Arithmetic of D-brane Superpotentials

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String-Math 2012, Bonn

based on: **arXiv:1201.6427**, **arXiv:1206.1787**, and *to appear*

I. Introduction

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Structure of Galois group $\mathrm{Gal}(K/\mathbb{Q})$ plays a fundamental rôle in relating GW invariants to integral (BPS, Gopakumar-Vafa) invariants of (X,L)

II. String Theory Setup

- Compactification of Type II string

$$M^{(10)} = \mathbb{R}^{3,1} \times M^{(6)}$$

with background D-branes, preserving 4 supercharges.

- Basic holomorphic observable: Space-time superpotential

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Φ: open and closed string "moduli"

We can study geometric origin of $\mathcal{W}$ and hope to calculate it in appropriate circumstances.

Superpotential serves two purposes:

1. Critical structure of $\mathcal{W}$ (in the sense of singularity theory) determines vacuum structure of the physical theory. The (only known?) holomorphic invariants are value of $\mathcal{W}$ at critical points. More precisely, tensions of BPS domain walls

$$\mathcal{T}_{ba} = \mathcal{W}(\Phi_b) - \mathcal{W}(\Phi_a)\,, \qquad \partial_\Phi \mathcal{W}|_{\Phi a,b} = 0$$

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2. In certain situations, and appropriate limits, $\mathcal{W}$ encodes degeneracy of BPS states (domain walls/solitons) (Ooguri-Vafa, Gopakumar-Vafa)

$$\mathcal{W} = \mathcal{W}(Q) = \sum_\beta n_\beta \mathcal{W}_\beta(Q)\,,$$

n_β : BPS degeneracy

$\mathcal{W}_\beta(Q) \sim \mathrm{Li}_2(Q^\beta)$

$\mathrm{Li}_2(z) = \sum_{k=1}^\infty \frac{z^k}{k^2} = -\int^z \frac{dw}{w} \ln(1-w)$

"Certain Situation:" In type II string, weak string coupling separates dynamics of open and closed strings $\rightsquigarrow$ We study

$$\mathcal{W}(z) = \mathcal{W}(z,u)|_{\partial_u \mathcal{W}=0}\,,$$

z: closed string modulus
u: open string "modulus"

"appropriate limit:" Large volume/large complex structure limit

A-model

• In type IIA on Calabi-Yau X, pick Lagrangian submanifold $L \subset X$ (or general object of Fukaya category), and wrap D4-brane on $L \times \mathbb{R}^{1,1} \subset X \times \mathbb{R}^{3,1}$.

• $\mathcal{W}$ has the form (Witten, Fukaya, Seidel)

$$\mathcal{W}(t,a) \sim \int_{L_0}^{L} (F_a + \omega)^2 + \int_{\mathcal{D}} \omega + \sum_{\text{hol. disks}} e^{-(\int_D \omega + \int_{\partial D} a)} + \sum_{\text{hol. spheres}} e^{-\int \omega}$$

$\rightsquigarrow$ Critical points of $\mathcal{W}_A$ are unobstructed objects of Fukaya category.

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• But: *hard to count holomorphic disks, especially in an invariant way*. Fundamental competition between "being able to count" (transversality and compactness) and "invariance" (under choices made)

• This talk: Mirror symmetry suggests that to count in invariant way, one might have to work over non-trivial extensions of $\mathbb{Q}$.

B-model

• In type IIB on mirror Calabi-Yau Y, D-branes are objects of derived category, in simplest case holomorphic vector bundle E. Superpotential is holomorphic Chern-Simons (Witten):

$$\mathcal{W}_B(z, A) = \int_Y \mathrm{Tr}\big(\frac{1}{2}A \wedge \bar{\partial}_0 A + \frac{1}{3}A \wedge A \wedge A\big) \wedge \Omega$$

