Holomorphic Blocks in 3d

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Message

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- Partition function on S^3_b, $\mathcal{Z}_b[\mathcal{T}]$. [Hama-Hosomichi-Lee]
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$N = S^3_b, S^2 \times_q S^1$ can be obtained by gluing solid tori. Similarly:

$$\mathcal{Z}[N] = \sum_{\alpha} \mathcal{B}^{\alpha}(x, q)\mathcal{B}^{\alpha}(\tilde{x}, \tilde{q}),$$

where $\mathcal{B}^{\alpha}(x, q)$ are partition functions on solid tori which we call holomorphic blocks.

SUSY vacua on $\mathbb{R}^2 \times S^1_\beta$

Consider class $\mathcal{R}$ theories: gauge group $U(1)^G$, flavor group $U(1)^{N_f}$.

On $\mathbb{R}^2 \times S^1_\beta$ we have an effective 2d $\mathcal{N} = (2, 2)$ theory with twisted superpotential $\widetilde{W}(\Sigma_a, M_i)$, with twisted chiral multiplets:

$$\Sigma_a = \sigma_a + \cdots \,, \qquad M_i = m_i + \cdots$$

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- Each chiral contributes to $\widetilde{W}$ as:

$$\sum_{n\in\mathbb{Z}} \left(m_\phi + \tfrac{2\pi i n}{\beta}\right)\left[\log(\beta m_\phi + 2\pi i n) - 1\right] \simeq \frac{\beta}{4} m_\phi^2 + \frac{1}{\beta}\mathrm{Li}_2(e^{-\beta m_\phi})\,.$$

- Chern-Simons terms contribute as: $\frac{\beta}{2} k_{ab}\Sigma_a\Sigma_b + \beta\, k_{ai}\Sigma_a M_i + \frac{\beta}{2} k_{ij} M_i M_j\,.$

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SUSY vacua $s_a^{(\alpha)}(x_i)$, $\alpha = 1 \cdots N$, are determined by:

$$\exp\left(\beta s_a \frac{\partial \widetilde{W}(s_a, x_i)}{\partial s_a}\right) = 1\,,$$

where $s_a = e^{\beta\sigma_a}$, $x_i = e^{\beta m_i}$. [Nekrasov-Shatashivili], [Nekrasov-Witten]

S_b^3 partition function

Partition functions on the ellipsoid:

$$S_b^3: \quad b^2|z_1| + \frac{1}{b^2}|z_2|^2 = 1, \qquad z_1, z_2 \in \mathbb{C},$$

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Consider $\mathcal{T}_\Delta$, single chiral with $k = -1/2$ flavor Chern-Simons:

$$Z_b[\mathcal{T}_\Delta] = e^{\frac{i\pi}{2}\left(\frac{iQ}{2}-m\right)^2} s_b(\frac{iQ}{2} - m),$$

the one-loop determinant is written in terms of the $s_b(x)$ function:

$$s_b(x) = \prod_{m,n\in\mathbb{Z}_{\geq 0}} \frac{mb + nb^{-1} + \frac{Q}{2} - ix}{mb + nb^{-1} + \frac{Q}{2} + ix}\,, \qquad Q = b + 1/b\,.$$

semiclassical limits

- for $b \sim 0$ we find:

$$S_b^3 \sim \mathbb{R}^2 \times S^1 \,, \qquad Z_b[\mathcal{T}_\Delta] \sim e^{\frac{1}{\log q}\mathrm{Li}_2(x^{-1})} = e^{\frac{1}{\log q}\widetilde{W}_\Delta(x)} \,,$$

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$\Rightarrow$ We get two copies of the effective 2d $(2,2)$ theory !

