$\mathcal{N}=2$ Gauge Theories, Half–Hypers, and Quivers

String–Math
Bonn, July 2012

- **S.C.**, arXiv:1203.6734.
- **S.C., & M. Del Zotto**, arXiv:1207.2275.

Powerful methods to compute BPS spectra of 4d $\mathcal{N}=2$ theories:

- 'Geometric' Methods **[Gaiotto, Moore, Neitzke]**
- 'Algebraic' Methods (Algebra/Quiver Representation Theory)

M. Alim, S.C., C. Cordova, S. Espahbodi, A. Rastogi, & C. Vafa, arXiv:1109.4941, 1112.3984

• The 'Algebraic' Method maps the computation of the BPS spectrum of a 4d $\mathcal{N}=2$ theory into a **canonical** problem in Representation Theory (RT)

• A lot of 'classical' RT facts are know (all with a direct physical interpretation) which make the problem 'easy' for most $\mathcal{N}=2$ models

• Interesting structures emerge which shed light on both physics and math

$$\{\text{BPS spectrum}\} \longrightarrow \left\{ \begin{array}{l} \text{representations of a quiver } Q, \\ \text{satisfying Jacobian relations } \partial\mathcal{W} = 0, \\ \textbf{stable} \text{ w.r.t. the central charge } Z \end{array} \right\}$$

- $\Gamma = \oplus_v \mathbb{Z} e_v$ lattice of conserved charges (electric, magnetic, flavor)
- $B_{uv} = \langle e_u, e_v \rangle_{\text{Dirac}} \in \mathbb{Z}$, *exchange matrix* of a 2–acyclic quiver Q (nodes $v \leftrightarrow e_v$, u, v connected by B_{uv} arrows $u \to v$)
radical of $B \equiv$ flavor charges
- a charge $\gamma \in \Gamma_+ \equiv \oplus_v \mathbb{Z}_+ e_v$ (positive cone) $\equiv$ dimension vector
- pair $(Q, \mathcal{W})$ not unique; Seiber duality $\equiv$ DWZ mutation classes
- central charge $Z \colon \Gamma \to \mathbb{C}$ linear with $Z(\Gamma_+) \subset \mathbb{H}$
- $X \in \text{rep}(Q, \mathcal{W})$ is **stable** iff $\forall$ subobject Y, $\arg Z(Y) < \arg Z(X)$
- X stable $\Rightarrow$ X is a *brick*: $\mathrm{End}\, X = \mathbb{C}$
- X belongs to a family of dimension $d \Rightarrow$

$$(\text{spin content of BPS supermultiplet}) = (0, \tfrac{1}{2}) \otimes \frac{d}{2}.$$

One needs the $(Q, \mathcal{W})$ class associated to the given $\mathcal{N} = 2$ theory

- arXiv:1112.3984 $G = ADE$ SQCD coupled to N_f fundamental hypers

Relatively easy: each hyper has a gauge invariant mass
$m_i \to \infty$ *decoupling limit*, $N_f \to N_f - 1$,

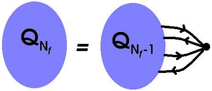
$$\text{rep}(Q_{N_f-1}, \mathcal{W}_{N_f-1}) \subset \text{rep}(Q_{N_f}, \mathcal{W}_{N_f}),$$

as a extension–closed, exact, full, controlled Abelian subcategory

Control function: $f_i : K_0\big(\text{rep}(Q_{N_f}, \mathcal{W}_{N_f})\big) \to \mathbb{Z}$: i–th flavor (dual to m_i)

- **if** $f_i(\Gamma_+) \geq 0$:

  Q_{N_f-1} full subquiver of Q_{N_f}
  (2–acyclic; map is restriction)



- Recursively we get to pure G SYM known from Type IIB engineering
- Process may be inverted using Dirac charge quantization $Q_{N_f-1} \to Q_{N_f}$

This strategy does not work for SYM coupled to **HALF**–hypermultiplets: **NO** flavor symmetry, **NO** mass parameter

Tricky theories, **on the verge of inconsistency**:
most of them quantum inconsistent,
few consistent owe their existence to peculiar *'miracles'*

If G simple and the half–hyper is in the fundamental irrepr. just one consistent example

$$G = E_7 \text{ coupled to } \tfrac{1}{2}\,\mathbf{56}$$

Other consistent half–hyper models

- $G = SU(6)$ & $\frac{1}{2}\,\mathbf{20}$
- $G = \mathrm{Spin}(12)$ & $\frac{1}{2}\,\mathbf{32}$
- $G = SU(2) \times SO(2n)$ & $\frac{1}{2}\,(\mathbf{2}, \mathbf{2n})$, $n = 2, 3, 4$
- ....

