Black Holes to Quivers

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http://www.hri.res.in/∼sen/stringmath.pdf
http://www.mri.ernet.in/∼sen/stringmath.pdf

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Main references

J. Manschot, B. Pioline, A.S., arXiv:1103.1887, arXiv:1207.2230

Related recent references

I. Bena, M. Berkooz, J. de Boer, S. El-Showk, D. Van den Bleeken, arXiv:1205.5023

S. -J. Lee, Z. -L. Wang, P. Yi, arXiv:1205.6511, arXiv:1207.0821

Related earlier references

J. de Boer, S. El-Showk, I. Messamah, and D. Van den Bleeken, arXiv:0807.4556, arXiv:0906.0011

J. Manschot, B. Pioline, A..S., arXiv:1011.1258

A.S., arXiv:1112.2515

S. Lee, P. Yi, arXiv:1102.1729

H. Kim, J. Park, Z. Wang, P. Yi, arXiv:1107.0723

Black holes

Our objects of study: Supersymmetric black holes in N=2 supersymmetric string theories

– e.g. the ones obtained by compactifying type IIA or IIB string theory on a Calabi-Yau 3-fold CY_3.

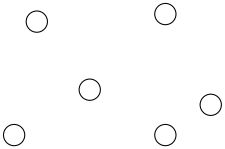
These theories have supersymmetric black hole solutions preserving 4 supersymmetries

They also have multi-centered black hole solutions preserving the same amount of supersymmetry.

Denef, hep-th/0005049; · · · Denef, Moore, hep-th/0702146

Goal: Use these black holes to learn about moduli spaces of quiver representations.

A crude picture of multi-centered black holes:



Each circle represents a single black hole.

Total charge = sum of the charges carried by all the black holes

Single black holes are like elementary particles

Multi-centered black holes are composite objects.

We define

$$\mathbf{Q}(\gamma;\mathbf{y}) = \mathbf{Tr}(-\mathbf{y})^{2J_3}$$

γ: total (electric, magnetic) charges carried by the black hole belonging to some charge lattice Γ

y: a parameter

J_3: third component of angular momentum

The trace is taken over all BPS states carrying total charge γ, after factoring out the trace over the center of mass bosonic and fermionic degrees of freedom.

$Q_S(\gamma;y)$: a similar trace over only the BPS states of a single centered black hole of charge γ.

We have a conjectured expression for $Q(\gamma; y)$ in terms of $Q_S(\alpha; y)$

Manschot, Pioline, A.S., arXiv:1103.1887

$$Q(\gamma; y) = \sum_{n\geq 1} \sum_{\substack{\alpha_1,\cdots\alpha_n\in\Gamma; m_1,\cdots m_n\geq 1 \\ \sum_i m_i\alpha_i=\gamma}} G(\alpha_1,\cdots\alpha_n; m_1,\cdots m_n; y) \prod_{i=1}^{n} Q_S(\alpha_i; y^{m_i})$$

$G(\alpha_1,\cdots\alpha_n; m_1,\cdots m_n; y)$: found by solving quantum mechanics of multiple black holes and is known (to be described later)

The dependence on the moduli of CY_3 is encoded in G.

This formula satisfies the Kontsevich-Soibelman motivic wall crossing formula with the wall crossing being controlled entirely by the function G.

$$Q(\gamma; y) = \sum_{n\geq 1} \sum_{\substack{\alpha_1,\cdots\alpha_n\in\Gamma;m_1,\cdots m_n\geq 1\\ \sum_i m_i\alpha_i=\gamma}} G(\alpha_1,\cdots\alpha_n; m_1,\cdots m_n; y) \prod_{i=1}^{n} Q_S(\alpha_i; y^{m_i})$$

Since the dependence on the moduli of CY_3 is encoded in the computable function G, this is useful for finding the dependence of $Q(\gamma; y)$ on the moduli of CY_3.

However for fixed values of the CY_3 moduli, this requires as many input variables as the ones we want to compute.

Input: $Q_S(\gamma; y) \Rightarrow$ A function of y for each γ.

Output: $Q(\gamma; y) \Rightarrow$ A function of y for each γ.

We can however do better.

Using the analysis of the near horizon geometry of black holes, and supersymmetry, one can argue that all microstates of a single centered black hole must have zero angular momentum.