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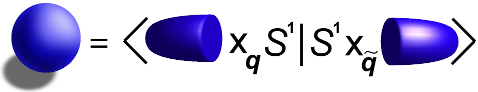
We stretch the interval out:



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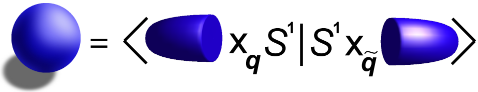

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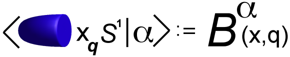

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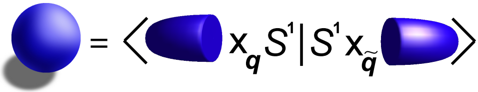

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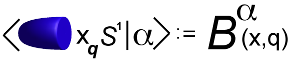
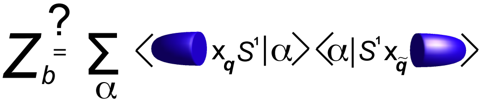


Then we should get:



$\rightarrow$ 3d version of tt^* [Cecotti-Vafa].

(vanilla) chiral factorization

The (Shintani) s_b representation:

$$Z_b[\mathcal{T}_\Delta] = \prod_{r=1}^{\infty} \frac{\left(1 - x^{-1} q^{r+1}\right)}{\left(1 - \tilde{x}^{-1} \tilde{q}^{-r}\right)}\,, \qquad |q| < 1\,,$$

is already factorized into Melvin Cigar partition functions with:

[Dimofte-Gukov-Hollands][Cecotti-Cordova-Vafa][S.P.][Dimofte-Gaiotto-GukovII]

$$\mathcal{B}_\Delta(x;q) := \prod_{r=1}^{\infty} \left(1 - x^{-1} q^{r+1}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n+1)}{2}} x^{-n}}{(q)_n}\,, \qquad |q| < 1\,.$$

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where $(q)_n := (1-q)\cdots(1-q^n)$. We then can write:

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The relation between $q = e^{2\pi i b^2} \equiv e^{2\pi i \tau}$ and $\tilde{q} = e^{2\pi i / b^2} \equiv e^{2\pi i \tilde{\tau}}$ is:

$$\tau \to \widetilde{\tau} = -S \cdot \tau = \frac{1}{\tau}\,.$$

$\rightarrow$ consistent with S-gluing + orientation reverse!

analytic properties

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$$\mathcal{B}_\Delta(x;q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n+1)}{2}} x^{-n}}{(q)_n} = \begin{cases} \prod_{n=0}^{\infty} \left(1 - q^{n+1}x^{-1}\right) & |q| < 1\,, \\ \prod_{n=0}^{\infty} \left(1 - q^{-n}x^{-1}\right)^{-1} & |q| > 1\,. \end{cases}$$

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- But of course partition functions:

$$\mathcal{Z}_b[\mathcal{T}] = ||\cdots||_S^2$$

  make perfect sense on the unit circle!

vortex-vortex (intermediate) factorization

For class $\mathcal{R}$ theories (experimentally) we find: [S.P.]

$$\mathcal{Z}_b[\mathcal{T}] = \sum_{\alpha=1}^{N} Z_{cl}^{\alpha} \cdot \left(Z_{1loop}^{\alpha}(x,q) \cdot Z_V^{\alpha}(x,q) \right) \cdot \left(Z_{1loop}^{\alpha}(\tilde{x},\tilde{q}) \cdot Z_V^{\alpha}(\tilde{x},\tilde{q}) \right)$$

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- α labels SUSY vacua of the effective 2d theory.

example: $T[SU(2)]$ factorization

$U(1)$ theory with $N_f = 2$, vector mass m, axial mass μ and FI ξ:

$$\mathcal{Z}_b[\mathcal{T}] = \frac{1}{s_b(\mu)} \int ds \; e^{2\pi i s \xi} \; \frac{s_b(s \pm m/2 + \mu/2 + iQ/4)}{s_b(s \pm m/2 - \mu/2 - iQ/4)}$$