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Spacetime

• Two-dimensional $\mathcal{N} = 2$ theory on $\mathbb{R}^{1,1} \subset \mathbb{R}^{3,1}$, coupled to $\mathcal{N} = 2$ supergravity in the bulk. Count: BPS solitons interpolating between critical points of $\mathcal{W}$. (Cecotti-Vafa, Ooguri-Vafa, Gaiotto-Moore-Neitzke)

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If (X, L) is mirror to (Y, E) under (homological) Mirror Symmetry, then, at the critical points,

$$\mathcal{W}_A|_{\partial\mathcal{W}_A=0} = \mathcal{W}_B|_{\partial\mathcal{W}_B=0}$$

We calculate in type IIB, and attempt enumerative interpretation in type IIA. *What if expansion coefficients turn out irrational?*

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In B-model:

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On general grounds,

$$\mathcal{W}_B|_{\partial \mathcal{W}_B = 0} = \int_\Gamma \Omega$$

where

$$\partial \Gamma = C \,, \qquad [C] = c_2(E) \in \mathrm{CH}^2(\mathrm{Y})$$

So, for purpose of computing holomorphic invariants, we may replace $D^b(Y)$ with $\mathrm{CH}^2(Y)$ (algebraic cycles modulo rational equivalence)

III. Mathematical Rationale

• Enumerative geometry of Calabi-Yau threefolds is very rich: The expected dimension of space of curves of genus g, class β, on X is 0 for all g and β.

• Mirror symmetry instructs us to relate this to *variation of Hodge structure* associated with family of different Calabi-Yau 3-folds $\mathcal{Y} \to B$

• Think of $\mathcal{Y}$ as a family of complex manifolds. Relevance of field of definition of $\mathcal{Y}$ is not clear, *a priori.*

• Non-trivial arithmetic appears in *degenerations* of $\mathcal{Y}$ and with *algebraic cycles* in higher co-dimension.

• Large complex structure ("maximal unipotent monodromy") degeneration is where mirror symmetry takes place. Algebraic cycles are provided by inclusion of D-branes.

Variation of Hodge structure

$\pi : \mathcal{Y} \to B$ family of Calabi-Yau threefolds over quasi-projective base B, extending to $\overline{\pi} : \overline{\mathcal{Y}} \to \overline{B}$ over a compactification $\overline{B}$ of B.

As Y varies over B, middle cohomology groups of Y fit together to local system $\mathcal{H}^3_{\mathbb{Z}} = R^3\pi_*\mathbb{Z}$, determining Gauss-Manin connection ∇ on $\mathcal{H}^3 = \mathcal{H}^3_{\mathbb{Z}} \otimes \mathcal{O}_B$. Hodge filtration

$$\mathcal{H}^3 = \mathcal{F}^0 \supset \mathcal{F}^1 \supset \mathcal{F}^2 \supset \mathcal{F}^3$$

is holomorphic and satisfies Griffiths transversality

$$\nabla(\mathcal{F}^p) \subset \mathcal{F}^{p-1} \otimes \omega_B$$

There is a natural extension of this structure over the boundary of B.

Maximal unipotent monodromy

Say B is one-dimensional. For $P \in \overline{B} \setminus B$ a boundary point, let M_P be monodromy of local system $\mathcal{H}^3_{\mathbb{Z}}$. P has maximal unipotent monodromy if

$$(1 - M_P)^3 \neq 0\,, \qquad (1 - M_P)^4 = 0$$

We let $\gamma_0, \gamma_1 \in H_3(Y, \mathbb{Z})$ be dual to a basis of $\mathrm{Im}((\log M_P)^3) \subset \mathrm{Im}((\log M_P)^2)$. This determines trivialization of $\overline{\mathcal{F}}^3 \to \overline{B}$ and a (multi-valued) *local coordinate*

$$t = \frac{\int_{\gamma_1} \Omega}{\int_{\gamma_0} \Omega}$$

where Ω is non-zero section of $\overline{\mathcal{F}}^3$.

