Homological mirror symmetry for Calabi-Yau hypersurfaces in projective space

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Context

- Mirror symmetry first came to the attention of mathematicians when Candelas-de la Ossa-Green-Parkes used it to predict counts of rational curves on the quintic three-fold (1991).
- Their predictions were mathematically verified by Givental and Lian-Liu-Yau (1996).
- Kontsevich introduced his 'Homological Mirror Symmetry' conjecture in 1994: mirror symmetry is an equivalence of triangulated categories.

Context

- Our main result is to prove homological mirror symmetry for a smooth hypersurface in $\mathbb{CP}^{n-1}$ of degree n.
- For $n = 3$ this is the elliptic curve (related to the result of Polishchuk-Zaslow).
- For $n = 4$ this is the quartic $K3$ (reproduces the result of Seidel).
- For $n = 5$ this is the quintic three-fold (also considered by Nohara-Ueda using my results).

The A-model

- Let $X^n \subset \mathbb{CP}^{n-1}$ be a smooth hypersurface of degree n.
- The **Fukaya category**, $\mathcal{F}(X^n)$, is a $\mathbb{Z}$-graded A_∞ category defined over the Novikov field

$$\Lambda := \left\{ \sum_{j=1}^{\infty} c_j r^{\lambda_j} : c_j \in \mathbb{C}, \lambda_j \in \mathbb{R}, \lambda_j \to \infty \right\}.$$

The B-model

- Define

$$\widetilde{Y}^n := \left\{ u_1 \dots u_n + r \sum_j u_j^n = 0 \right\} \subset \mathbb{P}^{n-1}_{\Lambda}$$
$$Y^n := \widetilde{Y}^n / \Gamma,$$

  where $\Gamma \cong (\mathbb{Z}/n)^{n-2}$ acts by multiplying coordinates by nth roots of unity.

- Consider the category of coherent sheaves on Y^n:

$$Coh(Y^n) \cong Coh^{\Gamma}\left(\widetilde{Y}^n\right).$$

Main result

Theorem (S.)

There is an equivalence of Λ-linear triangulated categories

$$D^{\pi}\mathcal{F}(X^n) \cong \Psi \cdot D^b Coh(Y^n),$$

where Ψ is an automorphism (the 'mirror map')

$$\begin{aligned} \Psi : \Lambda &\to \Lambda, \text{ sending} \\ r &\mapsto \psi(r)r, \end{aligned}$$

where $\psi(r) \in \mathbb{C}[\![r]\!]$ satisfies $\psi(0) = \pm 1$. We are not yet able to determine the higher-order terms in $\psi(r)$.

Description of the categories: $\mathrm{Perf}((\mathcal{A}\#G) \otimes \Lambda)$

- Let $R := \mathbb{C}[\![r_1, \ldots, r_n]\!]$.
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$$\begin{aligned} W &= u_1 \ldots u_n + \sum_{j=1}^{n} r_j u_j^n + \mathcal{O}(r^2) \\ &\in R[[u_1 \ldots u_n]] \otimes A \cong HH^*(A, A \otimes R). \end{aligned}$$

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- This characterizes $\mathcal{A}$ up to A_∞ quasi-isomorphism and a formal change of variables in R.
- Both A- and B-model categories are isomorphic to $\mathrm{Perf}((\mathcal{A}\#G)\otimes\Lambda)$ for such an $\mathcal{A}$, where $G\cong(\mathbb{Z}/n)^{n-1}\subset\mathbb{T}$.

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Main example: Fermat hypersurfaces

$$X_a^n := \left\{ \sum_j z_j^a = 0 \right\} \subset \mathbb{CP}^{n-1},$$

with the n coordinate divisors $D_j := \{z_j = 0\}$.

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Branched covers and Fukaya categories of orbifolds

- Given $\underline{a} = (a_1, \ldots, a_k)$, we define an ‘orbifold’ Fukaya category, $\mathcal{F}(X, D^{\underline{a}})$: same as $\mathcal{F}(X, D)$, but disks are required to have ramification of order a_j along D_j wherever they intersect.

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- If $\rho : (Y, E) \to (X, D)$ is a branched cover, with branching of degree a_j about divisor D_j, then

$$\mathcal{F}(Y, E) \cong \mathcal{F}(X, D^{\underline{a}}) \# G$$

where G is the character group of the covering group.