IIB engineering $\Rightarrow$ consistent QFT's & $(Q, \mathcal{W})$ exists [CS, Neitzke, Vafa]

Their existence related to Representation Theoretical 'miracles'

• Before discussing the RT 'miracles' which make HALF–hypers consistent, better to have a look to the 'ordinary' RT miracles *i.e.* the special properties of the category rep$(Q, \mathcal{W})$ corresponding to a QFT

Which categories rep$(Q, \mathcal{W})$ correspond to consistent $\mathcal{N} = 2$ QFT's?

For models having a corner in parameter space with a weakly coupled Lagrangian formulation[1], the physically most convincing argument: use $(Q, \mathcal{W})$ to compute the would–be BPS spectrum in the chamber(s) corresponding to the weak coupling corner; it should consists of two parts:

1. finitely many mutually–local states with bounded masses as $g_{\mathrm{YM}} \to 0$
2. states not local relatively to those in 1 with masses $O(1/g^2_{\mathrm{YM}})$ ('dyons')

The light states must consists of vector multiplets making *one* copy of the adjoint of G plus *finitely many* hypers in definite (quaternionic) reps. R_a of G.

If this is true the pair $(Q, \mathcal{W})$ corresponds to a theory which (in some S–duality frame) is G SQCD with quarks in the $\{R_a\}$ reprs.

[1] Assumption NOT needed, just to simplify the presentation

RT viewpoint $g_{\rm YM}\to 0$ is a decoupling limit, as was $m\to\infty$

There is an exact closed full Abelian category

$$\mathscr{L}(Q,\mathcal{W})\subset \mathrm{rep}(Q,\mathcal{W})$$

controlled by the magnetic charges

$$m\colon K_0\big(\mathrm{rep}(Q,\mathcal{W})\big)\to \mathbb{Z}^r\simeq \Gamma_{\rm coweights}(G)$$

s.t. stable objects of $\mathscr{L}(Q,\mathcal{W})\equiv$ light BPS states as $g_{\rm YM}\to 0$

Remarks & Properties

1. $\mathrm{rep}(Q,\mathcal{W})$ contains **many** ligh subcategories $\mathscr{L}$, one for each weakly coupled corner; *e.g.* $SU(2)$ $N_f=4$ a $SL(2,\mathbb{Z})$ orbit of such subcategories;
2. $m(\Gamma_+)\ngeq 0 \Rightarrow$ the light category is NOT the restriction to a subquiver, and its quiver is NOT necessarily 2–acyclic (as we shall see)
3. $\mathscr{L}(Q,\mathcal{W})$ is *tame*
4. **universality of the SYM sector**: for given gauge group G

$$\mathscr{L}(Q_{\rm SYM},\mathcal{W}_{\rm SYM})\subset\mathscr{L}(Q,\mathcal{W})$$

while finitely many bricks $X\in\mathscr{L}(Q,\mathcal{W})$ and $X\notin\mathscr{L}(Q_{\rm SYM},\mathcal{W}_{\rm SYM})$

As a warm–up, let us consider three classes of simple examples

Example 1: $SU(2)$ SQCD $N_f \leq 4$

- Full Abelian category (up to Seiberg equivalence) $\mathrm{Coh}(\mathbb{P}^1_{N_f})$ ($\mathbb{P}^1_{N_f} \equiv \mathbb{P}^1$ with N_f 'double points')
- Two quantum numbers, degree and rank

$$\text{rank} = \text{magnetic charge}, \qquad \text{degree} = \text{electric charge}$$

- light category $\mathscr{L}$ = sheaves of finite length ('skyskrapers')
- dyons = line bundles of various degree

Example 2: pure SYM with $G = ADE$

- Quiver exchange matrix fixed by Dirac charge quantization

$$B = C \otimes S, \qquad \begin{cases} C \text{ Cartan matrix of } G, \\ S \text{ modular } S\text{–matrix} \end{cases}$$

$$\text{Ex}: G = SU(6) \qquad \begin{array}{ccccccccc} \alpha_1^{(1)} & \longleftarrow & \alpha_2^{(2)} & \longrightarrow & \alpha_3^{(1)} & \longleftarrow & \alpha_4^{(2)} & \longrightarrow & \alpha_5^{(1)} \\ \Big\downarrow\!\Big\downarrow & & \Big\uparrow\!\Big\uparrow & & \Big\downarrow\!\Big\downarrow & & \Big\uparrow\!\Big\uparrow & & \Big\downarrow\!\Big\downarrow \\ \alpha_1^{(2)} & \longrightarrow & \alpha_2^{(1)} & \longleftarrow & \alpha_3^{(2)} & \longrightarrow & \alpha_4^{(1)} & \longleftarrow & \alpha_5^{(2)} \end{array}$$

- Consistency of Higgs $G \to SU(2)_i \times U(1)^{r-1}$ at weak coupling

$$X \in \mathscr{L}^{\rm YM}(G) \;\Rightarrow\; X\big|_{\upuparrows_i} \in \mathscr{L}^{\rm YM}(SU(2))$$

true mathematical theorem for the corresponding Abelian categories !!

