Large subgroups of M_{24} form overarching symmetry groups of K3

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How could M_{24} be acting?

Conjecture (following [Eguchi/Ooguri/Tachikawa10]):
M_{24} acts on $N=2$ half BPS-states of type II string theory on K3

[Aspinwall/Morrison94, Nahm/W01]:

$$\{\text{SCFTs on K3}\} \stackrel{1:1}{\longleftrightarrow} \sim\backslash\{\text{pos. def. or. 4-dim. } x \subset H_*(\mathrm{K3},\mathbb{R})\}$$

SYMMETRIES of a SCFT x:

$$\widehat{G}_x := \left\{ g \in \mathrm{Aut}\left(H_*(\mathrm{K3},\mathbb{Z})\right) \mid g_{|x} = \mathrm{id}_x \right\}$$

in general, $\widehat{G}_x \not\subset M_{24}$

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GEOMETRIC INTERPRETATION: a choice of grading

$$H_*(\mathrm{K3},\mathbb{R}) = H_0(\mathrm{K3},\mathbb{R}) \oplus H_2(\mathrm{K3},\mathbb{R}) \oplus H_4(\mathrm{K3},\mathbb{R})$$

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$$G_x := \left\{g \in \widehat{G}_x \mid g(H_k(\text{K3}, \mathbb{R})) \subset H_k(\text{K3}, \mathbb{R}), k = 0, 2, 4\right\}$$

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in general, $\widehat{G}_x \not\subset M_{24}$, but by **[Mukai88]:** $G_x \subset M_{24}$

$$|G_x| \leq 384 \ll 244.823.040 = |M_{24}|$$

LARGE SUBGROUPS OF M_{24} FORM OVERARCHING SYMMETRY GROUPS OF K3

[Nahm/W01] *A hiker's guide to K3 - Aspects of $N = (4,4)$ superconformal field theory with central charge $c = 6$*, Commun. Math. Phys. **216** (2001), 85-138; hep-th/9912067

[W01] *Consistency of orbifold conformal field theories on K3*, Adv. Theor. Math. Phys. **5** (2001), 429-456; hep-th/0010281

[Taormina/W11] *The overarching finite symmetry group of Kummer surfaces in the Mathieu group M_{24}*, arXiv:1107.3834 [hep-th]

Symmetry groups of K3: Terminology

always assume:

- X is a K3 SURFACE: a simply connected Calabi-Yau 2-fold, with a choice of complex structure
- X is ALGEBRAIC: $X \subset \mathbb{CP}^N$ for some N
- ω is a choice of POLARIZATION of X, fixing a particular Kähler structure: $\omega \in (H^{1,1}(X,\mathbb{C}))^* \cap H_2(X,\mathbb{Z}), \langle\omega,\omega\rangle > 0$

a group G of biholomorphic $f: \mathrm{K3} \longrightarrow \mathrm{K3}$ is finite

$\iff G$ FIXES some polarization ω

SYMMETRY f of X:

$f: X \longrightarrow X$ biholomorphic, s.th. $f_*: H_*(\mathrm{K3},\mathbb{R}) \longrightarrow H_2(\mathrm{K3},\mathbb{R})$ fixes ω

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if $X = \widetilde{T/\mathbb{Z}_2}$, a KUMMER K3,

$T = \mathbb{C}^2/\Lambda$ with $\Lambda = \mathrm{span}_{\mathbb{Z}}\{\lambda_1,\ldots,\lambda_4\} \subset \mathbb{C}^2$ a rank 4 lattice,

assume:

- cpx. structure and polarization of X are induced from $\mathbb{C}^2$ via T

Symmetry groups of K3: How do they act?

Torelli Theorem [Pjatecki-Šapiro/Šafarevič71, Looijenga/Peters81, Burns/Rapoport75,...]

$f: X \longrightarrow X$ a biholomorphic map

$$\overset{1:1}{\underset{\alpha = f_*}{\longleftrightarrow}} \alpha \in \mathrm{Aut}(H_*(X,\mathbb{Z}))$$

$+$ known cdts., e.g. fixing $(H_0 \oplus H_{2,0} \oplus H_{0,2} \oplus H_4)(X,\mathbb{C})$

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Theorem [Mukai88]
If G is a symmetry group of X, then G is isomorphic to a subgroup of the Mathieu group M_{24}.