First-order deformation classes

- Write the A_∞ structure maps in $\mathcal{F}(X, D)$ as

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- A_∞ equation for $\mu^* \Rightarrow \alpha_j^*$ is a Hochschild cochain in $HH^*(\mathcal{F}(X, D) \otimes_R \mathbb{C})$.
- We call the classes $[\alpha_j^*]$ the **first-order deformation classes.**

Behaviour of deformation classes under ramified covers

Proposition (S., cf. work of Pomerleano)

If the first-order deformation classes of $\mathcal{F}(X, D)$ are α_j, then the first-order deformation classes of $\mathcal{F}(X, D^{\underline{a}})$ are $\alpha_j^{a_j}$, where the power is taken with respect to the Yoneda product on Hochschild cohomology.

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- So (roughly), if the A_∞ structure of $\mathcal{F}(X, D)$ looks like

$$\mu_0 + \sum_j r_j \alpha_j + \mathcal{O}(r^2),$$

then the A_∞ structure of $\mathcal{F}(X, D^{\underline{a}})$ looks like

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The Fermat hypersurface as a branched cover

- There is a branched cover

$$\begin{aligned} X^n = \left\{ \sum_j z_j^n = 0 \right\} & \to \left\{ \sum_j z_j = 0 \right\} = \mathbb{CP}^{n-2} \subset \mathbb{CP}^{n-1}, \\ [z_1 : \ldots : z_n] & \mapsto [z_1^n : \ldots : z_n^n], \end{aligned}$$

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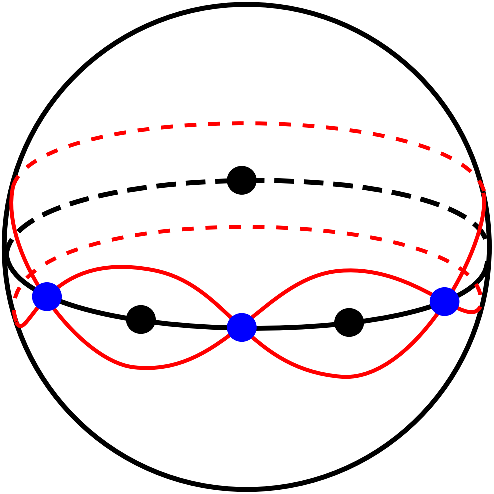
  with ramification of degree n about $D_j = \{z_j = 0\}$.
- $\mathbb{CP}^{n-2} \setminus D$ is called the ‘pair of pants’.
- We construct an (immersed) Lagrangian sphere $L \to \mathbb{CP}^{n-2} \setminus D$, and compute $CF^*(L, L)$ in $\mathcal{F}(\mathbb{CP}^{n-2}, D)$ to first order.

$L^1 \subset \mathbb{CP}^1 \setminus D$ as a 'pushoff' of $S^1 \to \mathbb{RP}^1 \subset \mathbb{CP}^1$



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- More generally, construct L^n as a pushoff of the immersion

$$S^{n-2} \xrightarrow{2:1} \mathbb{RP}^{n-2} \hookrightarrow \mathbb{CP}^{n-2}$$

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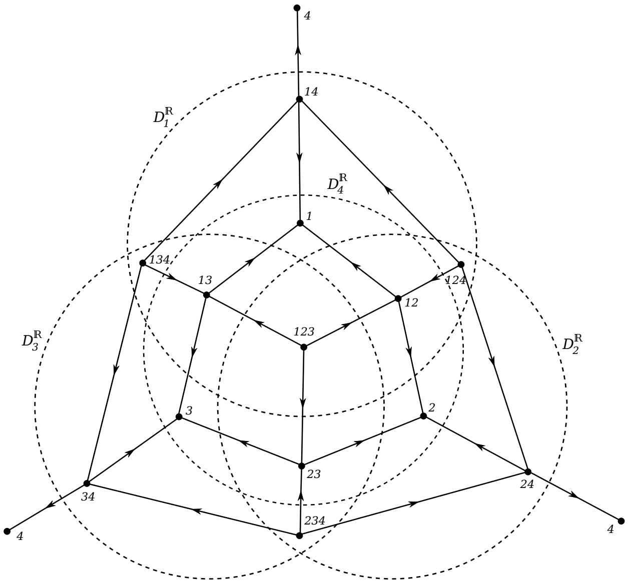
- ∇f must be transverse to the hypersurfaces

$$D_j^{\mathbb{R}} := D_j \cap \mathbb{RP}^{n-2}$$

  if L^n is to avoid the divisors D_j.