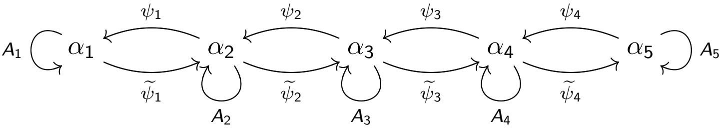
$\Rightarrow$ $X \in \mathscr{L}^{\rm YM}(G) \Rightarrow$ in each pair of $\upuparrows$ we set one arrow to 1

$$\Longrightarrow \quad \mathscr{L}^{\mathrm{YM}}(G) = \mathsf{rep}(Q', \mathcal{W}')$$

Q' **double** of the G Dynkin graph with loops Φ_v at the nodes (the '$\mathcal{N}=2$ quiver')

$$\mathcal{W}' = \sum_{a\text{: arrows}\in G} \mathrm{tr}\Big(\widetilde{\psi}_a \Phi_{t(a)} \psi_a - \psi_a \Phi_{h(a)} \widetilde{\psi}_a\Big)$$

$\mathrm{Ex}: \ G = SU(6)$

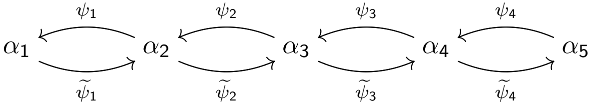


$$\ell\colon (X_{\alpha_1}, X_{\alpha_2}, \cdots, X_{\alpha_r}) \mapsto (A_1 X_{\alpha_1}, A_2 X_{\alpha_2}, \cdots, A_r X_{\alpha_r})$$

$\ell \in \mathrm{End}\, X$ hence X a brick $\Rightarrow A_i = \lambda \in \mathbb{C}\ \forall\, i$ (in facts, $\lambda \in \mathbb{P}^1$)

Fixing $\lambda \in \mathbb{P}^1$, X representation of the double $\overline{G}$ of the Dynkin graph

Ex : $\overline{A_5}$



with relations

$$\sum_{t(a)=v} \psi_a \widetilde{\psi}_a - \sum_{h(a)=v} \widetilde{\psi}_a \psi_a = 0$$

the Gelfand–Ponomarev **preprojective algebra** of the graph G, $\mathcal{P}(G)$

- [Gelfand & Ponomarev] L a graph
  $\dim \mathcal{P}(L) < \infty$ if and only if L is a Dynkin graph
- [Crawley–Boevey] $C_L = 2 - I_L$ Cartan matrix of the graph L,
  $X \in \operatorname{mod} \mathcal{P}(L)$ then

$$2 \dim \mathrm{End}\, X = (\dim X)^t C_L (\dim X) + \dim \mathrm{Ext}^1(X, X)$$

- [Lusztig] X indecomposable, $\dim \mathcal{M}(X) = \frac{1}{2} \dim \mathrm{Ext}^1(X, X)$

$\Rightarrow$ X a brick of $\mathcal{P}(G)$, $\dim X$ is a positive root of G and rigid

$\Rightarrow$ X a brick of $\mathscr{L}^{\mathrm{YM}}(G)$, $\dim X$ is a positive root of G and $\mathcal{M}(X) = \mathbb{P}^1$

$\Rightarrow$ the BPS states which are stable and have bounded mass as $g_{\mathrm{YM}} \to 0$ are **vector–multiplets in the adjoint of G**

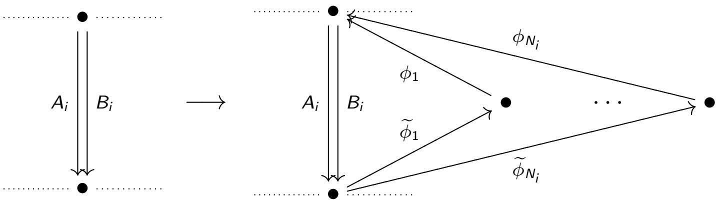
• a more detailed analysis shows that there is precisely one copy in ANY weakly coupled chamber

• in particular, this shows that the CNV identification of $(Q, \mathcal{W})$ is correct

Example 3: $G = ADE$ SQCD with N_i full hypers in the representation $F_i = [0, \cdots, 0, 1, 0, \cdots, 0]$ $(i = 1, 2, \ldots, r)$