- M_{24} is a simple sporadic group of order 244.823.040
- $M_{24} = \mathrm{Aut}(N)/(\mathbb{Z}_2)^{24}$ with N the Niemeier lattice of type A_1^{24}

Symmetry groups of K3: How do they act?

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Hence: Symmetry groups of X act on $II_{4,20} \cong H_*(X,\mathbb{Z})$ and on N.

The Niemeier lattice of type A_1^{24}

A NIEMEIER LATTICE $\widetilde{N}$ is an even and selfdual, positive definite lattice of rank 24.
Each $\widetilde{N}$ is uniquely determined by its ROOT SUBLATTICE $\widetilde{R}$,

$$\widetilde{R} = \mathrm{span}_{\mathbb{Z}} \{E \in \widetilde{N} \mid \langle E, E \rangle = 2\}, \quad \mathrm{rk}(\widetilde{R}) \in \{0, 24\}.$$

If $\mathrm{rk}(\widetilde{R}) = 24$, then $\widetilde{N}$ is called the Niemeier lattice OF TYPE $\widetilde{R}$.

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Niemeier lattice N of type A_1^{24}:

$R = \mathrm{span}_{\mathbb{Z}} \{f_1, \ldots, f_{24}\} \cong \mathbb{Z}^{24}(2)$ and $\langle f_i, f_j \rangle = 2\delta_{ij}$,

$$N = \left\{ \frac{1}{2} \sum_{i=1}^{24} \widehat{n}_i f_i \;\middle|\; \begin{array}{l} \widehat{n}_i \in \mathbb{Z} \text{ s.th. for } \; n_i := \widehat{n}_i \mod 2 \in \mathbb{F}_2, \\ \qquad (n_1, \ldots, n_{24}) \in \mathcal{G}_{24} \end{array} \right\},$$

$\mathcal{G}_{24} \subset \mathbb{F}_2^{24}$ the GOLAY CODE, i.e. a 12-dim. vector space s.th.

$$|n| := \#\{n_i \mid n_i \neq 0\} \in \{0, 8, 12, 16, 24\} \text{ for all } n \in \mathcal{G}_{24}.$$

$M_{24} = \mathrm{Aut}(\mathcal{G}_{24}) \subset S_{24}$ = permutation group on $\{1, \ldots, 24\}$,

$$\mathrm{Aut}(N) = M_{24} \ltimes (\mathbb{Z}_2)^{24}.$$

The Kummer lattice

KUMMER LATTICE Π:

an even, negative definite lattice of rank 16
with root sublattice $R_\Pi \subset \Pi$ of type A_1^{16}:

$$R_\Pi = \operatorname{span}_{\mathbb{Z}} \left\{ E_{\vec{a}}, \vec{a} \in \mathbb{F}_2^4 \right\}, \; \langle E_{\vec{a}}, E_{\vec{b}} \rangle = -2\delta_{\vec{a},\vec{b}};$$

$$\Pi = \operatorname{span}_{\mathbb{Z}} \left\{ E_{\vec{a}}, \vec{a} \in \mathbb{F}_2^4; \; \frac{1}{2} \sum_{\vec{b} \in H} E_{\vec{b}}, \; H \subset \mathbb{F}_2^4 \text{ a hyperplane} \right\}$$

Theorem [Pjatecki-Šapiro/Šafarevič71, Nikulin75]
Π can be primitively embedded in $II_{4,20} \cong H_*(X, \mathbb{Z})$,
$\Pi \hookrightarrow II_{4,20}$ is unique up to $\operatorname{Aut}(II_{4,20})$.

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for Kummer K3s $X = \widetilde{T/\mathbb{Z}_2}$:
$T/\mathbb{Z}_2$ (with $T = \mathbb{C}^2/\Lambda$) has 16 singular points of type A_1:

$$F_{\vec{a}} = \left[\tfrac{1}{2}\textstyle\sum_{i=1}^4 a_i\lambda_i\right], \ a_i \in \{0,1\}, \ \vec{a} \in \mathbb{F}_2^4 \cong \tfrac{1}{2}\Lambda/\Lambda$$

blow up each $F_{\vec{a}}$: the $E_{\vec{a}} \in H_2(X,\mathbb{Z})$ generate $R_\Pi \subset H_2(X,\mathbb{Z})$;
$\Pi \subset H_2(X,\mathbb{Z})$ is the smallest primitive sublattice containing R_Π.

