SUSY Gauge Theories, Quantized Moduli Spaces of Flat Connections, and Liouville Theory

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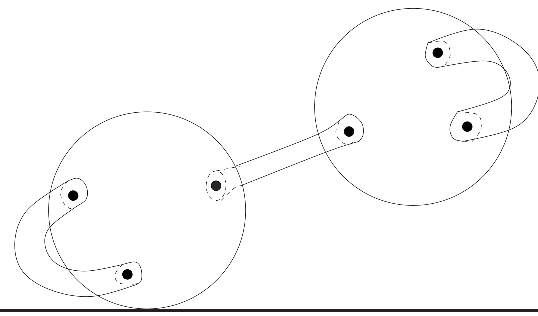
Based on joint work with G. Vartanov

$N = 2$ SUSY gauge theories of class $\mathcal{S}$

Class $\mathcal{S}$: large family of $N = 2$ SUSY gauge theories with gauge group $(U(2))^{3g-3+n}$ associated to Riemann surfaces (Gaiotto, Gaiotto, Moore, Neitzke):

Riemann surface C	Gauge theory $\mathcal{G}_C$
Pants decomposition 	Lagrangian description
r-th tube	r-th vector multiplet $(A_{r,\mu}, a_r, \dots)$
Punctures	Hypermultiplets
Change of pants decomposition	S-duality
Gluing parameters $q_r = e^{2\pi i \tau_r}$, $r = 1, \dots, 3g-3+n$	UV-couplings $\tau_r = \frac{4\pi i}{g_r^2} + \frac{\theta_r}{2\pi}$
Geodesic circumference of r-th tube	Coulomb branch modulus a_r

Loop observables – definition

– important probes of phase structure of gauge theories.

SUSY-versions:

- Wilson loop: $W_r = \mathrm{Tr}\,\mathcal{P}\exp\left(\int_{\mathcal{C}}(A_{r,\mu}dx^\mu + ia_r ds)\right)$,

- 't Hooft loop T_r: Expectation values defined by path integral

$$\langle\, T_r \,\rangle \;=\; \int_{(\mathrm{B.C.})_\mathrm{r}} [\mathcal{D}\Phi]\; e^{-S[\Phi]}\,,$$

  over all fields subject to boundary conditions

$$F_r \sim \frac{B_r}{4}\epsilon^{ijk}\frac{x^i}{|\vec{x}|^3}dx^k \wedge dx^j\,, \qquad a_r \sim \frac{B_r}{2|\vec{x}|}\,,$$

  near the contour $\mathcal{C}$. F_r and a_r are the field strength and the scalar associated to a $U(1)$-subgroup of the $SU(2)$ gauge group $SU(2)_r$.

Loop observables – exact results

Localization $\Rightarrow$ exact results on expectation values,

(Pestun; Gomis, Okuda, Pestun; Hama, Hosomichi)

$$\langle\, W_r^i \,\rangle_{E_\epsilon^4} = \int d\mu(a)\ \big|\, \mathcal{Z}^{\rm inst}(a,m,\tau,\epsilon)\,\big|^2\, 2\cosh(2\pi a_r/\epsilon_i)\,,$$

$$\langle\, T_r^i \,\rangle_{E_\epsilon^4} = \int d\mu(a)\ (\mathcal{Z}^{\rm inst}(a,m,\tau,\epsilon))^*\, \mathcal{D}_{a,r}^i \cdot \mathcal{Z}^{\rm inst}(a,m,\tau,\epsilon)\,,$$

- E_ϵ^4, $\epsilon=(\epsilon_1,\epsilon_2)$: Four-dimensional ellipsoid,

  $$E_\epsilon^4 := \{\,(x_0,\dots,x_4)\,|\,x_0^2+\epsilon_1^2(x_1^2+x_2^2)+\epsilon_2^2(x_3^2+x_4^2)=1\,\}\,,$$

  W_r^1, T_r^1 and W_r^2, T_r^2 loop observables supported on circles $S_{\epsilon_1}^1$ and $S_{\epsilon_2}^1$.

- $\mathcal{Z}^{\rm inst}(a,m,\tau,\epsilon)$ instanton partition function
  (Moore, Nekrasov, Shatashvili, Nekrasov, ...)
  $a=(a_1,\dots,a_{3g-3+n})$, $m=(m_1,\dots,m_n)$, $\tau=(\tau_1,\dots,\tau_{3g-3+n})$.