M. Alim, S.C., C. Cordova, S. Espahbodi, A. Rastogi, & C. Vafa, arXiv:1112.3984

One replaces the i–th subquiver $\downdownarrows$ of the pure G SYM quiver as



$$\mathcal{W} \longrightarrow \mathcal{W}_{\rm SYM} + \sum_{a=1}^{N_i} \mathrm{tr}\big[(\alpha_a\, A_i - \beta_a\, B_i)\phi_a\, \widetilde{\phi}_a\big],$$

$$(\alpha_a, \beta_a) \equiv \lambda_a \in \mathbb{P}^1 \text{ pairwise distinct}$$

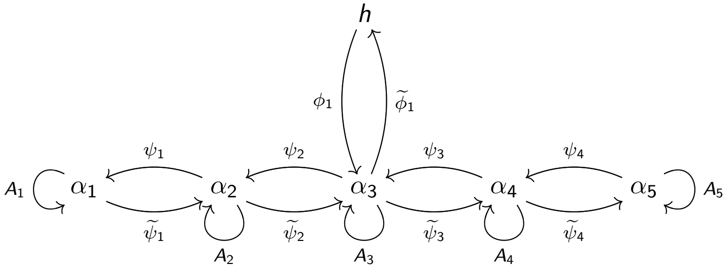
$\dim \ker B = N_i$: flavor charges (corresponding to the added nodes)

The light category $\mathscr{L} = \text{rep}(Q', \mathcal{W}')$ where

• Q' is the double of the graph $G[i, N_i]$ obtained by adding N_i extra nodes to the Dynkin graph G connected with a single hedge to the i–th node of G with loops only at all 'old' nodes of G

• $\mathcal{W}' = \mathcal{W}'_{\text{SYM}} + \sum_a \text{tr}\big[(\alpha_a A_i - \beta_a)\phi_a \widetilde{\phi}_a\big]$

Ex : $G = SU(6)$ with $N_3 = 1$ (one hyper h in the **20**)



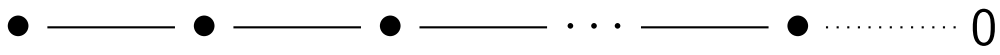
X a brick $\Rightarrow A_i = \lambda \in \mathbb{P}^1$,

• λ generic (*i.e.* $\lambda \neq \lambda_a$, $a = 1, 2, \dots, N_i$) Higgs fields $\phi_a, \widetilde{\phi}_a$ massive $\rightarrow$ integrate out
$\Rightarrow X$ is a brick of $\mathcal{P}(G) \Rightarrow$ its charge vector is a positive root of G
$\Rightarrow W$–bosons in the adjoint

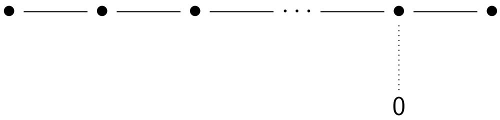
• $\lambda = \lambda_a$, then X is a brick of the preprojective algebra $\mathcal{P}(G[i,1])$. Right properties (finitely many, rigid, in right reprs. of G) if and only if $G[i,1]$ is also a Dynkin graph. Then

Theorem *(1) Consider $\mathcal{N} = 2$ SYM with simple simply–laced gauge group G coupled to a hyper in a representation of the form $F_i = [0, \cdots, 0, 1, 0, \cdots, 0]$. The resulting QFT is Asymptotically Free if and only if the augmented graph obtained by adding to the Dynkin graph of G an extra node connected by a single edge to the i–th node of G is also an ADE Dynkin graph*
(2) The model has a Type IIB engineering iff, in addition, the extra node is an extension node in the extended augmented Dynkin graph $\widehat{G[i,1]}$.

