Secret symmetries of AdS/CFT

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In progress with

(E. Sklyanin   O. Ohlsson Sax & B. Stefanski)

Reviews $\longrightarrow$ arXiv: 1012.3982 1012.4005 1104.2474

with Beisert *et al.*

Introduction

Hopf algebras in AdS/CFT

Non-abelian symmetries and the Yangian

Secret symmetries   Spectrum, Amplitudes, Pure spinors, Boundaries, q-deformations

ADS/CFT SPECTRAL CONJECTURE

$$4D\ \mathcal{N}=4\ \text{SYM} \quad \leftrightarrow \quad \text{IIB Superstrings in } AdS_5\times S^5$$

Superconformal algebra $\mathfrak{psu}(2,2|4)$

Single-trace composite operators $\mathcal{O}_{\{\alpha\}}(x) = \mathrm{tr}[\phi_{i_1}(x)\ldots\phi_{i_L}(x)]$

$\longrightarrow$ closed string states

Diag. of mixing matrix $\langle\mathcal{O}_{\{\alpha\}}(x),\mathcal{O}_{\{\beta\}}(y)\rangle = \frac{\delta_{\{\alpha\},\{\beta\}}}{|x-y|^{2\Delta_{\{\alpha\}}}}$

- Anomalous dimension $\leftrightarrow$ Energy of string excitations

Planar lim.: Equivalent to diag. the Hamiltonian of an *integrable* spin-chain ($\exists$ extra local integrals of motion)

[Minahan-Zarembo '02]

Example: $\mathfrak{su}(2)$ sector - one loop

$$Z = \phi_1 + i\phi_2 \qquad W = \phi_3 + i\phi_4$$

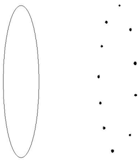
$$\mathcal{O}_{\{\alpha\}}(x) = \mathrm{tr}[Z \ldots Z W \ldots W Z \ldots W] \longrightarrow \downarrow \ldots \downarrow \uparrow \ldots \uparrow \downarrow \ldots \uparrow$$

Corresponding Hamiltonian $\longrightarrow$ $\mathfrak{su}(2)$ Heisenberg spin-chain ('isotropic')

WHY WE BELIEVE IT IS IMPORTANT

Spectrum of an interacting 4D QFT *via* 2D exactly solvable model

Chance of proving the AdS/CFT spectral conjecture



BETHE ANSATZ [Bethe '31]

Infinite spin-chain limit: 2-particle state

$$|\psi> = \sum_{n_1<n_2} \psi(n_1, n_2) \, |Z...Z \, V \, Z...Z \, W Z...\rangle$$

$$n_1{}^{\uparrow} \qquad n_2{}^{\uparrow}$$

$$\psi(n_1, n_2) = e^{ip_1 n_1 + ip_2 n_2} + S(p_1, p_2) \, e^{ip_2 n_1 + ip_1 n_2}$$

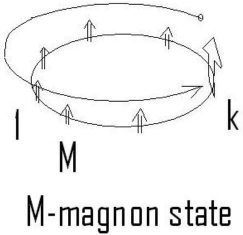
$S(p_1, p_2)$ S-matrix: *magnon* scattering

$$\downarrow ... \downarrow \; \Uparrow \; \downarrow ... \downarrow ... \downarrow ... \downarrow \; \Uparrow \; \downarrow ... \downarrow$$

Periodicity restored by Bethe Equations

$$e^{i p_k L} = \prod_{j=1}^{M} S_{kj}$$

$$S_{kj} = S(p_k, p_j)$$



AdS/CFT [Arutyunov-Frolov-Staudacher '04; Beisert-Staudacher '05]

EXACT S-MATRICES

{*for rev*} [P. Dorey '98]

2D integrable massive field theories (worldsheet counterpart)

- No particle production/annihilation

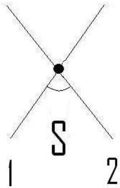
- Equality of initial and final sets of momenta

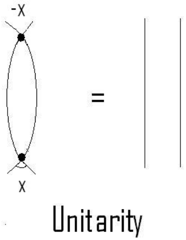
- Factorisation: $S_{M \longrightarrow M} = \prod S_{2 \longrightarrow 2}$