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$\rightarrow$ evaluate the integral by taking residues of s_b functions. There are two sets of poles corresponding to the two SUSY vacua ($\alpha = \pm$). The result is:

$$Z_{\text{cl}}^{\pm} = e^{\frac{1}{2\log q}\left(\log y \log z \pm \log z (\log y - \log z) + (\log y \pm \log x)(i\pi + \frac{\log q}{2})\right)},$$

$$Z_{1loop}^{\pm} = \frac{(q\; x^{\pm})_\infty}{(-q^{1/2}z)_\infty(-q^{1/2}z^{-1})_\infty(-q^{1/2}z^{-1}x^{\pm})_\infty},$$

$$Z_V^{\pm} = \sum_n \frac{(-q^{1/2}z^{-1})_n(-q^{1/2}z^{-1}x^{\pm})_n}{(q)_n(q\; x^{\pm})_n}\left(-q^{1/2}z\; y^{-1}\right)^n =$$
$$= {}_2F_1(-q^{1/2}z^{-1}, -q^{1/2}z^{-1}x^{\pm}, q\; x^{\pm}; -q^{1/2}z\; y^{-1}),$$

where:

$$q = e^{2\pi i b^2}, \quad (a)_n := \prod_{r=0}^{n-1}(1 - aq^r), \quad x = e^{2\pi b m}, \quad z = e^{2\pi b \mu}, \quad y = e^{2\pi b \xi}.$$

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$$e.g.: \qquad e^{\left(-\frac{(\log x)^2}{2\log q}+\frac{\log q}{24}-\frac{\pi^2}{6\log q}\right)} = \theta(x;q)\theta(\tilde{x};\tilde{q})$$

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This allows to write uniquely (up to elliptic functions):

$$Z_{cl}^{\alpha} = \left\| \frac{\theta(x,q)\cdots}{\theta(x',q)\cdots} \right\|_S^2 .$$

block-block factorization

Including theta functions (background Chern-Simons terms) we find:

$$\mathcal{Z}_b[\mathcal{T}] = \sum_{\alpha=1}^{N} \mathcal{B}_{\mathcal{T}}^{\alpha}(x, q)\mathcal{B}_{\mathcal{T}}^{\alpha}(\tilde{x}; \tilde{q}) = \sum_{\alpha=1}^{N} ||\mathcal{B}_{\mathcal{T}}^{\alpha}(x, q)||_S^2$$

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- In the semiclassical limit, $q = e^{\beta\epsilon}$, $\epsilon \to 0$, finite β, we find:

$$\mathcal{B}^{\alpha}_{\mathcal{T}}(x,q) \overset{\epsilon\to 0}{\sim} \exp\left[\frac{1}{\epsilon}\widetilde{W}(s^{\alpha}(x);x)\right]$$

  where $\widetilde{W}(s^{\alpha}(x);x)$ is the twisted superpotential on $\mathbb{R}^2 \times S^1_{\beta}$, so we see how holomorphic blocks are related to SUSY vacua.

block-block factorization

Including theta functions (background Chern-Simons terms) we find:

$$\mathcal{Z}_b[\mathcal{T}] = \sum_{\alpha=1}^{N} \mathcal{B}^{\alpha}_{\mathcal{T}}(x, q)\mathcal{B}^{\alpha}_{\mathcal{T}}(\tilde{x}; \tilde{q}) = \sum_{\alpha=1}^{N} ||\mathcal{B}^{\alpha}_{\mathcal{T}}(x, q)||^2_S$$

- In the semiclassical limit, $q = e^{\beta\epsilon}$, $\epsilon \to 0$, finite β, we find:

$$\mathcal{B}^{\alpha}_{\mathcal{T}}(x, q) \overset{\epsilon \to 0}{\sim} \exp\left[\frac{1}{\epsilon}\widetilde{W}(s^{\alpha}(x); x)\right]$$

  where $\widetilde{W}(s^{\alpha}(x); x)$ is the twisted superpotential on $\mathbb{R}^2 \times S^1_{\beta}$, so we see how holomorphic blocks are related to SUSY vacua.