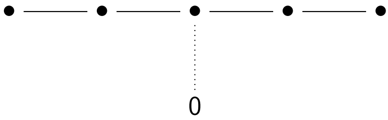
$SU(N)$ with $\mathbf{N}$



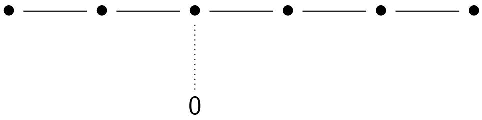
$SU(N)$ with $\mathbf{N(N-1)/2}$



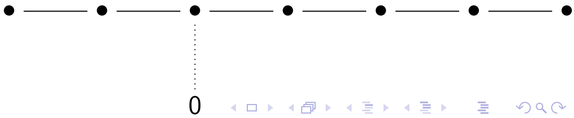
$SU(6)$ with $\mathbf{20}$



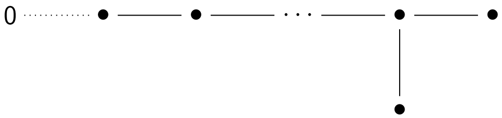
$SU(7)$ with $\mathbf{35}$



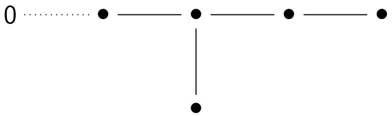
$SU(8)$ with $\mathbf{56}$



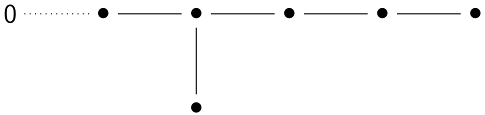
$SO(2n)$ with **2n**



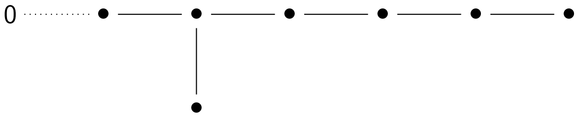
$SO(10)$ with **16**



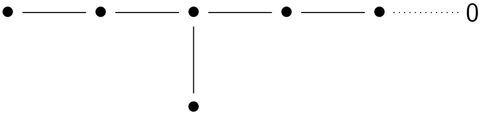
$SO(12)$ with **32**



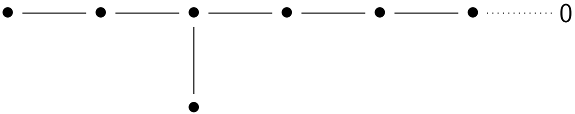
$SO(14)$ with **64**



E_6 with **27**



E_7 with **56**



- matter in the right representation of G since

i extension node in $\widehat{G[i,1]} \quad \Longrightarrow$

$$\mathrm{Ad}(G[i,1]) = \mathrm{Ad}(G) \oplus [0,\cdots,0,1,0,\cdots,0] \oplus \overline{[0,\cdots,0,1,0,\cdots,0]} \oplus \text{singlets}$$

end of warm–up

HALF–HYPERS

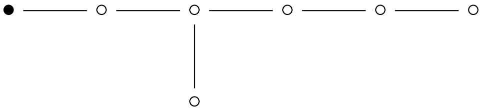
use yet another decoupling limit: *extreme Higgs*

- consider a $\mathcal{N}=2$ gauge theory with group G_r of rank r
- take a v.e.v. of the adjoint field $\langle\Phi\rangle \in \mathfrak{h}(G)$ s.t.

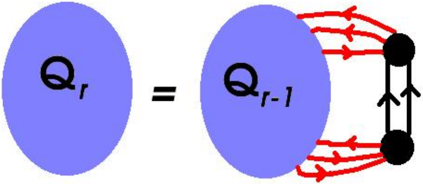
$$\alpha_a(\langle\Phi\rangle) = \begin{cases} t\, e^{i\phi}, \quad t \to +\infty & a = i \\ O(1) & \text{otherwise} \end{cases}$$

- states with electric weight ρ s.t. $\rho(\langle\Phi\rangle) = O(t)$ decouple and we remain with a gauge theory with gauge group G_{r-1} whose Dynkin diagram is obtained by deleting the i–th node from that of G (coupled to specific matter)

e.g. $G_7 = E_7$ & $\frac{1}{2}\,\mathbf{56}$ choosing $i=1 \longrightarrow G_6 = \mathrm{Spin}(12)$ & $\frac{1}{2}\,\mathbf{32}$



- again, the decoupling limit should correspond to a controlled Abelian *sub*category of rep$(Q_{G_r}, \mathcal{W}_{G_r})$
- one can choose $(Q_{G_r}, \mathcal{W}_{G_r})$ in its mutation–class and the phase ϕ so that $\lambda(\cdot)$ is *non–negative* in the positive–cone $K_0(\text{rep}(Q_G, \mathcal{W}_G))_+$
- $Q_{G_{r-1}}$ is a full subquiver of Q_{G_r} and $\mathcal{W}_{G_{r-1}}$ is just the restriction of $\mathcal{W}_{G_r}$
- the complementary subquiver is a Kronecker one
- quiver recursion of the form



- if we know the simpler quiver $Q_{G_{r-1}}$, to get Q_{G_r} we need just the (red) arrows connecting the Kronecker to $Q_{G_{r-1}}$

the red arrows are fixed by Dirac charge quantization

- by the recursion assumption, we know the representations X_{α_a} associated to all simple–root W–bosons of G_r
- under the maximal torus $U(1)^r \subset G$ they have charges

$$q_a(X_{\alpha_b}) = C_{ab}, \qquad \text{Cartan matrix}$$

- then the magnetic charges must be

$$m_a(X) = (C^{-1})_{ab} \left\langle \dim X, \dim X_{\alpha_b} \right\rangle_{\text{Dirac}}$$

