- B_{UV} 1-cocycle in Ω_{cl}^2

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- Courant algebroid $0 \rightarrow T^* \rightarrow E \rightarrow L \rightarrow 0$
- $0 \rightarrow 1 \rightarrow L \rightarrow T \rightarrow 0$
- L Lie algebroid (like $TP/U(1)$ for P a $U(1)$ -bundle over M)

- $[(A_{UV}, B_{UV})] \in H^1(M, \underline{\Omega}_{cl}^{2+1})$
- $0 \rightarrow \underline{\Omega}_{cl}^2 \rightarrow \underline{\Omega}_{cl}^{2+1} \rightarrow \underline{\Omega}_{cl}^1 \rightarrow 0$ sheaves
- $H^1(M, \underline{\Omega}_{cl}^2) \rightarrow H^1(M, \underline{\Omega}_{cl}^{2+1}) \rightarrow H^1(M, \underline{\Omega}_{cl}^1) \xrightarrow{\delta} H^2(M, \underline{\Omega}_{cl}^2)$

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- $$\begin{array}{ccccc}
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 \parallel & & & & \parallel & & \parallel \\
 H^3(M, \mathbf{R}) & & & & H^2(M, \mathbf{R}) & & H^4(M, \mathbf{R})
 \end{array}$$
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- $\delta(x) = x^2$
- $F \in \Omega_{cl}^2 \quad F^2 = -dH$

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- spinor bundle S , differential d
- or... split extension, and get
- $d + F\tau + H : \Omega^* \rightarrow \Omega^*$
- $(d + F\tau + H)^2 = 0$

WHAT IS THIS COHOMOLOGY?

- Suppose F = curvature of a $U(1)$ -connection
- connection form θ_U on U , $F = d\theta_U$
- $\theta_V - \theta_U = g_{UV}^{-1}dg_{UV}$
- on U transform by A-field θ_U on U

- $d + F\tau + H \mapsto d + H_U$
- $H_U = H + \theta_U \wedge d\theta_U$
- $dH + F^2 = 0 \Rightarrow dH_U = 0$
- twisted cohomology on U

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- twisted cohomology on U
- identify on $U \cap V$ by A-field $g_{UV}^{-1}dg_{UV}$

- $H_V - H_U = g_{UV}^{-1} dg_{UV} \wedge d\theta_U$
- $g_{UV} : U \cap V \rightarrow S^1$
- Generalization of twisted K-theory?

G_2^2 MANIFOLDS

- $G_2^2 \subset SO(4, 3)$
- stabilizer of a non-null spinor
- $\rho \in \Lambda^* T^*$, $\rho = \rho_0 + \rho_1 + \rho_2 + \rho_3$
- $\langle \rho, \rho \rangle = \rho_0 \rho_3 - \rho_1 \wedge \rho_2 \neq 0$

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- G_2^2 manifold: M^3 + closed form ρ with $\langle \rho, \rho \rangle \neq 0$

EXAMPLES

- $\rho = 1 + \rho_3$, ρ_3 volume form.
- $\rho_0 = 0 \Rightarrow \rho_1 \wedge \rho_2 \neq 0$
- non-vanishing closed 1-form $\rho_1 \Rightarrow M^3$ fibres over a circle.
- $p : M \rightarrow S^1$, $|\rho_1 - \lambda p^* d\theta| < \epsilon$, $\lambda \in \mathbb{Q}$

QUESTIONS

- IF ρ, ρ' ARE COHOMOLOGOUS AND SUFFICIENTLY CLOSE, ARE THEY EQUIVALENT UNDER A GENERALIZED DIFFEOMORPHISM?
- WHICH COHOMOLOGY CLASSES CONTAIN G_2^2 STRUCTURES?

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- IF ρ, ρ' ARE COHOMOLOGOUS AND SUFFICIENTLY CLOSE, ARE THEY EQUIVALENT UNDER A GENERALIZED DIFFEOMORPHISM? ✓

- WHICH COHOMOLOGY CLASSES CONTAIN G_2^2 STRUCTURES?

- $\rho_0 \neq 0$, use $A = \rho_1/\rho_0, B = \rho_2/\rho_0$
- transform to $\rho_0 + (\rho_3 - \rho_1 \wedge \rho_2/\rho_0)$
- $\Rightarrow \rho$ equivalent to $\rho_0(1 + vol)$

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- $\Rightarrow \rho$ equivalent to $\rho_0(1 + vol)$
- $\rho_0 = 0$: which classes in $H^1(M^3, \mathbf{R})$ are represented by non-vanishing 1-forms?
- Answer: cone on faces of the unit ball in the Thurston norm.

METRICS AND CONNECTIONS

METRICS

- generalized metric: reduction of $SO(n+1, n)$ structure of $T \oplus 1 \oplus T^*$ (or E) to $S(O(n+1) \times O(n))$
- \Leftrightarrow rank n subbundle $V \subset E$ on which induced metric is negative definite.
- $\Rightarrow V^\perp$ positive definite

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- $\Rightarrow V^\perp$ positive definite
- e.g. $V = \{X - gX \in T + 1 + T^*\}, V^\perp = \{X + \lambda + gX\}$
- in general, transform by (A, B) : metric $g + A\text{-field} + B\text{-field}$

CONNECTIONS

- generalized connection on $W = D : W \rightarrow E \otimes W$
- $D(fs) = fDs + df \otimes s$ ($df \in T^* \subset E$)
- = ordinary connection + section of $E \otimes \text{End } W$

CONNECTIONS

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- $D(fs) = fDs + df \otimes s$ ($df \in T^* \subset E$)
- = ordinary connection + section of $E \otimes \text{End } W$
- $u \in E, D_us = (Ds, u) \in W$
- frame bundle, principal bundle...