The 2-dimensional case



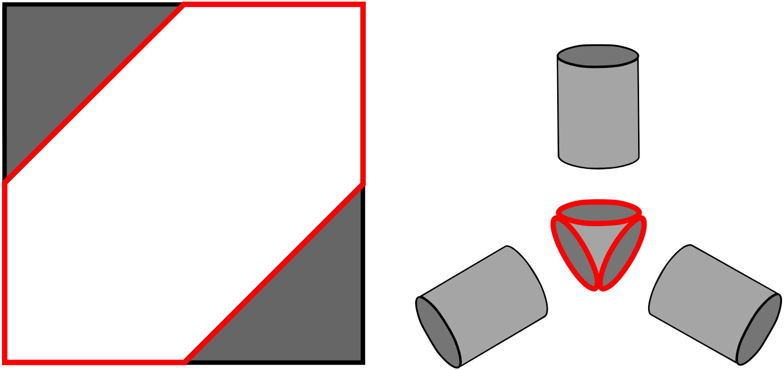
Amoeba and Coamoeba

- We have

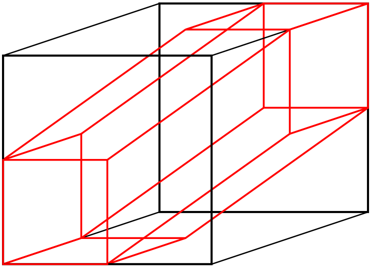
$$\begin{array}{ccccc} \mathbb{CP}^{n-2} \setminus D & \hookrightarrow & \mathbb{CP}^{n-1} \setminus \{z_j = 0\} & \xrightarrow{\mathrm{Log}} & \mathbb{R}^{n-1} \\ & & \Big\downarrow{\scriptstyle \mathrm{Arg}} & & \\ & & (S^1)^{n-1} & & \end{array}$$

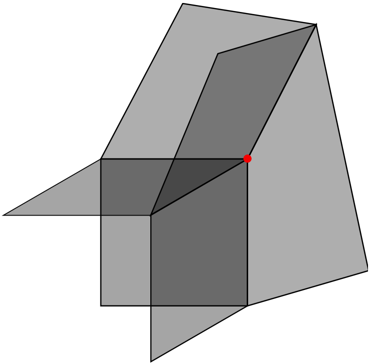
- $\mathrm{Log}(\mathbb{CP}^{n-2} \setminus D)$ is called the 'amoeba'.
- $\mathrm{Arg}(\mathbb{CP}^{n-2} \setminus D)$ is called the 'coamoeba'.

Coamoeba and ‘fibration’ of pants over the amoeba



Coamoeba and amoeba of two-dimensional pants

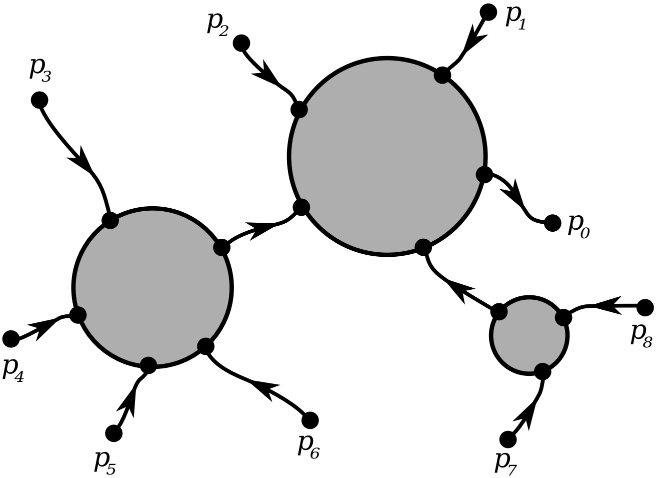




Morse-Bott model for $CF^*(L, L)$

- We constructed L as a pushoff of the immersion $S^{n-2} \to \mathbb{RP}^{n-2} \to \mathbb{CP}^{n-2}$ by a Morse function f.
- Pushoff by ϵf, and consider the limit $\epsilon \to 0$.
- Holomorphic disks with boundary on L degenerate to ‘pearly trees’, built out of Morse flowlines of f and holomorphic disks with boundary on $\mathbb{RP}^n$ (halves of real algebraic curves).