- $m_a(X) \in {}^L\Gamma_{\mathrm{root}}$ for all X for a *unique* choice of red arrows
- Q_{G_r} uniquely determined, $\mathcal{W}_{G_r}$ has some higher–order ambiguity which should be fixed in a different way

• taking a suitable sequence of such Higgs decouplings

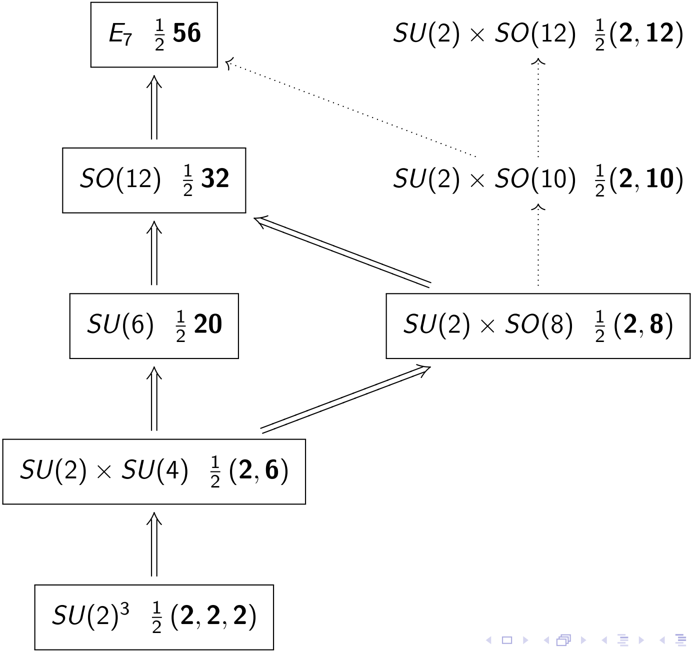
$$G_r \to G_{r-1} \to G_{r-2} \to \cdots\cdots \to G_k$$

we end up with a *complete* $\mathcal{N}=2$ having $G = SU(2)^k$ (they are essentially S–class theories of type A_1)

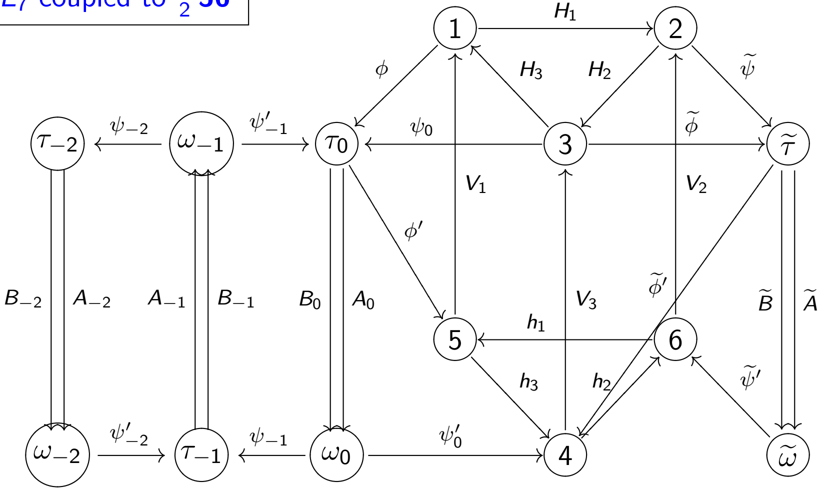
• all complete $\mathcal{N}=2$ quivers are *known* by classification (equivalently, by ideal triangulation of their Gaiotto surface)

• inverting the Higgs procedure, we get the pair $(Q_{G_r}, \mathcal{W}_{G_r})$ for the theory of interest by *'pulling back'* the pair $(Q_{\text{max comp}}, \mathcal{W}_{\text{max comp}})$ of their **maximal complete** (*i.e.* A_1) **subsector**