- At finite ϵ and β instead we have:

$$\mathcal{B}^{\alpha}_{\mathcal{T}}(x, q) = \exp\left[\beta\widetilde{W}(s^{\alpha}(x); x; q)\right]$$

  where $W(s^{\alpha}(x); x; q)$ is the twisted superpotential on $\mathbb{R}^2 \times_q S^1_{\beta}$, which includes gravitational (Ω-deformation) corrections.

mirror symmetry

3d SCFT's can have multiple weakly coupled UV descriptions $\mathcal{T}^{I}$'s.

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Going from mirror frame I to frame II, partition functions stay invariant while holomorphic blocks transform linearly:

$$\mathcal{B}^{II}_{\alpha} = (M_<)_{\alpha\beta}\,\mathcal{B}^{I}_{\beta}, \qquad |q| < 1\,,$$
$$\mathcal{B}^{II}_{\alpha} = (M_>)^{-1T}_{\alpha\beta}\,\mathcal{B}^{I}_{\beta}, \qquad |q| > 1\,.$$

with conjugate matrices:

$$(M_>) = (M_<)^{-1T}\ .$$

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This ensures that:

$$\mathcal{Z}_b[\mathcal{T}^{I}] = \sum_{\alpha} ||\mathcal{B}^{I}_{\alpha}||^2_S = \sum_{\alpha} ||\mathcal{B}^{II}_{\alpha}||^2_S = \mathcal{Z}_b[\mathcal{T}^{II}]\,.$$

example: $T[SU(2)]$ mirror symmetry

There are two mirror descriptions $\mathcal{T}_I, \mathcal{T}_{II}$, with Coulomb $\leftrightarrow$ Higgs branch and real masses exchanged as:

$$(x, y, z) \leftrightarrow (y, x, z^{-1}) \, .$$

Partition function is invariant [Hama-Hosomichi-Lee]:

$$Z_b[\mathcal{T}^{II}](x, y, z) = Z_b[\mathcal{T}^{I}](y, x, z^{-1}) = Z_b[\mathcal{T}^{I}](x, y, z) \, .$$

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In terms of blocks we have:

$$Z_b[\mathcal{T}^{I}](x, y, z) = ||\mathcal{B}_1^{I}(x, y, z)||_S^2 + ||\mathcal{B}_2^{I}(x, y, z)||_S^2 \, ,$$

$$Z_b[\mathcal{T}^{II}](x, y, z) = ||\mathcal{B}_1^{II}(x, y, z)||_S^2 + ||\mathcal{B}_2^{II}(x, y, z)||_S^2 \, .$$

- Blocks

$$\mathcal{B}^{I}_{1}(x,y,z) = \frac{\theta(-q^{1/2}y\ x^{-1};q)}{\theta(-q^{1/2}\ x^{-1};q)\theta(y\ z;q)} \times \frac{(q\ x^{-1})_\infty}{(-q^{1/2}(x\ z)^{-1})_\infty} \times {}_2F_1(-q^{1/2}z^{-1};-q^{1/2}(z\ x)^{-1};q\ x^{-1};-q^{1/2}z\ y^{-1}),$$

$$\mathcal{B}^{I}_{2}(x,y,z) = \frac{\theta(-q^{1/2}y;q)\theta(x^{-1}z;q)}{\theta(-q^{1/2}x;q)\theta(y\ z;q)\theta(z;q)} \times \frac{(q\ x)_\infty}{(-q^{1/2}x\ z^{-1})_\infty} \times {}_2F_1(-q^{1/2}z^{-1};-q^{1/2}x\ z^{-1};q\ x;-q^{1/2}z\ y^{-1}).$$