- $\mathcal{D}_{a,r}^i$: certain finite difference operators containing shifts $i\epsilon_i/2$ of variables a_r.

Quantum Teichmüller theory

Teichmüller space $\mathcal{T}(C)$: Space of all metrics on C with $R = -1$ modulo diffeomorphisms. $\dim(\mathcal{T}(C_{g,n})) = 6g-6+2n$. $\mathcal{T}(C)$ can be quantized $\Rightarrow$

- Algebra of observables $\mathcal{A}_b$ generated by **geodesic length operators** L_γ.

- Hilbert space $\mathcal{H}_\mathcal{T} \simeq L^2(\mathbb{R}_+^{3g-3+n})$. States represented by wave-fcts $\psi(l) = \langle\, l \,|\, \psi \,\rangle$

$$\mathsf{L}_r\psi(l) = 2\cosh(l_r/2)\psi(l)\,,$$

where $\mathsf{L}_r \equiv \mathsf{L}_{\gamma_r}$, γ_r: curves defining pants decomposition.

Relation to gauge theory: (J.T. '03, Drukker, Okuda, Gomis, J.T.)

There exists distinguished family of states $|\tau\rangle_\mathcal{T} \in \mathcal{H}_\mathcal{T}$ such that

$$\begin{aligned} \langle\, W_r^{\mathbf{1}} \,\rangle_{\mathcal{G}_C} &= \langle\, \tau \,|\, \mathsf{L}_r \,|\, \tau \,\rangle_\mathcal{T} \\ \langle\, T_r^{\mathbf{1}} \,\rangle_{\mathcal{G}_C} &= \langle\, \tau \,|\, \mathsf{L}_{\check{\gamma}_r} \,|\, \tau \,\rangle_\mathcal{T} \end{aligned} \qquad \mathcal{Z}_{\mathcal{G}_C}^{\text{inst}}(a,m,\tau,\epsilon) = \langle\, l \,|\, \tau \,\rangle_\mathcal{T}\,.$$

This follows from AGT-correspondence $\mathcal{Z}_{\mathcal{G}_C}^{\text{inst}} = \mathcal{F}_C^{\text{Liou}}$ since $\mathcal{F}_C^{\text{Liou}} = \langle\, l \,|\, \tau \,\rangle_\mathcal{T}$.

Gauge-theoretical intepretation I

Background (Nekrasov, Witten):

Gauge theory on $\mathbb{R}^4$ with Ω-background ($\mathbb{R}^4_{\epsilon_1\epsilon_2}$) can be topologically twisted. Resulting theory unchanged by replacement $\mathbb{R}^4_{\epsilon_1\epsilon_2} \to \mathbb{R} \times M^3_\epsilon$, where M^3_ϵ, $\epsilon = (\epsilon_1, \epsilon_2)$: three-manifold with $(U(1))^2$-isometries.

Utilize Hamiltonian framework with Hilbert space $\mathcal{H}(M^3_\epsilon)$ etc.. Ω-deformation preserves supercharges Q and Q†. Let

$$\mathcal{H}^{\rm top}(M^3_\epsilon) \subset \mathcal{H}(M^3_\epsilon) : \quad \text{Cohomology of Q.}$$

Consider $M^{4,-}_\epsilon$: Euclidean four-manifold with boundary M^3_ϵ which preserves Q.

$$\text{Path integral over } M^{4,-}_\epsilon \quad \Rightarrow \quad \text{State } |\,\tau\,\rangle\!\rangle^{\rm top}_{\mathcal{G}_C} \in \mathcal{H}^{\rm top}(M^3_\epsilon)\,.$$

The space $\mathcal{H}^{\rm top}(M^3_\epsilon)$ is acted on by **loop operators** W$_r$, T$_r$ commuting with Q. The loop operators generate algebra $\mathcal{A}^{\rm loop}_\epsilon = \mathcal{A}^{\rm loop}_{\epsilon_1} \times \mathcal{A}^{\rm loop}_{\epsilon_2}$.