• for the model of interest the *'pull back'* chain is in the next slide

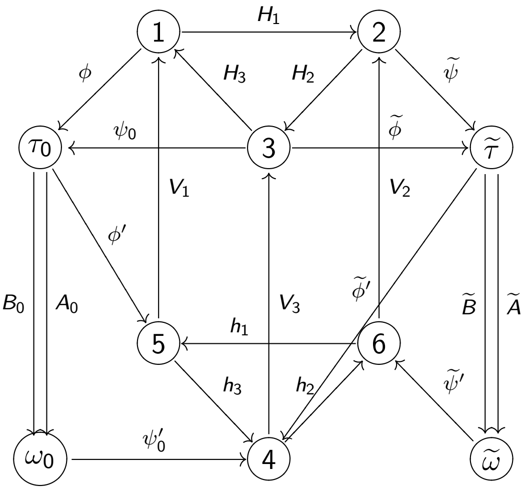

$E_7 \quad \frac{1}{2}\,\mathbf{56}$
$SU(2)\times SO(12) \quad \frac{1}{2}(\mathbf{2},\mathbf{12})$
$SO(12) \quad \frac{1}{2}\,\mathbf{32}$
$SU(2)\times SO(10) \quad \frac{1}{2}(\mathbf{2},\mathbf{10})$
$SU(6) \quad \frac{1}{2}\,\mathbf{20}$
$SU(2)\times SO(8) \quad \frac{1}{2}\,(\mathbf{2},\mathbf{8})$
$SU(2)\times SU(4) \quad \frac{1}{2}\,(\mathbf{2},\mathbf{6})$
$SU(2)^3 \quad \frac{1}{2}\,(\mathbf{2},\mathbf{2},\mathbf{2})$

E_7 coupled to $\frac{1}{2}$ **56**



$$\mathcal{W}_{E_7} = H_1H_3H_2 + h_3h_1h_2 + A\psi V_3\psi' + B\psi H_2V_2h_2\psi' + \phi V_1\phi' + \psi V_3h_3\phi' +$$
$$\phi H_3V_3\psi' B + \widetilde{A}\widetilde{\psi}V_2\widetilde{\psi}' + \widetilde{B}\widetilde{\psi}H_1V_1h_1\widetilde{\psi}' + \widetilde{\phi}V_3\widetilde{\phi}' + \widetilde{\psi}V_2h_2\widetilde{\phi}' + \widetilde{\phi}H_2V_2\widetilde{\psi}'\widetilde{B} +$$
$$+ A_0\psi'_{-1}B_{-1}\psi_{-1} - B_0\psi'_{-1}A_{-1}\psi_{-1} + A_{-1}\psi'_{-2}B_{-2}\psi_{-2} - B_{-1}\psi'_{-2}A_{-2}\psi_{-2}$$

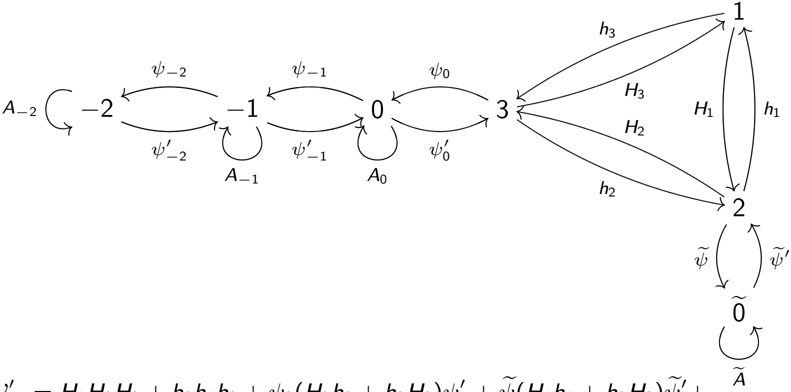
$SU(6)$ coupled to $\frac{1}{2}$ **20**



$$\mathcal{W}_{SU(6)} = H_1 H_3 H_2 + h_3 h_1 h_2 + A\psi V_3 \psi' + B\psi H_2 V_2 h_2 \psi' + \phi V_1 \phi' + \psi V_3 h_3 \phi' +$$
$$+ \phi H_3 V_3 \psi' B + \widetilde{A}\widetilde{\psi} V_2 \widetilde{\psi}' + \widetilde{B}\widetilde{\psi} H_1 V_1 h_1 \widetilde{\psi}' + \widetilde{\phi} V_3 \widetilde{\phi}' + \widetilde{\psi} V_2 h_2 \widetilde{\phi}' + \widetilde{\phi} H_2 V_2 \widetilde{\psi}' \widetilde{B}$$

• higher terms in $\mathcal{W}_G$ fixed by requiring $\mathrm{rep}(Q_G, \mathcal{W}_G)$ to contain the right light subcategory $\mathscr{L} = \mathrm{rep}(Q'_G, \mathcal{W}'_G)$ (at weak YM coupling, light vectors in one copy of $\mathrm{Ad}\, G$ plus light hypers in half the expected rep.)

e.g. E_7 & $\frac{1}{2}$ **56** Q'_{E_7}



$$\mathcal{W}'_{E_7} = H_1 H_3 H_2 + h_3 h_1 h_2 + \psi_0 (H_2 h_2 + h_3 H_3)\psi'_0 + \widetilde{\psi}(H_1 h_1 + h_2 H_2)\widetilde{\psi}' +$$
$$+ A_0 \psi_0 \psi'_0 + A_0 \psi'_{-1}\psi_{-1} - A_{-1}\psi_{-1}\psi'_{-1} + A_{-1}\psi'_{-2}\psi_{-2} - A_{-2}\psi_{-2}\psi'_{-2} + \widetilde{A}\widetilde{\psi}\widetilde{\psi}'.$$

comparison of G & $\mathbf{R}$ *vs.* G & $\frac{1}{2}\,\mathbf{R}$ (*e.g.* E_7 & **56** *vs.* E_7 & $\frac{1}{2}\,\mathbf{56}$)
shows the kind of RT 'miracles' needed for consistency at weak coupling