A.S., arXiv:0903.1477, arXiv:1008.4209, Dabholkar, Gomes, Murthy, A.S., arXiv:1009.3226

Thus $Q_S(\alpha; y) \equiv Tr(-y)^{2J_3}$ must be independent of y.

Now we have non-trivial information even for fixed values of the moduli of CY_3.

Input: $Q_S(\gamma) \Rightarrow$ A single number for each γ.

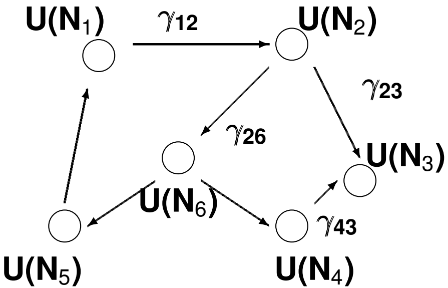
Output: $Q(\gamma; y) \Rightarrow$ A function of y for each γ.

At present we do not have an independent way of computing Q(γ; y).

As a result we cannot test our conjecture on black holes.

Quiver quantum mechanics provides a way out.

Quiver quantum mechanics



An N=4 supersymmetric gauge theory in 0+1 dimensions with $\prod_{j=1}^{K} U(N_j)$ gauge group.

γ_{jk}: number of arrows from j-th node to the k-th node describing the number of matter supermultiplets in the $(N_j, \bar{N}_k)$ representation of $U(N_j) \times U(N_k)$.

A quiver is characterized by

1. A gauge invariant superpotential W.

2. A set of real numbers $(c_1, c_2, \cdots)$ satisfying

$$\sum_j c_j N_j = 0$$

– coefficients of the Fayet-Ilioupoulos terms.

Some convention:

Instead of considering a single quiver we consider a family of quivers with a fixed set of γ_{jk}'s but different gauge groups.

Represent a quiver with gauge group $U(n_1) \times \cdots \times U(n_K)$ by a K-dimensional vector $(n_1, \cdots n_K)$.

Γ: Collection of vectors $(n_1, \cdots n_K)$ with

$$n_j \in \mathbb{Z}, \qquad n_j \geq 0$$

Basis vectors: $\gamma_j = (0, 0, \cdots 0, 1, 0, \cdots 0)$.

For any vector $\gamma \in \Gamma$, we define:

$$\mathbf{Q}(\gamma; \mathbf{y}) \equiv \mathbf{Tr}(-\mathbf{y})^{2\mathbf{J}_3}$$

for the corresponding quiver quantum mechanics.

$(\mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3)$: generators of SU(2) R-symmetry group of the quantum mechanics.

We can compute this using two different descriptions.

Higgs branch description: Effective theory of the matter multiplets, with vector multiplets integrated out.

$\mathcal{M}$: space spanned by the matter multiplet scalars $X_{jk;a}$ subject to D-term and F-term constraints:

$$\sum_{\substack{k,a \\ \gamma_{jk}>0}} X^\dagger_{jk,a}\, T^s_{(j)}\, X_{jk,a} - \sum_{\substack{k,a \\ \gamma_{kj}>0}} X^\dagger_{kj,a}\, T^s_{(j)}\, X_{kj,a} = c_j\, \mathrm{Tr}(T^s_{(j)}) \quad \forall\, j$$

$$\partial W / \partial X_{jk,a} = 0$$

$T^s_{(j)}$: generators of $U(N_j)$.

In this description, $Q(\gamma, y) = \sum_p b_p (-y)^{p-d}$

d: dimension, b_p: Betti numbers of $\mathcal{M}$.

$\mathcal{M}$ **can also be described as the moduli space of quiver representations.**

We shall assume the superpotential is generic and the space $\mathcal{M}$ is compact.

Coulomb branch description

Effective theory of vector multiplets, with matter multiplets integrated out.

Dynamics of this system is identical to that of a multi-black hole system in N=2 supersymmetric string theory with the following identification: Denef, hep-th/0206072

1. Charge lattice of the black holes $\Leftrightarrow$ the set Γ containing all quivers $(n_1, \cdots n_K)$ in a given family.

2. $Q_S(\gamma_j; y) = 1$ for the basis vectors $\gamma_j \in \Gamma$.

3. For any other $\gamma \in \Gamma$, $Q_S(\gamma; y)$ is independent of y and is an input parameter.

(In order for $Q_S(\gamma)$ to be non-zero, γ needs to satisfy certain conditions.)