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Result [Taormina/W10]
$\Pi(-1)$ can be primitively embedded in N,
$\Pi(-1) \hookrightarrow N$ is unique up to $\mathrm{Aut}(N)$.

Symmetry groups of Kummer K3s

X: an algebraic polarized Kummer K3, $X = \widetilde{T/\mathbb{Z}_2}$
G: its symmetry group, $\Pi \subset H_2(X,\mathbb{Z})$ its Kummer lattice

Note:
For $f \in G$: $f_*(\Pi) = \Pi$; $\exists A \in \mathrm{Aff}(\mathbb{F}_2^4)$: $f_*(E_{\vec{a}}) = E_{A(\vec{a})} \forall \vec{a} \in \mathbb{F}_2^4$;
A uniquely determines f.

The action of G is completely determined by its action on Π.

$$(\mathbb{Z}_2)^4 \rtimes G_T = G \subset \mathrm{Aff}(\mathbb{F}_2^4) = (\mathbb{Z}_2)^4 \rtimes GL_4(\mathbb{F}_2)$$

($(\mathbb{Z}_2)^4$ is translational, and G_T is induced from symmetries of T),

$$\overset{\text{[Fujiki88]}}{\Longrightarrow} \quad G \subset (\mathbb{Z}_2)^4 \rtimes A_7 \subset M_{24}.$$

Moveover: $e := \frac{1}{2} \sum_{\vec{a} \in \mathbb{F}_2^4} E_{\vec{a}}$ is invariant under G.

$(\mathbb{Z}_2)^4 \rtimes A_7$: the OVERARCHING symmetry group of Kummer K3s

Note: $|M_{24}| = 244.823.040 \gg |(\mathbb{Z}_2)^4 \rtimes A_7| = 40.320 \gg 384 \geq |G|$

Mukai's theorem and Kondo's proof revisited

X: an algebraic polarized K3 surface, G: its symmetry group;
$L^G := H_*(X,\mathbb{Z})^G$, $L_G := (L^G)^\perp \cap H_*(X,\mathbb{Z})$;

Theorem **[Mukai88]**
G is isomorphic to a subgroup of the Mathieu group M_{24}.

steps of a proof [Kondo98]:

1. $L_G \oplus \mathrm{span}_\mathbb{Z}\{E\} \hookrightarrow \widetilde{N}(-1)$ for some Niemeier lattice $\widetilde{N}$,
   E: a root, i.e. $\langle E, E\rangle = -2$.
2. G acts on $\widetilde{N}$ such that $L_G \hookrightarrow \widetilde{N}(-1)$ is G-equivariant,
   $(L_G)^\perp \cap \widetilde{N}$ is G-invariant.
3. G acts on $\widetilde{R}^*/\widetilde{R}$ and thus (...) $G \subset M_{24}$.

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Problems:

- which $\widetilde{N}$?
- what is E ?
- for Kummer K3s: $e = \frac{1}{2}\sum_{\vec{a}\in\mathbb{F}_2^4} E_{\vec{a}} \in L^G$, so $\Pi \not\subset L_G$

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For Kummer K3s X:

Result **[Taormina/W12]**
Let $\upsilon_0 \in H_0(X,\mathbb{Z}), \upsilon \in H_4(X,\mathbb{Z})$, such that $\langle \upsilon, \upsilon_0 \rangle = 1$.
Then:

$$M_G := L_G \oplus \text{span}_\mathbb{Z}\{e, \upsilon_0 - \upsilon\} \supset \Pi, \qquad M_G \hookrightarrow N(-1).$$

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- which $\widetilde{N}$? $\widetilde{N} = N$!
- what is E ? $E = \upsilon_0 - \upsilon$!
- $\Pi \not\subset L_G$, but $\Pi \subset M_G, \text{rk}(M_G) = \text{rk}(L_G) + 2$!

G acts on N such that $i_G : M_G \hookrightarrow N(-1)$ is G-equivariant,
$(M_G)^\perp \cap N$ is G-invariant.