Mirror Map

These structures define a q-expansion principle (the mirror map) for geometric quantities on B.

$$q := \exp(2\pi i t)$$

• For example, classical mirror symmetry is concerned with q-expansion of the "Yukawa coupling"

$$\int \nabla^3 \Omega \wedge \Omega \rightsquigarrow \kappa_B \in \mathrm{Symm}^3(T_B^*) \otimes (\mathcal{F}^3)^{-2}$$

After mirror map:

$$\widehat{\kappa}_A = \cdots + \sum \tilde{N}_d d^3 q^d$$

with *rational* genus 0 Gromov-Witten invariants $\tilde{N}_d$. Integrality:

$$\sum \tilde{N}_d q^d = \sum_d N_d \mathrm{Li}_3(q^d)\,, \qquad N_d \in \mathbb{Z}_{(\geq 0)}$$

(Kontsevich-Schwarz-Vologodsky)

Normal functions

Holomorphic, transverse sections of (Griffiths) intermediate Jacobian fibration

$$\mathcal{J}^2 = \mathcal{H}^3/(\mathcal{F}^2 + \mathcal{H}^3_{\mathbb{Z}}) \cong (\mathcal{F}^2)^*/H_3(Y,\mathbb{Z}) \to B$$

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When $\mathcal{C} \to B$ is algebraic cycle of co-dimension 2, and restricting to $\mathcal{F}^3$, we obtain truncated normal function

$$\mathcal{W}_B(z) = \int^{C_z} \Omega(z)$$

To any object in $D^b(Y)$ that deforms with Y, we can associate such a truncated normal function, then apply mirror map.

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This is precisely D-brane contribution to space-time superpotential, when open strings are on-shell. *We expect A-model interpretation.*

Conjecture

Recap

- *Around point of maximal unipotent monodromy*, mirror map gives q-expansion principle for *Hodge theoretic invariants* associated to $\mathcal{Y} \to B$

- *Mirror principle:* The coefficients of this q-expansion are symplectic invariants of X, with enumerative meaning.

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• *Mirror principle:* The coefficients of this q-expansion are symplectic invariants of X, with enumerative meaning.

• According to *Homological Mirror Symmetry*

$$D^b(Y) \cong \mathrm{Fuk}(X)$$

• Composition of Chern class map, $c_2 : D^b(Y) \to \mathrm{CH}^2(Y)$ with Abel-Jacobi $\nu : \mathrm{CH}^2(Y) \to J^2(Y)$ associates *truncated normal function* to objects in $D^b(Y)$ that deform with Y (and hence to unobstructed objects in $\mathrm{Fuk}(X)$)

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The question

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• In practice, we study *differential equations* satisfied by period $\varpi = \int_\gamma \Omega$ for $\gamma \in H_3(Y, \mathbb{Z})$,$\Omega \in \Gamma(\mathcal{F}^3)$. Picard-Fuchs equation,

$$\mathcal{L}\varpi(z) = 0$$

has *rational coefficients* over $B \ni z$. $\rightsquigarrow$ $\varpi(z)$ are *transcendental functions*, but expansion coefficients around large complex structure are a priori *rational*.

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with $f(z)$ *algebraic* over B. This extension splits around large complex structure in groups each governed by an algebraic number field (Newton-Puiseux expansion).

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• Fourier coefficients of $\mathcal{W}_A$ will take values in algebraic number field. What is their enumerative meaning?

Conjecture: *The A-model (q-)expansion of the truncated normal function associated with an algebraic cycle takes the form*

$$\widehat{\mathcal{W}}_A(q) = const. + \sum_{d>0} \tilde{n}_d q^{d/r}$$

where $r \in \mathbb{Z}_{>0}$ *(Puiseux index) and* $\tilde{n}_d \in K$ *for some algebraic number field* K*, with* $d^2\tilde{n}_d$ *singular at most at the discriminant of* K*.*

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In the expansion

$$\widehat{\mathcal{W}}_A(q) = \sum_d L_D(n_d; q^{d/r})$$

the $n_d \in K$ *are singular at most at the discriminant. Here, the D-logarithm*

$$L_D(n; z) = \sum_{k>0} \frac{\sigma_k(n)}{k^2} z^k$$

is an analytic function that may be attached to any such algebraic number n*. The coefficients* $\sigma_k(n)$ *are determined* $\bmod k^2$ *by the action of the Galois group on* n*.*

The D-logarithm

• Let $n \in \mathcal{O}_{\overline{\mathbb{Q}}}$ be an algebraic integer. Think of n as root of polynomial $P \in \mathbb{Z}[n]$, with leading coefficient 1. Let $K = \mathbb{Q}(n)$.