  (all info in 2-body processes)

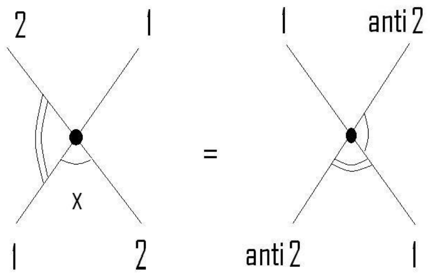
Assume relativistic invariance (for now)

$$S_{2\longrightarrow 2} = S(x_1 - x_2) \equiv S(x) \qquad [E_i = m\cosh(x_i),\ p_i = m\sinh(x_i)]$$

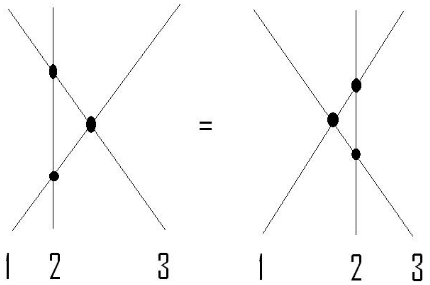




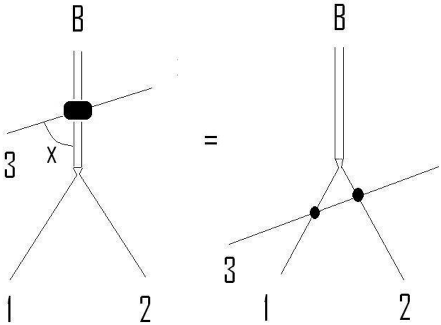
$$S_{12}(x)\ S_{21}(-x) = 1$$



Crossing symmetry $S_{12}(x) = S_{\bar{2}1}(i\pi - x)$



Yang-Baxter Equation (YBE) $S_{12}\,S_{13}\,S_{23} \;=\; S_{23}\,S_{13}\,S_{12}$



Bootstrap $S_{3B}(x) = S_{32}\left(x + i(\pi - x_{2B}^{1})\right) S_{31}\left(x - i(\pi - x_{1B}^{2})\right)$

[Zamolodchikov-Zamolodchikov '79]

S-MATRIX and HOPF ALGEBRA

{*for rev*} [Delius '95]

Algebraic treatment $\longrightarrow$ relativistic & non (spin chains)

$R \; : \; V_1 \otimes V_2 \longrightarrow V_1 \otimes V_2$ R is S-matrix S up to a subtelty which I spare you from

V_i module for a representation of (super)algebra A

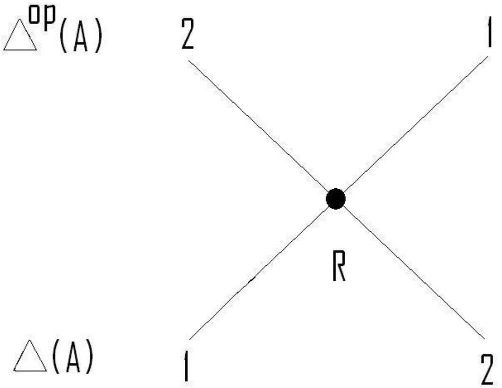
Symmetry on 'in' states: coproduct

$$\Delta \; : \; A \longrightarrow A \otimes A$$

$[\Delta(a), \Delta(b)] = \Delta([a, b])$ (homomorphism)

$$(P\Delta)\,R \;=\; R\,\Delta$$

P (graded) permutation $\quad$ $P\Delta$ 'opposite' coproduct Δ^{op} ('out')



Lie (super)algebras can have 'trivial' coproduct

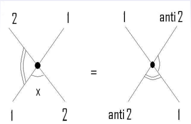
$$\Delta^{op}(Q) = \Delta(Q) = Q \otimes 1 + 1 \otimes Q \qquad \forall\, Q \in A$$

non trivial $\rightarrow$ quantum groups

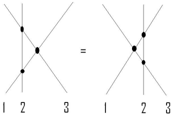
Hopf algebra: coproduct + extra algebraic structures
e.g. antipode (antiparticles) + list of axioms