An overarching bijection between lattices

Result **[Taormina/W11]**

construction of a linear bijection

$$\Theta\colon H_*(\mathrm{K3},\mathbb{Z}) \longrightarrow N(-1),$$

such that

1. for X_{D_4} the TETRAHEDRAL KUMMER SURFACE (i.e. T is the torus with D_4-lattice):
   - $G = \mathcal{T}_{192} \cong (\mathbb{Z}_2)^4 \rtimes A_4$
   - $\Theta_{|M_{\mathcal{T}_{192}}} = i_{\mathcal{T}_{192}}$, i.e. $\Theta_{|M_{\mathcal{T}_{192}}}$ gives a $\mathcal{T}_{192}$-equivariant isometry between lattices (of rank 20)
2. for X_0 the SQUARE KUMMER SURFACE (i.e. $T = \mathbb{C}^2/\mathbb{Z}^4$):
   - $G = \mathcal{T}_{64} \cong (\mathbb{Z}_2)^4 \rtimes (\mathbb{Z}_2)^2$
   - $\Theta_{|M_{\mathcal{T}_{64}}} = i_{\mathcal{T}_{64}}$, i.e. $\Theta_{|M_{\mathcal{T}_{64}}}$ gives a $\mathcal{T}_{64}$-equivariant isometry between lattices (of rank 20)

The overarching symmetry group of Kummer K3s

Result [Taormina/W11]
The induced actions of $\mathcal{T}_{192}$ and $\mathcal{T}_{64}$ on the Niemeier lattice N generate the action of the full overarching symmetry group of Kummer K3s $(\mathbb{Z}_2)^4 \rtimes A_7$.

Discussion

[Taormina/W10–12]

- develop a technique to express symmetry groups of K3 surfaces explicitly as subgroups of the Mathieu group M_{24}, improving Mukai's and Kondo's techniques for all Kummer surfaces that are polarized by their underlying torus
- develop a device to combine symmetry groups of different polarized K3 surfaces to larger subgroups of the Mathieu group M_{24}
- recover the full overarching symmetry group of Kummer surfaces $(\mathbb{Z}_2)^4 \rtimes A_7$,

  $|G| \leq 384 \ll |(\mathbb{Z}_2)^4 \rtimes A_7| = 40.320 \ll 244.823.040 = |M_{24}|$

The End

Thank you for your attention!

The overarching bijection Θ

Result [Taormina/W10]

For the Niemeier lattice N with roots $\{f_i\}$ and $\langle f_i, f_j\rangle = 2\delta_{ij}$:

- the map $f_i \mapsto E_{I(i)}$ with

$$I:\left\{\begin{array}{llll} 1\mapsto(0,0,0,0), & 2\mapsto(0,0,1,1), & 4\mapsto(0,1,1,0), & 7\mapsto(1,1,1,1),\\ 8\mapsto(1,0,0,1), & 10\mapsto(1,1,0,1), & 11\mapsto(1,0,0,0), & 12\mapsto(0,1,1,1),\\ 13\mapsto(0,1,0,0), & 14\mapsto(0,0,1,0), & 16\mapsto(1,0,1,0), & 17\mapsto(0,0,0,1),\\ 18\mapsto(0,1,0,1), & 20\mapsto(1,1,1,0), & 21\mapsto(1,1,0,0), & 22\mapsto(1,0,1,1)\end{array}\right.$$

  induces an isometry $\Pi \cong \widetilde{\Pi}(-1)$, $\widetilde{\Pi} \subset N$ primitive, which extends to

  $\mathcal{P} := \Pi \oplus \mathrm{span}_{\mathbb{Z}}\{\upsilon_0 - \upsilon\}$, $\widetilde{\mathcal{P}} := \widetilde{\Pi} \oplus \mathrm{span}_{\mathbb{Z}}\{f_5\}$, $\mathcal{P} \cong \widetilde{\mathcal{P}}(-1)$