Given this correspondence, we can use the general formula:

$$Q(\gamma;y)=\sum_{n\geq 1}\sum_{\substack{\alpha_1,\cdots\alpha_n\in\Gamma;m_1,\cdots m_n\geq 1\\ \sum_i m_i\alpha_i=\gamma}} G(\alpha_1,\cdots\alpha_n;m_1,\cdots m_n;y)\prod_{i=1}^{n}Q_S(\alpha_i;y^{m_i})$$

to compute the index $Q(\gamma;y)$ of quiver quantum mechanics.

– a generalization of Reineke formula for quivers with superpotentials.

This can then be compared with the result from the Higgs branch analysis.

$$Q(\gamma; y) = \sum_{n\geq 1} \sum_{\substack{\alpha_1,\cdots\alpha_n\in\Gamma;m_1,\cdots m_n\geq 1\\ \sum_i m_i\alpha_i=\gamma}} G(\alpha_1,\cdots\alpha_n; m_1,\cdots m_n; y) \prod_{i=1}^{n} Q_S(\alpha_i; y^{m_i})$$

$Q_S(\alpha)$: represent states of the quiver quantum mechanics which appear to be elementary in the Coulomb branch analysis.

Since their origin cannot be understood in the Coulomb branch, they are called ‘pure Higgs states’.

Explicit examples show that like single centered black holes, pure Higgs states are SU(2) singlets, leading to y-independence of $Q_S(\gamma; y)$.

Bena, Berkooz, de Boer, El-Showk, Van den Bleeken, arXiv:1205.5023; Lee, Wang, Yi, arXiv:1205.6511

We shall assume that this holds in general.

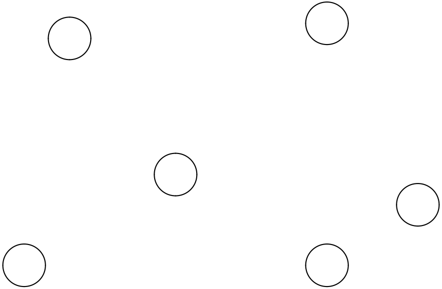
$$Q(\gamma; y) = \sum_{n \geq 1} \sum_{\substack{\alpha_1, \cdots \alpha_n \in \Gamma; m_1, \cdots m_n \geq 1 \\ \sum_i m_i \alpha_i = \gamma}} G(\alpha_1, \cdots \alpha_n; m_1, \cdots m_n; y) \prod_{i=1}^{n} Q_S(\alpha_i; \not{y}^{m_i})$$

This determines the index $Q(\gamma; y) = \sum_p b_p (-y)^{p-d}$ for the quiver in terms of the parameters $Q_S(\alpha)$.

It remains to describe the algorithm for computing the function G.

We shall begin with the most naive guess and then describe how to correct it.

Recall the picture of multi-centered black hole configuration.



If the individual centers carry charges $\alpha_1, \alpha_2, \cdots \alpha_n$ then the total charge is

$$\gamma = \alpha_1 + \alpha_2 + \cdots \alpha_n$$

Naive guess

$$Q(\gamma; y) = \sum_{n\geq 1} \sum_{\substack{\alpha_1,\cdots\alpha_n\in\Gamma \\ \sum_i \alpha_i=\gamma}} g(\alpha_1,\cdots\alpha_n; y)\prod_{i=1}^{n} Q_S(\alpha_i; \not{y})$$

$g(\alpha_1,\cdots\alpha_n; y)$: $Tr(-y)^{2J_3}$ associated with multi-centered black hole quantum mechanics.

For n=1, $g(\alpha_1; y) = 1$.

For $n \geq 2$, $g(\alpha_1,\cdots\alpha_n; y)$ can be computed using localization.

1. Introduce a symplectic product $\langle \alpha, \beta \rangle$ between elements of Γ using

$$\langle \gamma_j, \gamma_k \rangle = \gamma_{jk} = -\langle \gamma_k, \gamma_j \rangle$$

2. Find the extrema 'p' of

$$V(\{x_i\}) = -\sum_{1 \le i < j \le n} \alpha_{ij}\, \mathrm{sign}(x_j - x_i) \ln |x_j - x_i| - \sum_{i=1}^{n} b_i x_i, \;\; x_i \in \mathbb{R}$$

w.r.t. $x_2, \cdots x_n$ at fixed x_1.

$$b_i = \sum_j A_j^{(i)} c_j \text{ if } \alpha_i = \sum_j A_j^{(i)} \gamma_j, \quad \alpha_{ij} \equiv \langle \alpha_i, \alpha_j \rangle \,.$$