The Yangian is an ∞-dim non-abelian Hopf algebra

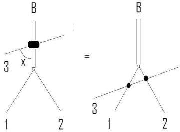
{*books* } [Chari-Pressley '94; Kassel '95; Etingof-Schiffmann '98]



$$(\Sigma \otimes 1)R = R^{-1} = (1 \otimes \Sigma^{-1})R$$



$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$



$$(\Delta \otimes 1)R = R_{13}\, R_{23}$$
$$(1 \otimes \Delta)R = R_{13}\, R_{12}$$

AdS/CFT

Vacuum $\downarrow = Z = \phi_1 + i\phi_2$

$$\dots \downarrow \dots \downarrow \dots \downarrow \ \Uparrow \ \downarrow \dots \downarrow \dots \downarrow \dots \downarrow \dots \downarrow \dots$$

$\mathfrak{psu}(2,2|4) \longrightarrow \mathfrak{psu}(2|2)$ (excitations on the vacuum)

2 bosonic and 2 fermionic d.o.f.; even part $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$

{*rep theory of* $\mathfrak{psu}(2|2)$} [Götz-Quella-Schomerus '05]

$\mathfrak{psu}(2|2)$ only simple basic classical Lie superalgebra admitting up to 3 central extensions

[Iohara-Koga '01]

Repr. theory with triple c.e. (and q-def thereof) not much studied

[Kac priv comm]

AdS/CFT S-matrix: A is centrally-extended $\mathfrak{psu}(2|2)$

[Beisert '05]

$$
\begin{aligned}
&[\mathbb{L}_a{}^b, \mathbb{J}_c] = \delta_c^b \mathbb{J}_a - \tfrac{1}{2}\delta_a^b \mathbb{J}_c \\
&[\mathbb{L}_a{}^b, \mathbb{J}^c] = -\delta_a^c \mathbb{J}^b + \tfrac{1}{2}\delta_a^b \mathbb{J}^c \\
&\{\mathbb{Q}_\alpha{}^a, \mathbb{Q}_\beta{}^b\} = \epsilon_{\alpha\beta}\epsilon^{ab}\, \mathbb{C} \\
&\{\mathbb{Q}_\alpha^a, \mathbb{G}_b^\beta\} = \delta_b^a \mathbb{R}_\alpha{}^\beta + \delta_\alpha^\beta \mathbb{L}_b{}^a + \tfrac{1}{2}\,\delta_b^a \delta_\alpha^\beta\, \mathbb{H}
\end{aligned}
$$

$$
\begin{aligned}
&[\mathbb{R}_\alpha{}^\beta, \mathbb{J}_\gamma] = \delta_\gamma^\beta \mathbb{J}_\alpha - \tfrac{1}{2}\delta_\alpha^\beta \mathbb{J}_\gamma \\
&[\mathbb{R}_\alpha{}^\beta, \mathbb{J}^\gamma] = -\delta_\alpha^\gamma \mathbb{J}^\beta + \tfrac{1}{2}\delta_\alpha^\beta \mathbb{J}^\gamma \\
&\{\mathbb{G}_a{}^\alpha, \mathbb{G}_b{}^\beta\} = \epsilon^{\alpha\beta}\epsilon_{ab}\, \mathbb{C}^\dagger \\
&a, b, c = 1, 2 \quad \alpha, \beta, \gamma = 3, 4
\end{aligned}
$$

- Fundam rep: *Dynamical Spin-Chain*

$$\mathbb{H}\,|p\rangle = \epsilon(p)\,|p\rangle$$

$$\mathbb{C}\,|p\rangle = c(p)\,|p\,Z^-\rangle \qquad \mathbb{C}^\dagger\,|p\rangle = \bar{c}(p)\,|p\,Z^+\rangle$$

$Z^{+(-)}$: one site of the chain is added (removed)