A pearly tree contributing to $\mu^8(p_8, \ldots, p_1)$



Calculating $CF^*(L, L)$ to 0th order

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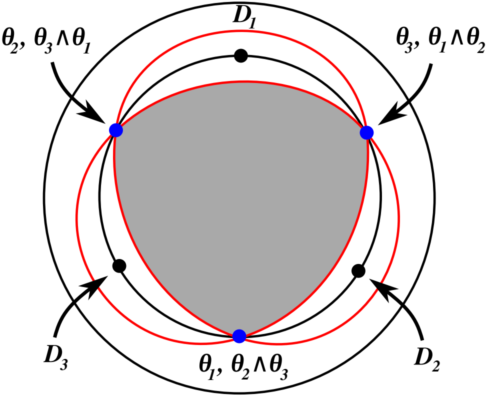
- The higher A_∞ products $\mu_0^{>2}$ correspond to the class

$$u_1 \dots u_n \in HH^*(A) \cong \mathbb{C}[\![u_1, \dots, u_n]\!] \otimes A \text{ (HKR).}$$

- This comes from a single holomorphic disk, which gives

$$\mu_0^n(\theta_1, \theta_2, \dots, \theta_n) = 1.$$

Calculating $CF^*(L, L)$ to 0th order



Calculating $CF^*(L, L)$ to first order

- The first-order deformation class of $CF^*(L, L)$ in $\mathcal{F}(\mathbb{CP}^{n-2}, D)$ is

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- So, it has all the properties so we can call it $\mathcal{A}$!

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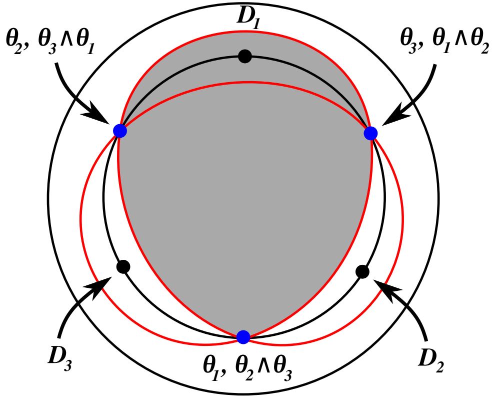
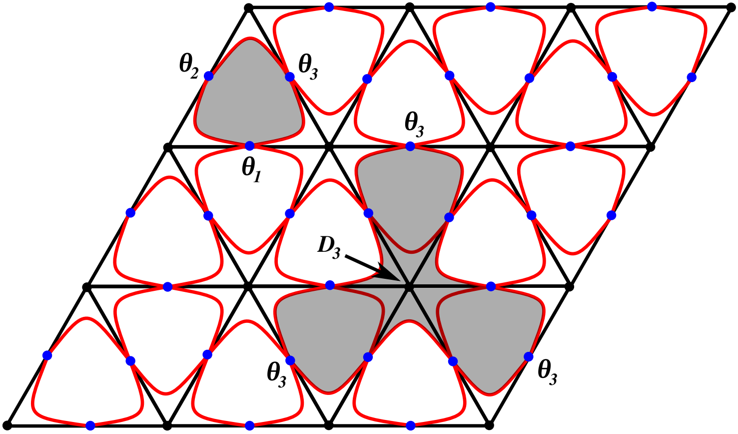


Lifts of L to X^1 = elliptic curve



A full subcategory of $\mathcal{F}(X^n)$

- The full subcategory of $\mathcal{F}(X^n, D)$ whose objects are lifts of L is isomorphic to $\mathcal{A}\#G$.
- We denote the image of this subcategory after embedding into $\mathcal{F}(X^n)$ by
$$\widetilde{\mathcal{A}} \cong (\mathcal{A}\#G) \otimes_R \Lambda.$$
- We will show that this subcategory $\widetilde{\mathcal{A}}$ split-generates the Fukaya category.