3. Then

$$g(\alpha_1, \ldots, \alpha_n; y) = (-1)^{\sum_{i<j} \alpha_{ij} + n - 1} \left[(y - y^{-1})^{1-n} \sum_p s(p)\, y^{\sum_{i<j} \alpha_{ij}\, \mathrm{sign}[x_j - x_i]} \right]$$

$$s(p) = \mathrm{sign}\, \det M, \quad M \equiv \partial^2 V / \partial x_i \partial x_j$$

Example: Consider the case $n = 3$ with

$$\alpha_{12} = -2, \quad \alpha_{23} = 3, \quad \alpha_{31} = -3, \quad b_1 = 1, \quad b_2 = 2 \quad b_3 = -3$$

One finds that extrema of V exist for the arrangement:

$$x_2 < x_1 < x_3, \quad x_3 < x_1 < x_2, \quad x_1 < x_3 < x_2, \quad x_2 < x_3 < x_1$$

Corresponding $s(p): \quad +, \quad +, \quad -, \quad -,$ leading to

$$\begin{aligned} g(\alpha_1, \alpha_2, \alpha_3; y) &= (y - y^{-1})^{-2}(y^8 + y^{-8} - y^2 - y^{-2}) \\ &= (y^2 + 1 + y^{-2})(y^4 + y^2 + 1 + y^{-2} + y^{-4}) \end{aligned}$$

– gives $\mathrm{Tr}(-y)^{2J_3}$ for the 3-centered quantum mechanics.

But life is not always so simple.

Consider the case

$$\alpha_{12} = a > 0, \quad \alpha_{23} = b > 0, \quad \alpha_{31} = c > 0, \quad b_1 = 1, \quad b_2 = 2 \quad b_3 = -3$$

with a,b,c satisfying triangle inequality.

One finds that extrema of V exist for the arrangement:

$$x_1 < x_2 < x_3, \quad x_3 < x_2 < x_1$$

Corresponding $s(p)$: $+, \quad +$, leading to

$$g(\alpha_1, \alpha_2, \alpha_3; y) = (-1)^{a+b+c}(y - y^{-1})^{-2}(y^{a+b-c} + y^{-a-b+c})$$

This is not a polynomial in y, y^{-1} and hence cannot be interpreted as $Tr(-y)^{2J_3}$ for the supersymmetric quantum mechanics of 3 centers.

Source of the problem:

There exist singular extrema of V where all the centers approach each other.

– scaling solution. Denef, Moore, hep-th/0702146

Thus we must correct the previous result by including the contribution from this singular configuration.

Prescription: Include in the sum some additional terms of the form $\sum_k A_k y^k$ with least possible power of y that repairs the damage.

e.g. for $a + b - c$ even

$$g(\alpha_1, \alpha_2, \alpha_3; y) \Rightarrow (-1)^{a+b+c}(y - y^{-1})^{-2}(y^{a+b-c} + y^{-a-b+c} - 2)$$

For $a + b - c$ odd

$$g(\alpha_1, \alpha_2, \alpha_3; y) \Rightarrow (-1)^{a+b+c}(y - y^{-1})^{-2}(y^{a+b-c} + y^{-a-b+c} - y - y^{-1})$$

Proposed systematic modification: First step

$$\mathbf{Q}(\gamma;\mathbf{y}) = \sum_{n\geq 1} \sum_{\substack{\alpha_1,\cdots\alpha_n\in\Gamma \\ \sum_i \alpha_i=\gamma}} \mathbf{g}(\alpha_1,\cdots\alpha_n;\mathbf{y}) \prod_{i=1}^{n} \left[\mathbf{Q_S}(\alpha_i;\not{\mathbf{y}}) + \mathbf{Q_{scaling}}(\alpha_i;\mathbf{y})\right]$$

$$\mathbf{Q_{scaling}}(\alpha;\mathbf{y}) = \sum_{m} \sum_{\beta_i\in\Gamma,\sum_i\beta_i=\alpha} \mathbf{H}(\beta_1,\cdots\beta_m;\mathbf{y}) \prod_{i=1}^{m} \mathbf{Q_S}(\beta_i;\not{\mathbf{y}})$$

$\mathbf{H}(\beta_1,\cdots\beta_m;\mathbf{y})$: A function which is fixed uniquely be demanding that

1. $\mathbf{H}(\beta_1,\cdots\beta_m;\mathbf{y}) = \mathbf{H}(\beta_1,\cdots\beta_m;\mathbf{y}^{-1})$

2. $\mathbf{H}(\beta_1,\cdots\beta_m;\mathbf{y}) \to 0$ as $\mathbf{y} \to 0,\infty$ (minimal modification).