- Mirror Blocks

$$\mathcal{B}^{II}_{1}(x,y,z) = \frac{\theta(-q^{1/2}y\ x^{-1};q)}{\theta(-q^{1/2}x^{-1};q)\theta(y\ z;q)} \times \frac{(q\ y^{-1})_\infty}{(-q^{1/2}z\ y^{-1})_\infty} \times {}_2F_1(-q^{1/2}z;-q^{1/2}z\ y^{-1};q\ y^{-1};-q^{1/2}(z\ x)^{-1}),$$

$$\mathcal{B}^{II}_{2}(x,y,z) = \frac{1}{\theta(z;q)} \times \frac{(q\ y)_\infty}{(-q^{1/2}y\ z^{-1})_\infty} \times {}_2F_1(-q^{1/2}z;-q^{1/2}y\ z;q\ y;-q^{1/2}(z\ x)^{-1}).$$

blocks transformation

Combining several hypergeometric identities (vital role of theta functions!) we found:

$$\begin{pmatrix} \mathcal{B}_1^{II}(x,y,z) \\ \mathcal{B}_2^{II}(x,y,z) \end{pmatrix} = \begin{pmatrix} \mathcal{B}_1^{I}(x,y,z) \\ \mathcal{B}_1^{I}(x,y,z) + \mathcal{B}_2^{I}(x,y,z) \end{pmatrix} , \qquad |q| < 1 ,$$

$$\begin{pmatrix} \mathcal{B}_1^{II}(x,y,z) \\ \mathcal{B}_2^{II}(x,y,z) \end{pmatrix} = \begin{pmatrix} \mathcal{B}_1^{I}(x,y,z) - \mathcal{B}_2^{I}(x,y,z) \\ \mathcal{B}_2^{I}(x,y,z) \end{pmatrix} , \qquad |q| > 1 .$$

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This ensures that:

$$\begin{aligned} Z_b[\mathcal{T}^{II}](x,y,z) &= ||\mathcal{B}_1^{II}(x,y,z)||_S^2 + ||\mathcal{B}_2^{II}(x,y,z)||_S^2 = \\ &= ||\mathcal{B}_1^{I}(x,y,z)||_S^2 + ||\mathcal{B}_2^{I}(x,y,z)||_S^2 = Z_b[\mathcal{T}^{I}](x,y,z) \end{aligned}$$

the superconformal index

The partition function on $S^2 \times_q S^1$ computes the superconformal index:
[Imamura-Yokoyama], [Kapustin-Willett]

$$\mathcal{I}[\mathcal{T}](m, \zeta; q) = \mathrm{Tr}_{\mathcal{H}_m} (-1)^F e^{-\beta(E-R-j_3)} q^{\frac{E+j_3}{2}} \zeta^e \,.$$

- $m = (m_1, \cdots, m_N) \in \mathbb{Z}^N$ are magnetic fluxes on S_2.
- $\zeta_1, \cdots, \zeta_N$ are fugacities for flavor charges $e = (e_1, \cdots, e_N) \in \mathbb{Z}^N$ and $\zeta^e = \zeta_1^{e_1} \cdots \zeta_N^{e_N}$.
- q is the angular momentum fugacity.

It was predicted that index factorizes too! [Dimofte-Gukov-GaiottoII]

index factorization

Indeed (experimentally) we found:

$$\mathcal{I}[\mathcal{T}](m,\zeta;q) = \sum_{\alpha} B^{\alpha}_{\mathcal{T}}(x;q) B^{\alpha}_{\mathcal{T}}(\widetilde{x},\widetilde{q}) =: \sum_{\alpha} \left\| B^{\alpha}_{\mathcal{T}}(x;q) \right\|^2_{id},$$

where:

$$x = q^{\frac{m}{2}}\zeta, \qquad \widetilde{x} = q^{\frac{m}{2}}\zeta^{-1}, \qquad \widetilde{q} = q^{-1}.$$

Now the relation between $q \equiv e^{2\pi i \tau}$ and $\tilde{q} \equiv e^{2\pi i \tilde{\tau}}$ is:

$$\tau \to \widetilde{\tau} = -id \cdot \tau = -\tau\,.$$

$\rightarrow$ consistent with *id*entity gluing + orientation reverse.