- construction of a linear bijection $\Theta\colon H_*(\mathrm{K3},\mathbb{Z}) \longrightarrow N(-1)$, induced by $E_{I(i)} \mapsto f_i$, and by

$$\begin{array}{lcllcl} \pi_*(\lambda_1\vee\lambda_2) & \mapsto & f_3+f_6-f_{15}-f_{19}, & \pi_*(\lambda_3\vee\lambda_4) & \mapsto & f_6+f_9-f_{15}-f_{19},\\ \pi_*(\lambda_1\vee\lambda_3) & \mapsto & -f_6+f_{15}-f_{23}+f_{24}, & \pi_*(\lambda_2\vee\lambda_4) & \mapsto & -f_{15}+f_{19}+f_{23}-f_{24},\\ \pi_*(\lambda_1\vee\lambda_4) & \mapsto & f_3-f_9-f_{15}+f_{24}, & \pi_*(\lambda_2\vee\lambda_3) & \mapsto & f_3-f_9-f_{15}+f_{23},\\ \upsilon_0-\upsilon & \mapsto & f_5, & \multicolumn{3}{l}{\upsilon_0 \mapsto \tfrac12(f_3+f_5+f_6+f_9-f_{15}-f_{19}-f_{23}-f_{24})} \end{array}$$

The tetrahedral Kummer K3

D_4 TORUS: $T_{D_4} = \mathbb{C}^2/\Lambda_{D_4}$,

$\Lambda_{D_4} := \text{span}_{\mathbb{Z}}\{\lambda_1 = (1,0),\ \lambda_2 = (i,0),\ \lambda_3 = (0,1),\ \lambda_4 = (\frac{i+1}{2}, \frac{i+1}{2})\}$;

TETRAHEDRAL KUMMER SURFACE: $X_{D_4} := T_{D_4}/\mathbb{Z}_2$

SYMPLECTIC AUTOMORPHISMS of T_{D_4} [Fujiki88]:

$$\begin{array}{llll} \gamma_1: & (z_1, z_2) & \mapsto & (iz_1, -iz_2), \\ \gamma_2: & (z_1, z_2) & \mapsto & (-z_2, z_1), \\ \gamma_3: & (z_1, z_2) & \mapsto & \frac{i+1}{2}(i(z_1 - z_2), -(z_1 + z_2)) \end{array}$$

generate the BINARY TETRAHEDRAL GROUP $\mathbb{T} \subset SU(2)$, $|\mathbb{T}| = 24$

polarization preserving symmetry group $\mathcal{T}_{192}$ of X_{D_4}:
gen. by the TETRAHEDRAL $\mathbb{T}/\mathbb{Z}_2 = A_4$ and the GENERIC $(\mathbb{Z}_2)^4$;

$|\mathcal{T}_{192}| = 192$

The square Kummer K3

SQUARE TORUS: $T_0 = \mathbb{C}^2/\Lambda_0$,

$$\Lambda_0 := \text{span}_{\mathbb{Z}}\{\lambda_1 = (1,0),\ \lambda_2 = (i,0),\ \lambda_3 = (0,1),\ \lambda_4 = (0,i)\};$$

SQUARE KUMMER SURFACE: $X_0 := T_0/\mathbb{Z}_2$

POLARIZATION PRESERVING SYMMETRY GROUP $\mathcal{T}_{64}$ of X_0:

induced by

$$\alpha_1: \quad (z_1, z_2) \mapsto (iz_1, -iz_2),$$
$$\alpha_2: \quad (z_1, z_2) \mapsto (-z_2, z_1)$$

along with the GENERIC $(\mathbb{Z}_2)^4$

A Mathieu moonshine phenomenon?

elliptic genus

- for K3 surfaces (here: $\tau, z \in \mathbb{C}, \mathrm{Im}(\tau) > 0$):

$$\mathcal{E}_{K3}(\tau, z) = \frac{2}{\eta^6(\tau)} \left\{ \vartheta_2^2(\tau, z)\vartheta_3^2(\tau, 0)\vartheta_4^2(\tau, 0) + cycl. \right\}$$

- for $N = 4$ superconformal field theories associated to K3: (here: $\mathcal{H}$: Hilbert space of (Ramond-) states, $y = e^{2\pi iz}$, $q = e^{2\pi i\tau}$)

$$\mathcal{E}_{K3}(\tau, z) = \mathrm{sTr}_{\mathcal{H}} \left(y^{J_0} q^{L_0 - 1/4} \overline{q}^{\overline{L}_0 - 1/4} \right)$$

with $N = 4$ superconformal characters χ_0, $\chi_{1/2}$, $q^h \chi$ ($h \in \mathbb{R}_{>0}$):

$$\mathcal{E}_{K3}(\tau, z) = -2\chi_0(\tau, z) + 20\chi_{1/2}(\tau, z) + 2e(\tau) \cdot \chi(\tau, z)$$