3. In $\mathbf{Q}(\gamma;\mathbf{y})$, the coefficient of $\prod_i \mathbf{Q_S}(\alpha_i;\not{\mathbf{y}})$ is a polynomial in $\mathbf{y},\mathbf{y}^{-1}$ for every $\{\alpha_i\}$.

Example

$$
\begin{aligned}
Q(\alpha_1+\alpha_2+\alpha_3; y) = & \Big[Q_S(\alpha_1+\alpha_2+\alpha_3; \not{y}) \\
& + H(\alpha_1, \alpha_2, \alpha_3; y) \prod_{i=1}^{3} Q_S(\alpha_i; \not{y}) \Big] \\
& + g(\alpha_1, \alpha_2, \alpha_3; y) \prod_{i=1}^{3} Q_S(\alpha_i; \not{y}) \\
& + \cdots
\end{aligned}
$$

$H(\alpha_1, \alpha_2, \alpha_3; y)$ has to compensate for the lack of polynomiality of g.

Complication 2: Identity of particles

If among the constituents some α_i's are identical then we have to take into account the effect of symmetrization.

e.g. a system of two identical bosons, each of which exists in Q states, has $Q(Q+1)/2$ states.

A system of two identical fermions, each of which exists in Q states, has $Q(Q-1)/2$ states.

In dealing with this problem we shall first ignore the effect of scaling configurations.

Results

Manschot, Pioline, A..S., arXiv:1011.1258

We can use Maxwell-Boltzman statistics instead of Bose or Fermi statistics if we replace $Q(\gamma; y)$ and $Q_S(\gamma; \not{y})$ by

$$\bar{Q}(\gamma; y) \equiv \sum_{m\geq 1, m|\gamma} \frac{1}{m} \frac{y - y^{-1}}{y^m - y^{-m}} Q(\gamma/m, y^m)$$

$$\bar{Q}_S(\gamma; y) \equiv \sum_{m\geq 1, m|\gamma} \frac{1}{m} \frac{y - y^{-1}}{y^m - y^{-m}} Q_S(\gamma/m; \not{y}^m)$$

Modified formula:

$$\bar{Q}(\gamma; y) = \sum_{n\geq 1} \sum_{\substack{\alpha_1, \cdots \alpha_n \in \Gamma \\ \sum_i \alpha_i = \gamma}} \left\{ \prod_k \frac{1}{s_k!} \right\} g(\alpha_1, \cdots \alpha_n; y) \prod_{i=1}^{n} \bar{Q}_S(\alpha_i; y)$$

if among the set $\{\alpha_1, \ldots \alpha_n\}$, s_1 are identical to each other, another s_2 are identical to each other etc.

We get the final formula by combining the two.

$$\bar{Q}(\gamma; y) \equiv \sum_{m\geq 1, m|\gamma} \frac{1}{m} \frac{y - y^{-1}}{y^m - y^{-m}} Q(\gamma/m, y^m)$$

$$= \sum_{n\geq 1} \sum_{\{\alpha_i \in \Gamma\}, \sum_{i=1}^n \alpha_i = \gamma} \frac{1}{\prod_k s_k!} g\left(\alpha_1, \alpha_2, \cdots, \alpha_n; y\right)$$

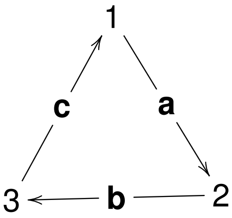
$$\prod_{i=1}^{n} \left\{ \sum_{\substack{m_i \geq 1 \\ m_i | \alpha_i}} \frac{1}{m_i} \frac{y - y^{-1}}{y^{m_i} - y^{-m_i}} \left[Q_S(\alpha_i/m_i; \not y^{m_i}) + Q_{scaling}(\alpha_i/m_i; y^{m_i}) \right] \right\},$$

$$Q_{scaling}(\alpha; y) = \sum_{\substack{\{\beta_i \in \Gamma\}, \{m_i \geq 1\} \\ m_i \geq 1, \sum_i m_i \beta_i = \alpha}} H(\{\beta_i\}; \{m_i\}; y) \prod_i Q_S(\beta_i; \not y^{m_i})$$

$H(\{\beta_i\}; \{m_i\}; y)$: determined by requiring the coefficient of $\prod_i Q_S(\beta_i; \not y^{m_i})$ in the full expression for $Q(\gamma; y)$ to be a polynomial in y, y^{-1}.