Exactly the same holomorphic blocks that appear in $\mathcal{Z}_b$!!

blocks & BPS counting

Holomorphic blocks can also be defined as BPS indices:

$$B^{\alpha}_{\mathcal{T}}(x;q) \sim \mathrm{Tr}_{\mathcal{H}(\mathbb{R}^2;\alpha)}(-1)^F e^{-\beta H} q^{\frac{R}{2}-J} x^e \,,$$

with fugacity q for the spin on $\mathbb{R}^2$ and x for flavor symmetry.

$\rightarrow$ so we are expressing the index $\mathcal{I}[\mathcal{T}]$ in terms of another index $B^{\alpha}_{\mathcal{T}}$!

Sometimes $B^{\alpha}_{\mathcal{T}}(x;q)$ can be realized with a brane construction and counts BPS states of branes.

geometric engineering

Realize $\mathcal{T}$ in M-theory by wrapping M5 branes on a Lagrangian cycle $\mathcal{L}$ inside a non-compact Calabi-Yau $X_{\mathcal{T}}$:

$$\mathrm{M5}: \ \mathbb{R}^2 \times S^1 \times \mathcal{L} \subset \mathbb{R}^4 \times S^1 \times X_{\mathcal{T}} \,.$$

The open A-model partition function $Z^{\alpha}_{\mathrm{top}}(X_{\mathcal{T}}, \mathcal{L})$ is a BPS index computing the spectrum of M2's ending on M5's. [Ooguri-Vafa]

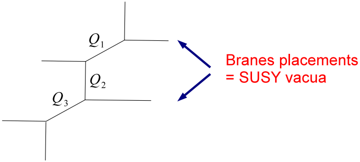
We expect:

$$\begin{array}{ccc} B^{\alpha}_{\mathcal{T}} & \sim & Z^{\alpha}_{\mathrm{top}}(X_{\mathcal{T}}, \mathcal{L}) \\ \text{SUSY vacua} & \leftrightarrow & \text{Phases of branes} \end{array}$$

$\rightarrow$ we checked that this is the case in several (toric) cases, modulo semi-classical prefactors.

example: T[SU(2)] engineering

The CY geometry $X_{\mathcal{T}}$ engineering $T[SU(2)]$ is the strip geometry:



$\rightarrow$ using the topological vertex [Aganagic-Klemm-Marino-Vafa]:

$$Z^{+}_{\mathrm{top}} = \sum_n \frac{(Q_1;q)_n (Q_1 Q_2 Q_3;q)_n}{(q)_n (Q_1 Q_2;q)_n} u^n \,, \quad Z^{-}_{\mathrm{top}} = \sum_n \frac{(Q_2^{-1};q)_n (Q_3;q)_n}{(q)_n ((Q_1 Q_2)^{-1};q)_n} u^n \,.$$

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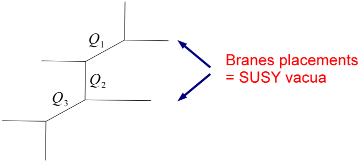

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It is easy to see that (with an appropriate dictionary):

$$Z_V^{\pm} = Z_{\rm top}^{\pm}$$

This is just the vortex partition function engineering [Dimofte-Gukov-Hollands].

Holomorphic blocks (encoding background Chern-Simons terms) provide a non-perturbative completion of $Z_{\rm top}$.

summary

We have seen how holomorphic blocks:

- arise from the factorization of $\mathcal{Z}_b[\mathcal{T}]$ and $\mathcal{I}[\mathcal{T}]$
- transform under mirror symmetry
- are related to BPS counting

Holomorphic blocks can be defined and computed directly on $\mathbb{R}^2 \times_q S^1$ and have many more interesting properties !! (see Tudor's talk)