Gauge-theoretical intepretation II – Instanton partition functions

Claim:

There exists a representation for $\mathcal{H}^{\rm top}(M^3_\epsilon)$ in which all Wilson loops are diagonal, wave-fcts $\psi(a) = \langle\!\langle\, a \,|\, \psi \,\rangle\!\rangle^{\rm top}_{\mathcal{G}_C}$,

$$\mathsf{W}^i_r\, \psi(a) = 2\cosh(2\pi a_r/\epsilon_i)\, \psi(a)\,.$$

We have

$$\boxed{\mathcal{Z}^{\rm inst}_{\mathcal{G}_C}(a,m,\tau,\epsilon) \;=\; \langle\!\langle\, a \,|\, \tau \,\rangle\!\rangle^{\rm top}_{\mathcal{G}_C} \;=\; \langle\, l \,|\, \tau \,\rangle_{\mathcal{T}}\,,}$$

provided that

$$\frac{a_r}{\epsilon_2} \;=\; \frac{l_r}{4\pi}\,.$$

Gauge-theoretical intepretation III – Expectation values

Preparation:

Gauge theory expectation values on E_ϵ^4 with observables inserted on equator $x_0 = 0$.

Near equator, $E_\epsilon^4 \sim \mathbb{R} \times E_\epsilon^3 \Rightarrow$ Utilize Hamiltonian framework with Hilbert space $\mathcal{H}(E_\epsilon^3)$ etc.. There exist generators Q, Q† of unbroken SUSY such that H $= \{\,\mathsf{Q}\,,\,\mathsf{Q}^\dagger\,\}$. Consider $E_\epsilon^{4,-}$: half-ellipsoid with boundary E_ϵ^3.

$$\text{Path integral over } E_\epsilon^{4,-} \quad \Rightarrow \quad \text{State } |\,\tau\,\rangle\!\rangle_{\mathcal{G}_C} \in \mathcal{H}(E_\epsilon^3)\,.$$

Let $\mathcal{H}_0(E_\epsilon^3) \subset \mathcal{H}(E_\epsilon^3)$: Subspace of zero energy states.

The space $\mathcal{H}_0(E_\epsilon^3)$ is acted on by **loop operators** W_r, T_r commuting with Q

$\Rightarrow \mathcal{H}_0(E_\epsilon^3)$ module for $\mathcal{A}_\epsilon^{\text{loop}}$.

Gauge-theoretical intepretation IV – Expectation values

Claim: We have $\mathsf{H}\,|\,\tau\,\rangle\!\rangle_{\mathcal{G}_C} = 0 \quad \Leftrightarrow \quad |\,\tau\,\rangle\!\rangle_{\mathcal{G}_C} \in \mathcal{H}_0(E^3_\epsilon)\,.$

There exists a representation for $\mathcal{H}_0(E^3_\epsilon)$ in which all Wilson loops are diagonal, states $|\,\psi\,\rangle$ represented by wave-fcts $\psi(a) = \langle\!\langle\, a\,|\,\psi\,\rangle\!\rangle^0_{\mathcal{G}_C}$, such that

$$\mathsf{W}^i_r\,\psi(a) = 2\cosh(2\pi a_r/\epsilon_i)\,\psi(a)\,.$$

We have

$$\boxed{\langle\!\langle\, a\,|\,\tau\,\rangle\!\rangle^0_{\mathcal{G}_C} = \langle\, l\,|\,\tau\,\rangle_{\mathcal{T}}\,,} \qquad \frac{a_r}{\epsilon_2} = \frac{l_r}{4\pi}\,.$$

It follows immediately that

$$\boxed{\begin{aligned} \langle\, W^1_r\,\rangle_{\mathcal{G}_C} &= \langle\!\langle\,\tau\,|\,\mathsf{W}^1_r\,|\,\tau\,\rangle\!\rangle_{\mathcal{G}_C} = \langle\,\tau\,|\,\mathsf{L}_r\,|\,\tau\,\rangle_{\mathcal{T}} \\ \langle\, T^1_r\,\rangle_{\mathcal{G}_C} &= \langle\!\langle\,\tau\,|\,\mathsf{T}^1_r\,|\,\tau\,\rangle\!\rangle_{\mathcal{G}_C} = \langle\,\tau\,|\,\mathsf{L}_{\check{\gamma}_r}\,|\,\tau\,\rangle_{\mathcal{T}} \end{aligned}}$$

In other words:

Pestun's localization $\Rightarrow$ effective quantum mechanics on $\mathcal{H}_0(E^3_\epsilon) \simeq \mathcal{H}_{\mathcal{T}}$.

On the derivations I

Key input (`Nekrasov, Witten`):

$$\mathcal{A}_\epsilon^{\text{loop}} \simeq \mathcal{A}_\epsilon^{\text{flat}}\,, \qquad \mathcal{H}^{\text{top}}(M_\epsilon^3) \simeq \mathcal{H}_{\mathcal{T}}\,,$$

where $\mathcal{A}_\epsilon^{\text{flat}}$: quantized algebra of functions on $\mathcal{M}_{\text{flat}}(C) \supset \mathcal{T}(C)$.