We have checked this in many examples by comparing the Higgs branch and Coulomb branch results for $\mathbf{Q}(\gamma;\mathbf{y})$.

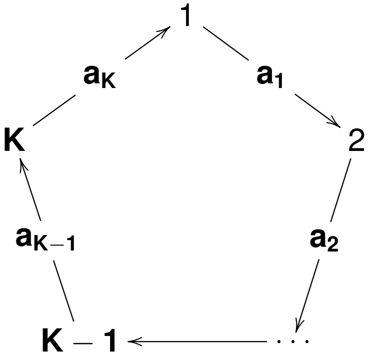
1. Arbitrary $U(1)^3$ quiver



$$-\infty < \mathbf{a}, \mathbf{b}, \mathbf{c} < \infty$$

Also in Bena, Berkooz, de Boer, El-Showk, Van den Bleeken, arXiv:1205.5023

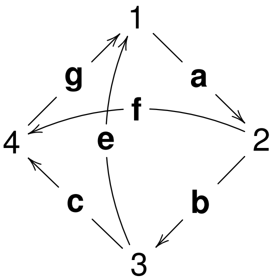
2. Arbitrary cyclic $U(1)^K$ quivers



$$0 < a_1, a_2, \cdots a_K < \infty$$

Also in Lee, Wang, Yi, arXiv:1205.6511, arXiv:1207.0821

3. A family of $U(1)^4$ quivers with two loops.

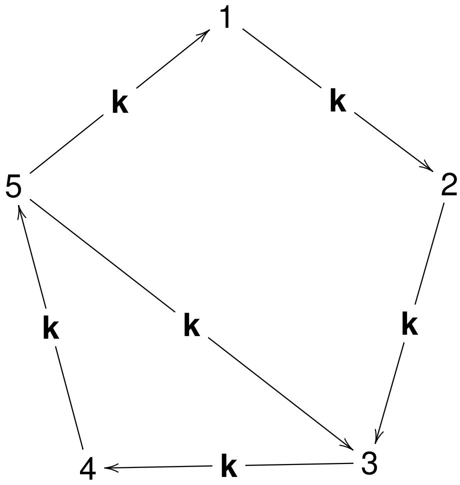


$\text{Case 1} : a = 3k, b = 4k, c = 7k , g = 4k , e = 5k, f = 4k$

$\text{Case 2} : a = 15k, b = 20k, c = 35k , g = 10k , e = 5k, f = 2k$

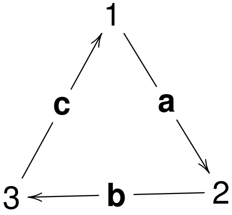
k is an arbitrary positive integer.

4. A family of $U(1)^5$ quivers with two loops.



k is an arbitrary positive integer.

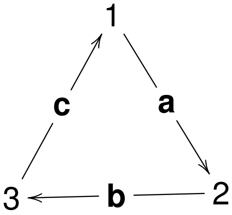
5. A family of $U(1) \times U(1) \times U(N)$ quivers



$$a, b, c > 0, \quad (c-b)N < a < (c-b)N + 2b\ , \quad b < c$$

Explicit comparison with the Higgs branch result has been done only for the rank (1,1,2) case.

6. A family of $U(1) \times U(2) \times U(2)$ quiver



$$a, b, c > 0, \quad a = 3k, \quad b = 2k, \quad c = 3k$$

Lessons

The agreement with the quiver results vindicates the general relation between the single centered index and the full index including contribution from multi-centered black holes.

Furthermore the y-independence of Q_S for the quiver provides us with an indirect evidence for y-independence of Q_S for black holes.

$\Rightarrow$ black holes are structureless at the horizon scale carrying strictly zero angular momentum.

For quivers this analysis gives a unified way of computing the betti numbers

The unknown constants $Q_S(\alpha_i)$'s can be calculated in terms of the Euler character of the quiver moduli space for which there exist simple methods.

A conjecture, generalizing this formula for computing Hodge numbers, also exists.

Perhaps eventually this result will be applicable to more general mathematical objects going beyond quivers.