Interpreted as a non-trivial Hopf algebra:

$$\mathbb{C} \otimes 1 \, |p_1\rangle \otimes |p_2\rangle =$$

$$\mathbb{C} \otimes 1 \sum_{n_1 << n_2} e^{i\, p_1\, n_1 + i\, p_2\, n_2} \, | \cdots Z \, Z \, V \, \underbrace{Z \cdots Z}_{n_2 - n_1 - 1} \, W \, Z \cdots \rangle$$

$$(\textit{rescale } n_2) \quad = c(p_1) \, e^{ip_2} \, |p_1\rangle \otimes |p_2\rangle$$

$$S \, \Delta(\mathbb{C}) = S \, [\mathbb{C} \otimes 1 + 1 \otimes \mathbb{C}] = S \, [e^{ip_2} \mathbb{C}_{local} \otimes 1 + 1 \otimes \mathbb{C}_{local}]$$

$$\Delta(\mathbb{C}_{local}) = \mathbb{C}_{local} \otimes e^{ip} + 1 \otimes \mathbb{C}_{local}$$

[Gomez-Hernandez '06; Plefka-Spill-AT '06]

Similar coproduct arises for the other (super)charges

$$\Delta(Q) = Q \otimes e^{i[[Q]]p} + 1 \otimes Q \qquad \underline{\neq \ \Delta^{op}(Q)}$$

$[[\mathbb{R}, \mathbb{L}, \mathbb{H}]] = 0$, $[[\mathbb{Q}]] = \frac{1}{2}$, $[[\mathbb{Q}^\dagger]] = -\frac{1}{2}$, $[[\mathbb{C}]] = 1$, $[[\mathbb{C}^\dagger]] = -1$

With central coproducts $\Delta(C)$, *consistency requires*

$$\Delta^{op}(C)\, R \ = \ R\, \Delta(C) \ = \ \Delta(C)\, R$$

R invertible, hence

$$\Delta^{op}(C) \ = \ \Delta(C) \qquad [co - commutativity]$$

Solved by $\quad e^{ip} = ig\,\mathbb{C} \ + \ 1 \quad$ physicality condition

$[\ \longrightarrow \quad \Delta(\mathbb{C}) = ig\,\mathbb{C} \otimes \mathbb{C} + \mathbb{C} \otimes 1 + 1 \otimes \mathbb{C} \ ]$

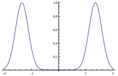
STRING WORLDSHEET PICTURE

Coproduct reproduced using [Bernard-LeClair '92] procedure by [Klose-McLoughlin-Roiban-Zarembo '06]. Alternative classical argument insp. by [Lüscher-Pohlmeyer '78; MacKay '92]. LC w.sheet susys are non-local

$$Q = \int_{-\infty}^{\infty} d\sigma\, J(\sigma)\, e^{i \int_{-\infty}^{\sigma} d\sigma' \,\partial x^{-}(\sigma')}$$

[Arutyunov-Frolov-Plefka-Zamaklar '06]

Imagine two well-separated solitonic excitations



$$\begin{aligned} Q_{|profile} &= \int_{-\infty}^{0} d\sigma\, J(\sigma) e^{i \int_{-\infty}^{\sigma} d\sigma' \,\partial x^{-}(\sigma')} \\ &+ \int_{0}^{\infty} d\sigma\, J(\sigma)\, e^{i \int_{-\infty}^{0} d\sigma' \,\partial x^{-}(\sigma')}\, e^{i \int_{0}^{\sigma} d\sigma' \,\partial x^{-}(\sigma')} \end{aligned}$$

$$\sim \quad Q_1 + e^{ip_1} Q_2 \quad \longrightarrow \quad \Delta(Q) = Q \otimes 1 + e^{ip} \otimes Q$$

YANGIANS

∃ Lie (super)algebra Q^A. Suppose ∃ additional charges $\hat{Q}^A$

$$[Q^A, Q^B] = i f^{AB}_C \, Q^C \qquad\qquad [Q^A, \hat{Q}^B] = i f^{AB}_C \, \hat{Q}^C$$

(plus Serre) with coproducts

$$\Delta(Q^A) = Q^A \otimes 1 + 1 \otimes Q^A$$

$$\Delta(\hat{Q}^A) = \hat{Q}^A \otimes 1 + 1 \otimes \hat{Q}^A + \frac{i}{2} f^A_{BC} Q^B \otimes Q^C$$

[Drinfeld '86; {*rev*} MacKay'04]

(Infinite) Spin-Chain Dolan-Nappi-Witten '03; Agarwal-Rajeev '04; Zwiebel '06; Beisert-Zwiebel '07]