• Propose to define

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• For $k = p$ (unramified) prime, we define $\sigma_p(n)$ as the lift $\mathrm{mod} p^2$ of Frobenius at p (in extension $K/\mathbb{Q}$) acting on n. (Namely, $\sigma_p(n)$ is root of $P \bmod p^2$ that agrees with $n^p \bmod p$.)

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• For k co-prime with discriminant of $K/\mathbb{Q}$, $\sigma_k(n)$ is multiplicative $\bmod k^2$.

• If $K/\mathbb{Q}$ is Galois, and $\mathfrak{p}_i$ prime above p,

$$\sigma_p(n) = \mathrm{Frob}_{\mathfrak{p}_i/p}(n) \bmod \mathfrak{p}_i$$

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• When $K/\mathbb{Q}$ is abelian extension,

$$L_D = \sum_{k=1}^{\infty} \frac{\chi(k)}{k^2} z^k$$

is (essentially) a linear combination of ordinary di-logarithms, determined by Dirichlet character χ.

• For example, when $n = \omega$ is a root of unity,

$$L_D(\omega; z) = \mathrm{Li}_2(\omega z)$$

Evidence from Examples

• First examples of open string mirror symmetry: Local (toric) models. Aganagic-Vafa, Katz-Liu, Aganagic-Klemm-Vafa, Lerche-Mayr-Warner, etc., cmp. topological vertex. Open strings not on-shell. Framing ambiguity. But: Open/closed duality might explain (rational) integrality in all cases studied so far.

• A number of compact examples have been worked out since 2007 (W., Krefl-W., Knapp-Scheidegger, Jockers-Soroush, Alim-Hecht-Mayr-Mertens, Aganagic-Beem, Klemm-Grimm-Klevers, Shimizu-Suzuki, . . .)

• All examples so far defined over rationals. Conspicuously, integrality of on-shell superpotential involves

$$\sum_{p\nmid k} \frac{1}{k^2} q^k$$

simple instance of D-logarithm (principal Dirichlet character)

The Real Quintic

- $\mathcal{Y} : \{x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 - 5\psi x_1 x_2 x_3 x_4 x_5 = 0\}/(\mathbb{Z}/5)^3$

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- (Pandharipande-Solomon-W.) $\widehat{\mathcal{W}}_A = 15 q^{1/2} + \cdots$

expansion coefficients are open (real) Gromov-Witten invariants counting holomorphic disks ending on real locus $L = \{x_i = \bar{x}_i\} \subset X$ inside generic quintic defined over $\mathbb{R}$.

- Example fits consistently into global picture of mirror symmetry.

Van Geemen lines

• More systematic approach on the (mirror) quintic: $\mathrm{CH}^2(Y)$ is filtered by degree of representative curve. $\rightsquigarrow$ Proceed degree by degree

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• Albano-Katz, van Geemen, Mustaţă: Generic member of the Dwork family of quintics has
$*$ 375 isolated coordinate lines (with $\mathcal{W} = 0$)
$*$ 2 families of lines parametrized by smooth $g = 626$ curve

• 5000 special members of family: "van Geemen lines"

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Inhomogeneity (J.W.): $\mathcal{L}\mathcal{W}_B = \frac{\sqrt{-3}}{4\pi^2}\cdot\frac{32}{45}\cdot\frac{\frac{63}{\psi^5}+\frac{1824}{\psi^{10}}-\frac{512}{\psi^{15}}}{\left(1-\frac{128}{3\psi^5}\right)^{5/2}}$