Split-generation of the Fukaya category

Theorem* (Abouzaid-Fukaya-Oh-Ohta-Ono, in preparation)

Suppose that X^{2d} is a $2d$-dimensional Calabi-Yau, and $\mathcal{C} \subset \mathcal{F}(X^{2d})$ is a full subcategory. Then, if the top-degree part of the closed-open string map

$$\mathcal{CO} : QH^{2d}(X^{2d}) \to HH^{2d}(\mathcal{C})$$

is non-zero, then $\mathcal{C}$ split-generates $\mathcal{F}(X^{2d})$.

Split-generation of $\mathcal{F}(X^n)$

- We have

$$HH^* \left(\widetilde{\mathcal{A}}\right)^G \cong \mathrm{Jac}(W)^G \cong \Lambda[u]/u^{n-1}$$

- The map $QH^* \to HH^*$ sends

$$[\omega] \mapsto r\frac{\partial \mu^*}{\partial r} = r\frac{\partial W}{\partial r} = c \cdot u, \text{ where } c \neq 0.$$

- It is a Λ-algebra homomorphism, so it sends

$$[\omega]^{n-2} \mapsto (c \cdot u)^{n-2} \neq 0,$$

so the top class does not get sent to zero, and our collection of Lagrangians split-generates $\mathcal{F}(X^n)$.

The B-model: matrix factorizations

- We make computations in $D^b Coh(Y^n)$ using $\mathbb{T}$-equivariant matrix factorizations of the superpotential

$$W = u_1 \dots u_n + \sum_{j=1}^{n} r_j u_j^n$$

  in the ring $S = R[u_1, \dots, u_n]$.
- Consider $\mathcal{O}_0$, the matrix factorization corresponding to $S/(u_1, \dots, u_n)$.
- $\mathrm{End}_{MF(W)}(\mathcal{O}_0)$ is an A_∞ deformation of

$$\mathrm{End}_{D(\mathbb{C}[u_1,\dots,u_n])}(\mathcal{O}_0) \cong \Lambda^* \mathbb{C}^n.$$

The B-model: equivalence with the A-model

- Homological perturbation lemma $\Rightarrow$ the deformation classes of $\mathcal{B} := \mathrm{End}_{MF(W)}(\mathcal{O}_0)$ are given by the coefficients of

$$W = u_1 \dots u_n + \sum_{j=1}^{n} r_j u_j^n$$

- It follows that $\mathcal{A} \cong \mathcal{B}$ up to a formal change of variables in R.

The B-model: equivariant matrix factorizations

- $G \cong (\mathbb{Z}/n)^{n-1} \subset \mathbb{T}$ acts on S, preserving W.
- The subcategory of $MF^G(S, W)$ whose objects are twists of $\mathcal{O}_0$ by characters of G is isomorphic to $\tilde{\mathcal{B}} \cong \mathcal{B}\#G$.
- The image of this subcategory under embedding into $MF^G(S \otimes_R \Lambda, W \otimes 1)$ is isomorphic to

$$\widetilde{\mathcal{B}} \cong (\mathcal{B}\#G) \otimes_R \Lambda.$$

The LG-CY correspondence

- $\widetilde{\mathcal{B}}$ embeds in $D^bCoh(Y^n)$, by Orlov's theorem:

$$\begin{aligned} MF^{\mathbb{Z}_n}(S,W) &\cong D^bCoh\left(\widetilde{Y}^n\right) \text{ (as } \widetilde{Y}^n = \{W=0\} \subset \mathbb{P}^{n-1}_{\Lambda}) \\ \Rightarrow MF^{G}(S,W) &\cong D^bCoh^{G/\mathbb{Z}_n}\left(\widetilde{Y}^n\right) \\ &\cong D^bCoh(Y^n). \end{aligned}$$

- In fact, the corresponding objects of $D^bCoh^G(\tilde{Y}^n)$ are the twists of the restricted Beilinson exceptional collection $\Omega^j(j)$, for $j = 0, 1, \ldots, n-1$, by characters of G, and in particular generate.