TORSION

- generalized connection $D : E \rightarrow E \otimes E$
- torsion for affine connection ∇ : $\nabla_X Y - \nabla_Y X - [X, Y]$
- Courant bracket $[u, fv] = f[u, v] + (Xf)v - (u, v)df$
- $$T(u, v, w) = (D_u v - D_v u - [u, v], w) + \frac{1}{2}(D_w u, v) - \frac{1}{2}(D_w v, u)$$
- if D preserves $(,)$, $T \in \Lambda^3 E$

- $T^* \xrightarrow{\nabla} T^* \otimes T^* \xrightarrow{\pi} \Lambda^2 T^*$

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- spinors S , $D : S \rightarrow E \otimes S$
- “Dirac” operator: $S \xrightarrow{D} E \otimes S \xrightarrow{\text{Cliff}} S$
- torsion $= \text{Cliff} \circ D - d$

PROBLEM:

FIND A TORSION-FREE GENERALIZED CONNECTION WHICH
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- $E = V \oplus V^\perp$
- $V \cong T, V^\perp \cong 1 \oplus T$
- take Levi-Civita ∇ and trivial connection on 1 and add on terms so that $\text{Cliff} \circ D = d$

A Coimbra, C Strickland-Constable and D Waldram, *Supergravity as generalized geometry I: Type II theories*, arXiv: 1107.1733

- $V_- = \{X^- = X - gX\}, V_+ = \{X^+ = X + gX\}, V_0 = 1$
- add terms in $\Lambda^2 V_- \otimes (V_- + V_0 + V_+)$ and
 $\Lambda^2(V_0 + V_+) \otimes (V_- + V_0 + V_+)$
- so that $\text{Cliff} \circ D = d + F\tau + H$

- $$d\varphi = gX_i \wedge \nabla_i \varphi = \frac{1}{2}(X_i^+ - X_i^-) \cdot \nabla_i \varphi$$

- $$H \wedge \varphi = \frac{1}{48} H_{ijk} (X_i^+ - X_i^-) \cdot (X_j^+ - X_j^-) \cdot (X_k^+ - X_k^-) \cdot \varphi$$

- $$F \wedge \tau\varphi = \frac{1}{8} F_{ij} (X_i^+ - X_i^-) \cdot (X_j^+ - X_j^-) \cdot X_0 \cdot \varphi$$

- $\Lambda^2 V_- \otimes (V_- + V_0 + V_+)$

- $(H \wedge \varphi)^{---} = -\frac{1}{48} H_{ijk} X_i^- \cdot X_j^- \cdot X_k^- \cdot \varphi$

$$(H \wedge \varphi)^{- - +} = \frac{1}{16} H_{ijk} X_i^- \cdot X_j^- \cdot X_k^+ \cdot \varphi$$

- $(F \wedge \tau\varphi)^{- - 0} = \frac{1}{8} F_{ij} X_i^- \cdot X_j^- \cdot X_0 \cdot \varphi$

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CURVATURE

- $D_u D_v - D_v D_u - D_{[u,v]}$?
- Courant bracket $[u, fv] = f[u, v] + (Xf)v - (u, v)df$
- OK if $(u, v) = 0$
- Curvature $\in \text{End } E \otimes V \otimes V^\perp$

RICCI CONTRACTIONS

- $(\text{End } V \oplus \text{End } V^\perp) \otimes V \otimes V^\perp$
- $\text{End } V \otimes V \otimes V^\perp \mapsto V \otimes V^\perp$
 $\text{End } V^\perp \otimes V \otimes V^\perp \mapsto V \otimes V^\perp$

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- $\text{End } V \otimes V \otimes V^\perp \mapsto V \otimes V^\perp$
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- generalized metric $V \subset E$
- tangent vector $\in \text{Hom}(V, V^\perp)$
- define generalized Ricci flow

RICCI CURVATURES

- $\text{End } V \otimes V \otimes V^\perp \mapsto V \otimes V^\perp$
- $V \otimes V^\perp \cong T \otimes (1 + T) \cong \text{Sym}^2 T^* + \Lambda^2 T^* + T^*$

$$\text{Sym}^2 T^* : R_{il} - \frac{1}{4} H_{ijk} H_\ell^{jk}$$

$$\Lambda^2 T^* : \nabla^i H_{ijk}$$

RICCI CURVATURES

- $\text{End } V^\perp \otimes V \otimes V^\perp \mapsto V \otimes V^\perp$

$$\text{Sym}^2 T^* : R_{il} - \frac{1}{4} H_{ijk} H_\ell^{jk} - F_{ij} F^{j\ell}$$

$$\Lambda^2 T^* : -\nabla^i H_{ijk}$$

$$T^* : \nabla^i F_{ij} + \frac{1}{2} F^{ki} H_{jki}$$

RICCI CURVATURES

- $\text{End } V^\perp \otimes V \otimes V^\perp \mapsto V \otimes V^\perp$

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$$T^* : \nabla^i F_{ij} + \frac{1}{2} F^{ki} H_{jki}$$

heterotic supergravity + $U(1)$ gauge field