Classical String [Mandal-Suryanarayana-Wadia '02; Bena-Polchinski--Roiban '03; Hatsuda-Yoshida '04; Das-Maharana-Melikyan-Sato '04]

S-matrix Yangian [Beisert '07]

We know modification

$$\Delta(Q^A) = Q^A \otimes 1 + e^{i\,[[A]]\,p} \otimes Q^A$$

Additionally, $\exists$ centrally-extended $\mathfrak{psu}(2|2)$ Yangian

$$\Delta(\hat{Q}^A) = \hat{Q}^A \otimes 1 + e^{i\,[[A]]\,p} \otimes \hat{Q}^A + \frac{i}{2}\, f^A_{BC}\, Q^B\, e^{i\,[[C]]\,p} \otimes Q^C$$

SIGMA MODEL PICTURE

Take a 2D classical field theory, with local currents

$$J_\mu = J_\mu^A \, T_A \qquad \partial^\mu J_\mu^A = 0 \qquad Q^A = \int_{-\infty}^{\infty} dx \, J_0^A$$

satisfying flatness (Lax pair)

$$\partial_0 J_1 - \partial_1 J_0 + [J_0, J_1] = 0$$

The following non-local current is conserved

$$\hat{J}_\mu^A(x) = \epsilon_{\mu\nu} \, \eta^{\nu\rho} \, J_\rho^A(x) + \frac{i}{2} f_{BC}^A \, J_\mu^B(x) \int_{-\infty}^{x} dx' \, J_0^C(x')$$

$$\frac{d}{dt} \hat{Q}^A = \frac{d}{dt} \int_{-\infty}^{\infty} dx \, \hat{J}_0^A(x) = 0$$

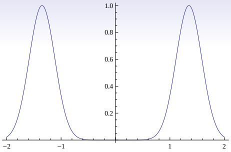
Prototype: Principal Chiral Model

$$L = Tr[\partial_\mu g^{-1} \partial^\mu g] \qquad g \in Lie$$

(left,right) global symmetry $\qquad g \longrightarrow e^{i\lambda} g,\ g\, e^{i\lambda}$

Flat currents $\qquad J^{L,R} = (\partial_\mu g) g^{-1},\ g^{-1} (\partial_\mu g) \ \in lie$

String $\longrightarrow$ $\frac{PSU(2,2|4)}{SO(4,1)\times SO(5)}$ coset $\qquad$ [Metsaev-Tseytlin action]



Classical argument [Lüscher-Pohlmeyer '78; MacKay '92]:

$$\hat{Q}^A = \int_{-\infty}^{\infty} dx\, J_1^A(x) + \frac{i}{2} f^A_{BC} \int_{-\infty}^{\infty} dx\, J_0^B(x) \int_{-\infty}^{x} dx'\, J_0^C(x')$$

Evaluating on profile

$$\hat{Q}^A_{|profile} = \int_{-\infty}^{0} J_1^A + \frac{i}{2} f^A_{BC} \int_{-\infty}^{0} J_0^B \int_{-\infty}^{x} J_0^C$$

$$+ \int_{0}^{\infty} J_1^A + \frac{i}{2} f^A_{BC} \int_{0}^{\infty} J_0^B \int_{0}^{x} J_0^C + \frac{i}{2} f^A_{BC} \int_{0}^{\infty} J_0^B \int_{-\infty}^{0} J_0^C$$

$$\longrightarrow \Delta(\hat{Q}^A) = \hat{Q}^A \otimes 1 + 1 \otimes \hat{Q}^A + \frac{i}{2} f^A_{BC}\, Q^B \otimes Q^C$$

REMARKS

- Evaluation representation:

$$\hat{Q}^A = u\, Q^A = ig\,(x^+ + \frac{1}{x^+} - \frac{i}{2g})\, Q^A$$

$x^\pm = x^\pm(p)$ also parameterize algebra representation

- Yangian symmetry in evaluation rep $\to$ *difference form*

$$S = S(u_1 - u_2)$$

S-matrix does not to possess this symmetry because $u = u(x^\pm)$

(see however [AT '09])

• Coproduct needs raising index f_C^{AB} with inverse Killing form, but...