- bricks of $\mathrm{rep}(Q_G, \mathcal{W}_G)$ should be labelled by $\lambda \in \mathbb{P}^1$
- for $\lambda \neq 0$ isomorphic to those of $\mathcal{P}(G)$ (universality of the SYM sector)

technically

$$\mathscr{L} = \bigvee_{\lambda \in \mathbb{P}^1} \mathscr{L}_\lambda, \qquad \mathscr{L}_\lambda \simeq \mathscr{L}^{\mathrm{SYM}}(G)_\lambda, \ \text{for } \lambda \neq 0$$

- for $\lambda = 0$ 'half' the bricks of $\mathcal{P}(G[i,1])$ (matter in the $\frac{1}{2}\,\mathbf{R}$)

technically

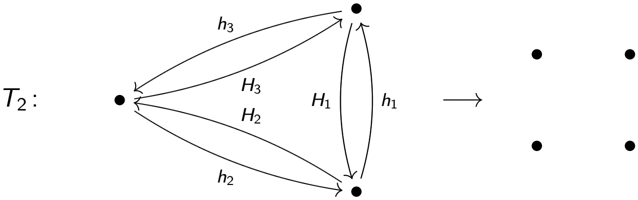
$$\mathrm{mod}\,\mathcal{P}(G[i,1]) \xrightarrow{\text{projection}} \mathscr{L}_{\lambda=0}$$

$\longrightarrow$ **superficially impossible**: the quiver Q'_G has **one less node** than $G[i,1]$, dimension vectors different rank; superficially $G[i,1] \not\subset Q'_G$

$\longrightarrow$ **consistency requires a RT 'miracle'** $G[i,1] \subset Q'_G$

'miracle' pull–back of a 'miracle' already in the complete subsector

- all quivers of the light subcategory $\mathscr{L}$ for G & $\frac{1}{2}$ **R** models contain



which corresponds to the Gaiotto A_1 theory on S^2 with three punctures

M. Alim, S.C., C. Cordova, S. Espahbodi, A. Rastogi, & C. Vafa, arXiv:1112.3984

- T_2 is **4** free hypers so the disconnected quiver on **4** nodes on the right

- this 'T_2 *duality*' produces the **extra node** we need at $\lambda = 0$

- 'T_2 *duality*' plus some very special properties of Dynkin graphs G and $G[i, 1]$ imply that — for our choice of $\mathcal{W}_G$ — the pair $(Q_G, \mathcal{W}_G)$ **has the right BPS spectrum** (and physics) at weak coupling

STRONG COUPLING

having determined the mutation class of $(Q_G, \mathcal{W}_G)$, we may study the non–perturbative physics in any regime, in particular at Strong Coupling

natural question:
'Given a G & $\frac{1}{2}\mathbf{R}$ model find its finite BPS chambers (if any)'

By the 'mutation algorithm' [M. Alim, S.C., C. Cordova, S. Espahbodi, A. Rastogi, & C. Vafa, arXiv:1112.3984] this is a purely combinatoric problem for Q_G

At the moment answers for $G = SU(2) \times SO(2n)$ coupled to $\frac{1}{2}(\mathbf{2}, \mathbf{2n})$

- they all have finite chambers

e.g. $SU(2) \times SO(6)$ & $\frac{1}{2}(\mathbf{2}, \mathbf{6})$ chambers with 21 and 27 hypers
$SU(2) \times SO(8)$ & $\frac{1}{2}(\mathbf{2}, \mathbf{8})$ chamber with 48 hypers

however based on combinatorial identities different in nature with respect to the ones for the **full**–hyper case: in that case they are 'classical' identities, whereas in the **half**–hyper we have new 'miracolous' identities

CONCLUSIONS

- the 'algebraic' approach to the 4d $\mathcal{N} = 2$ BPS spectra is an effective computational tool
- it gives explicit answers even for the trickiest theories as the half–hyper ones
- modulo some (non trivial) technicalities, once one has understood the A_1 theories, all other (quiver) $\mathcal{N} = 2$ models are also understood
- the dictionary $\mathcal{N} = 2$ QFT $\longleftrightarrow$ RT transforms well–known facts in physics into deep RT theorems, most of which unknown to the math literature. In Greg Moore's terminology, it is more *'Physical Mathematics'* than *'Mathematical Physics'*