Integrality: $\mathcal{W}_A = \sum_{d>0}\tilde{n}_d q^d = \sum_{d,k>0} n_d \frac{\left(\frac{k}{3}\right)}{k^2} q^{dk}, \qquad 3^d\frac{n_d}{\sqrt{-3}}\in\mathbb{Z}$

Conics on the mirror quintic

• The generic quintic contains 609250 conics (Katz). The mirror quintic contains several families (Mustață), and many isolated conics. see arXiv:1201.6427.

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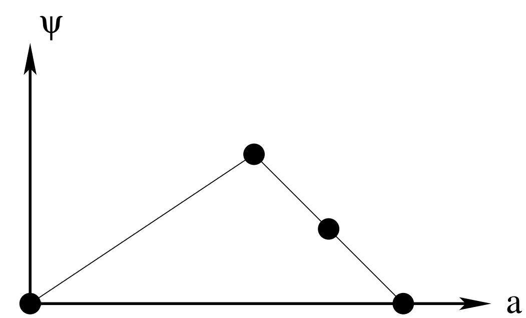
• "First component" $(a, b$ parameterize conics in $\mathbb{P}^4)$

$$\begin{aligned} 64 + 5a^3\psi^2 - 40a^4\psi + 12a^5 &= 0 \\ -128 - 5a^2\psi^3 + 40a^3\psi^2 - 12a^4\psi + 64b^2 &= 0 \end{aligned} \qquad (*)$$

As $\psi \to \infty$ the 10 branches of solutions of $(*)$ split into 2 groups, with asymptotic exponent for a given by the negative slope of the two upper segments of the Newton polygon. $a \sim \psi^\alpha$



Conics on the mirror quintic

• The generic quintic contains 609250 conics (Katz). The mirror quintic contains several families (Mustaţă), and many isolated conics. see arXiv:1201.6427.

• "First component" (a, b parameterize conics in $\mathbb{P}^4$)

$$\left.\begin{aligned} 64 + 5a^3\psi^2 - 40a^4\psi + 12a^5 &= 0 \\ -128 - 5a^2\psi^3 + 40a^3\psi^2 - 12a^4\psi + 64b^2 &= 0 \end{aligned}\right\} \quad (*)$$

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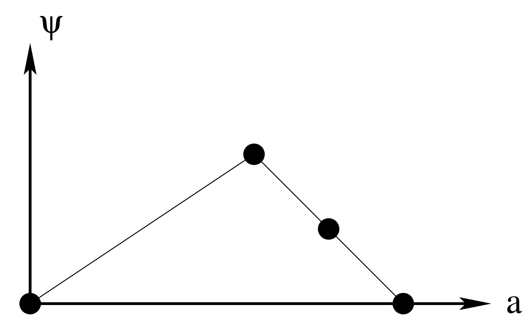

Expansion governed by $K = \mathbb{Q}(\lambda)$ with
First group: $\lambda^6 = 5^4$, ($\alpha = -2/3$); Second group: $5\lambda^4 + 20\lambda^2 - 48 = 0$, ($\alpha = 1$)
Conjecture checked!

Inhomogeneity

$$\mathcal{L} \int^{C} \Omega = f(z)$$

$$f(z) = \frac{1}{4\pi^2} \frac{b}{320\,(-128+3\psi^5)^3\,(-5308416+26104832\psi^5+459\psi^{10})^3} \cdot$$