...for $\mathfrak{psu}(2|2)$, this does not exist, yet table of coproducts can be fully determined (cf. extension by automorphisms

[Spill 'dipl.thesis, Beisert '06])

• For higher bound-states, either YBE or Yangian symmetry have to be used to completely fix S-matrix

[Arutyunov-Frolov '08, de Leeuw '08, MacKay-Regelskis '10]

Drinfeld's second realization

[Drinfeld '88]

$$[\kappa_{i,m}, \kappa_{j,n}] = 0, \quad [\kappa_{i,0}, \xi^{\pm}_{j,m}] = \pm a_{ij}\, \xi^{\pm}_{j,m},$$
$$[\xi^{+}_{i,m}, \xi^{-}_{j,n}] = \delta_{ij}\, \kappa_{j,m+n},$$

$$[\kappa_{i,m+1}, \xi^{\pm}_{j,n}] - [\kappa_{i,m}, \xi^{\pm}_{j,n+1}] = \pm\frac{1}{2} a_{ij} \{\kappa_{i,m}, \xi^{\pm}_{j,n}\}, \quad a_{ij} \text{ Cartan matrix}$$
$$[\xi^{\pm}_{i,m+1}, \xi^{\pm}_{j,n}] - [\xi^{\pm}_{i,m}, \xi^{\pm}_{j,n+1}] = \pm\frac{1}{2} a_{ij} \{\xi^{\pm}_{i,m}, \xi^{\pm}_{j,n}\},$$

$$i \neq j, \quad n_{ij} = 1 + |a_{ij}|, \quad Sym_{\{k\}} [\xi^{\pm}_{i,k_1}, [\xi^{\pm}_{i,k_2}, \ldots [\xi^{\pm}_{i,k_{n_{ij}}}, \xi^{\pm}_{j,l}] \ldots]] = 0.$$

Serre relations

AdS/CFT has central deformation

[Spill, AT '08]

$$[\kappa_{i,m}, \kappa_{j,n}] = 0, \quad [\kappa_{i,0}, \xi^{\pm}_{j,m}] = \pm a_{ij}\, \xi^{\pm}_{j,m},$$
$$[\xi^{+}_{i,m}, \xi^{-}_{j,n}] = \delta_{ij}\, \kappa_{j,m+n},$$

$$[\kappa_{i,m+1}, \xi^{\pm}_{j,n}] - [\kappa_{i,m}, \xi^{\pm}_{j,n+1}] = \pm\frac{1}{2} a_{ij}\{\kappa_{i,m}, \xi^{\pm}_{j,n}\}, \quad a_{ij} \text{ Cartan matrix}$$
$$[\xi^{\pm}_{i,m+1}, \xi^{\pm}_{j,n}] - [\xi^{\pm}_{i,m}, \xi^{\pm}_{j,n+1}] = \pm\frac{1}{2} a_{ij}\{\xi^{\pm}_{i,m}, \xi^{\pm}_{j,n}\},$$

$$(i,j) = (1,2), (2,1), (1,3), (3,1)$$
$$n_{ij} = 1 + |a_{ij}|, \quad Sym_{\{k\}}[\xi^{\pm}_{i,k_1}, [\xi^{\pm}_{i,k_2}, \ldots [\xi^{\pm}_{i,k_{n_{ij}}}, \xi^{\pm}_{j,l}] \ldots]] = 0$$
$$[\xi^{\pm}_{2,m}, \xi^{\pm}_{3,n}] = C^{\pm}_{m+n}, \qquad [C^{\pm}_{n}, \cdot] = 0$$

UNIVERSAL R-MATRIX

Given H non *co*-commutative Hopf algebra ($\Delta^{op} \neq \Delta$),
suppose $\exists$ abstract solution $R \in \mathrm{H} \otimes \mathrm{H}$ of

$$\Delta^{op} R = R \Delta$$

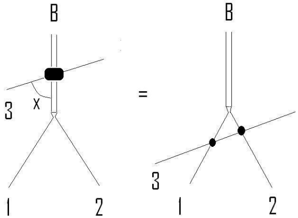
Universal: independent of reps in each factors of $\otimes$

Standard Yangian is one such H

Theorem (Drinfeld): if R satisfies Quasi-Triangularity

(rep-independent version of bootstrap principle)

$$(\Delta \otimes 1)R = R_{13}\, R_{23}$$
$$(1 \otimes \Delta)R = R_{13}\, R_{12}$$



then it also satisfies YBE and Crossing

Direct proof of properties of the S-matrix

Complete solution to scattering problem reduces to:

find the abstract tensor R given H, and then project it into your favorite (bound-state) reps

Define an integrable model and its correlation functions purely *via* its quantum group symmetry

[LeClair-Smirnov '92]

Q-operator [Matthias' talk]

Can we do it for AdS/CFT?