Sketch of derivation:

- View M_ϵ^3 as circle fibration $S^1_{\epsilon_1} \times S^1_{\epsilon_2} \to I$, where I: interval.
- View $\mathcal{G}_C$ as coming from six-dimensional $(2,0)$-theory on $M^4 \times C$ by compactification on C.
- Compactify first on $S^1_{\epsilon_1} \times S^1_{\epsilon_2}$, then on $C \Rightarrow$ 2d sigma model on $\mathbb{R} \times I$ with target $\mathcal{M}_H(C)$: Hitchin moduli space. $\mathcal{M}_H(C) \simeq \mathcal{M}_{\text{flat}}(C) \supset \mathcal{T}$.
- Effect of Ω-deformation $\Rightarrow$ boundary conditions $\mathcal{B}_{\epsilon_1}$, $\mathcal{B}_{\epsilon_2}$ at ends of interval I.
- $\mathcal{A}_{\epsilon_i}^{\text{loop}} \simeq \text{Hom}(\mathcal{B}_{\epsilon_i}, \mathcal{B}_{\epsilon_i})$, $\mathcal{H}^{\text{top}}(M_\epsilon^3) \simeq \text{Hom}(\mathcal{B}_{\epsilon_1}, \mathcal{B}_{\epsilon_2}) \simeq \mathcal{H}_{\mathcal{T}}$.

On the derivations II

We know for a while how to quantize $\mathcal{T}(C) \subset \mathcal{M}_{\text{flat}}(C)$ using Penner-Fock-Goncharov coordinates (Fock, Chekhov; Kashaev).

What we really need is quantization of **Fenchel-Nielsen** (Nekrasov-Rosly-Shatashvili) coordinates – quantization of these coordinates $\mapsto$ generators L_γ of $\mathcal{A}_\epsilon$.

Quantization worked out with G. Vartanov. **Example:** $C = C_{0,4}$. Relations:

$$
\begin{aligned}
\text{(i)}\quad & e^{\pi i b^2} L_s L_t - e^{-\pi i b^2} L_t L_s \\
& \qquad = (e^{2\pi i b^2} - e^{-2\pi i b^2}) L_u + (e^{\pi i b^2} - e^{-\pi i b^2})(L_1 L_3 + L_2 L_4)\,. \\
\text{(ii)}\quad & W_{0,4}(L_s, L_t, L_u) = 0,
\end{aligned}
$$

(– generalized **'t Hooft commutation relations** –), where

$$
\begin{aligned}
W_{0,4}(L_s, L_t, L_u) = & -e^{\pi i b^2} L_s L_t L_u \\
& + e^{2\pi i b^2} L_s^2 + e^{-2\pi i b^2} L_t^2 + e^{2\pi i b^2} L_u^2 \\
& + e^{\pi i b^2} L_s (L_3 L_4 + L_1 L_2) + e^{-\pi i b^2} L_t (L_2 L_3 + L_1 L_4) + e^{\pi i b^2} L_u (L_1 L_3 + L_2 L_4) \\
& + L_1^2 + L_2^2 + L_3^2 + L_4^2 + L_1 L_2 L_3 L_4 - \big(2\cos \pi b^2\big)^2\,.
\end{aligned}
$$

On the derivations III

Main claim,

$$\psi_\tau(l) := \langle\!\langle\, a \,|\, \tau \,\rangle\!\rangle^0_{\mathcal{G}_C} \;=\; \langle\, l \,|\, \tau \,\rangle_{\mathcal{T}}\,,$$

follows from the following **physically motivated** assumptions

(A) **Analyticity**: The wave-function $\langle\!\langle\, a \,|\, \tau \,\rangle\!\rangle^0_{\mathcal{G}_C}$ is holomorphic in τ,

(B) **proper realization of electric-magnetic duality**,

(C) **perturbative limits**.

On (B): Analytic continuation along closed path μ in coupling constant space $\mathcal{C}(\mathcal{G}_C)$
$\Rightarrow$ automorphism of the algebra $\mathcal{A}_\epsilon^{\rm loop} = \mathcal{A}_\epsilon^{\rm flat}$.

Group of automorphisms of $\mathcal{A}_\epsilon^{\rm flat}$: Mapping class group $\mathrm{MCG}(C)$.