$$\cdot[-3529208202219460015329116160-5917959309462446377556508672\,a^4\psi-2408017425167911269332680704 0\,a^3\psi^2$$
$$-3710297974941369036177408000 0\,a^2\psi^3+5322140674208202106664386560\,a\psi^4$$
$$+3770136142774746429737926656 00\,\psi^5-2236733164787881063481176227 84\,a^4\psi^6+2316204250227303666522941030 40\,a^3\psi^7$$
$$+5771733650837854501741579468 80\,a^2\psi^8+1161971462867073400022583214 080\,a\psi^9$$
$$+1138625829170016488325937889 280\,\psi^{10}-1624268140610607304875662376 96\,a^4\psi^{11}+4622000367473942874930177638 40\,a^3$$
$$+1968616622508632980846962278 40\,a^2\psi^{13}-1985672891439418898762851942 40\,a\psi^{14}$$
$$+3856789576252600100435315916 80\,\psi^{15}-1884759026743731950632336097 28\,a^4\psi^{16}+3973005574366601397250136473 60\,a^3\psi$$
$$+4688135192639453261856558284 80\,a^2\psi^{18}+4797235286751406202472628224 00\,a\psi^{19}$$
$$+3527524759284915305107685376 00\,\psi^{20}-3926307658648803777806598144 0\,a^4\psi^{21}+1107774985973213972838481920 00\,a^3\psi^2$$
$$+4223363264559961264273489920 0\,a^2\psi^{23}+1669593291399098681744424960 0\,a\psi^{24}$$
$$+5506564481958675778539356160\,\psi^{25}-2790927025434491767939399 68\,a^4\psi^{26}+8847700783212377501230694 40\,a^3\psi^{27}$$
$$+3425159727240604239790080 0\,a^2\psi^{28}-1218027340623898031947776 0\,a\psi^{29}$$
$$-7891860706457745044275200\,\psi^{30}+557447463014026659692544\,a^4\psi^{31}-1763923787950883886858240\,a^3\psi^{32}$$
$$-712237630506382474444 80\,a^2\psi^{33}+4711857482247092305920\,a\psi^{34}$$
$$+1639504965244195307520\,\psi^{35}-341394338368327357 44\,a^4\psi^{36}+1108445732793926553 60\,a^3\psi^{37}$$
$$-4645064401757907840\,a^2\psi^{38}-375748813003714560\,a\psi^{39}$$
$$-14770116391956480\,\psi^{40}+66315921005988\,a^4\psi^{41}-220588897640760\,a^3\psi^{42}+26084392488495a^2\psi^{43}+193405158000a\psi^{44}]$$

Summary

Mirror Principle:
Algebraic statements on Y translated to symplectic statements on X.

Cycle (classes) related to (classes) of Lagrangians.

Extrapolation from proven case: The real quintic

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Mirror Principle:
Algebraic statements on Y translated to symplectic statements on X.

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How do we understand **irrational** $\tilde{n}_d$, n_d *as enumerative invariants?*

Potential "Explanation"

Sigma-model path integral:

$$\mathcal{W} = \langle \cdots \rangle_{\text{disk}} = \int \mathcal{D}\phi \mathcal{D}\psi \exp(\bar{\psi}\partial\psi + R(\psi,\bar{\psi},\psi,\bar{\psi}) + \cdots),$$

receives instanton contributions

With zero modes around instanton

$$\sim \int_{\substack{\text{bosonic}\\ \text{moduli}}} (curvature)^{\#} \rightsquigarrow \text{intersection theory}, \quad \tilde{n}_d \overset{?}{=} \int_{[\overline{\mathcal{M}}]^{\text{virt}}} \mathbf{1}$$

Basic problem: Moduli space of discs has boundaries. $\rightsquigarrow$ No good intersection theory. However, path-integral will still be well-defined, and $\tilde{n}_d$ could be (symplectic) "volume" of moduli space.

Conclusions

New calculations touch upon several areas of symplectic, algebraic, arithmetic geometry, and physics

1. Algebraic cycles, $\mathrm{CH}^2(Y)$

2. Arithmetic in degeneration of VHS

3. A new arithmetic twist of di-logarithm

4. Predict existence and properties of Lagrangian submf. (objects of $\mathrm{Fuk}(X)$).

5. Believe in enumerative interpretation of invariants n_d.

6. In "physics": Structure of Galois group appears essential to understand spectrum of (2d-4d) BPS states.