The realization of $\mathrm{MCG}(C)$ on $\mathcal{H}_{\mathcal{T}}$ is **canonical** (!), represented by integral operators

$$(\mathsf{M}_m\psi)(l) \;=\; \int d\mu(l')\; M_m(l,l')\,\psi(l') \qquad m\in\mathrm{MCG}(C)\,.$$

The kernels can be constructed combinatorially from explicitly known building blocks.

On the derivations IV

The condition (B) therefore implies

$$\psi_{m.\tau}(l) \;=\; (\mathsf{M}_m\psi_\tau)(l) \;=\; \int d\mu(l')\; M_m(l,l')\,\psi_\tau(l')\,.$$

This fixes the monodromies of $\phi_l(\tau) \equiv \psi_\tau(l)$. (C) fixes asymptotic behavior

$$\phi_l(\tau) \;\sim\; \prod_{r=1}^{3g-3+n} q_r^{a_r^2}\,, \qquad q_r = e^{2\pi i \tau_r}\,.$$

Note that $\mathrm{MCG}(C)$ is generated from monodromies around $\partial\mathcal{T}(C)$. All monodromies can be transformed to standard form $\rightsquigarrow$ all asymptotic regions in coupling constant space $\mathcal{C}(\mathcal{G}_C)$ have a perturbative description as quiver gauge theories of class $\mathcal{S}$.

Note: Asymptotic regions of $\mathcal{C}(\mathcal{G}_C) \leftrightarrow$ pants decompositions σ.

The AGT-correspondence

Taken together: (A)-(C) $\Rightarrow$ **Riemann-Hilbert problem**:

Find functions $\phi_l(\tau)$ on $\mathcal{T}(C)$ such that

$$\left\{ \begin{aligned} &(a)\;\; \phi_l(\tau) \text{ is analytic on } \mathcal{T}(C), \\ &(b)\;\; \phi_l(m.\tau) \,=\, \int d\mu(l')\, M_m(l,l')\, \phi_{l'}(\tau)\,, \\ &(c)\;\; \phi_l(\tau) \,\sim\, \prod_r e^{2\pi i \tau_{\sigma,r} a_{\sigma,r}^2} \quad \forall \text{ pants dec. } \sigma\,. \end{aligned} \right\} \qquad \text{(RH)}$$

It remains to notice (J.T. '03) that the Riemann-Hilbert problem (A)-(C) has a unique solution,

$$\phi_l(\tau) \,=\, \mathcal{F}^{\rm Liou}(a,m,\tau,\epsilon)\,.$$

So, putting everything together, we have outlined a derivation of the AGT-correspondence in the spirit of Seiberg and Witten,

$$\mathcal{Z}^{\rm inst}(a,m,\tau,\epsilon) \,=\, \phi_l(\tau) \,=\, \mathcal{F}^{\rm Liou}(a,m,\tau,\epsilon)\,.$$

On the assumptions

The input $\mathcal{A}_\epsilon^{\text{loop}} \simeq \mathcal{A}_\epsilon^{\text{flat}}$ from (`Nekrasov, Witten`) is a powerful statement.

Supported by calculations of `Gomis, Okuda, Pestun; Ito, Okuda, Taki`

$\rightsquigarrow$ explicit results for the realization of 't Hooft loops on $\mathcal{H}^{\text{top}}(E_\epsilon^3)$.

$\mathcal{A}_\epsilon^{\text{loop}} \simeq \mathcal{A}_\epsilon^{\text{flat}}$ can probably be replaced by these results <u>and</u>

(B') **Electric-magnetic duality exchanges** $W_r^i \leftrightarrow T_r^i$.

Main lessons

- **Pestun's localization** $\rightsquigarrow$

  $\rightsquigarrow$ **effective quantum mechanics** on $\mathcal{H}_0(E^3_\epsilon) \simeq \mathcal{H}_{\mathcal{T}}$.

- Algebra $\mathcal{A}^{\mathrm{loop}}_\epsilon$ of loop operators acting on $\mathcal{H}_0(E^3_\epsilon) \sim$

  $\sim$ low-energy **"skeleton"** of gauge theory,

  $\mathcal{A}^{\mathrm{loop}}_\epsilon$ captures information on **nonperturbative effects** like dualities.

- **Representation theory** of $\mathcal{A}^{\mathrm{loop}}_\epsilon$

  $\rightsquigarrow$ **derivation** of AGT-correspondence à la Seiberg-Witten.