Yangian in short reps relatively well understood, *e.g.* all bound state S- and transfer matrices [Arutyunov-de Leeuw-(+Suzuki)-AT '09]

Classical r-matrix [AT '07, Moriyama-AT '07, Beisert-Spill '07] has nice classical double structure, but no quantization yet

Long reps show the need of extending the algebra, or else universal R-matrix does not $\exists$ [Arutyunov-de Leeuw-AT '10]

$\longrightarrow$ SECRET YANGIAN SYMMETRY [Matsumoto-Moriyama-AT '07, Beisert-Spill '07] with $\mathfrak{gl}(2|2)$ signature may be the key

$$\Delta(\hat{B}) = \hat{B} \otimes 1 + 1 \otimes \hat{B} + \frac{i}{2g}(\mathbb{G}_a^\alpha \otimes \mathbb{Q}_\alpha^a + \mathbb{Q}_\alpha^a \otimes \mathbb{G}_a^\alpha)$$

$$\Sigma(\hat{B}) = -\hat{B} + \frac{2i}{g}\mathbb{H}$$

$$\hat{B} = \frac{1}{4}(x^+ + x^- - 1/x^+ - 1/x^-)\, diag(1,1,-1,-1)$$

[Exact in g, not a strong coupling exp.]

Universal R-matrix should look stg. like [Khoroshkin-Tolstoy '96]

$$R = R_E\, R_H\, R_F$$

$$R_H = \exp\left\{ \mathrm{Res}_{u=v} \left[\sum_{i,j} \frac{\mathrm{d}}{\mathrm{d}u} (\log H_i^+(u)) \otimes D_{ij}^{-1} \log H_j^-(v) \right] \right\}$$

$$D_{ij} = -(T^{\frac{1}{2}} - T^{-\frac{1}{2}})\, a_{ij}(T^{\frac{1}{2}}), \quad a_{ij}(q) = \frac{q^{a_{ij}} - q^{-a_{ij}}}{q - q^{-1}}, \quad Tf(u) = f(u+1)$$

$$\mathrm{Res}_{u=v} \left(A(u) \otimes B(v) \right) = \sum_k a_k \otimes b_{-k-1}$$

for $A(u) = \sum_k a_k u^{-k-1}$ and $B(u) = \sum_k b_k u^{-k-1}$

$$H_i^{\pm}(u) = 1 \pm \sum_{\substack{n \geq 0 \\ n < 0}} h_n u^{-n-1}$$

Problem

$$R \sim \prod_{\alpha \in \Delta_+} \exp\left[e_\alpha \otimes e_{-\alpha}\right] \cdot \exp\left[a_{ij}^{-1} h^i \otimes h^j\right] \cdot \cdot \prod_{\alpha \in \Delta_+} \exp\left[e_{-\alpha} \otimes e_\alpha\right]$$

(Centrally extended) $\mathfrak{psl}(2|2)$ has degenerate Cartan matrix

Khoroshkin-Tolstoy: go to the next non-degenerate one ($\mathfrak{gl}(2|2)$)

$\mathfrak{gl}(2|2)$ contains B which makes a_{ij} non-degenerate

More problems

The Yangian cannot be $Y(\mathfrak{gl}(2|2))$ since at level zero

$$\Delta(B) = B \otimes 1 + 1 \otimes B \qquad\qquad B \propto diag(1,1,-1,-1)$$

is not a symmetry, *e.g* $\quad R\,|\phi_1\rangle \otimes |\phi_2\rangle \propto \ldots + |\psi_3\rangle \otimes |\psi_4\rangle$

It starts at level one

Indentation: Perhaps new type of quantum groups

[Etingof, priv comm]

SECRET (BONUS) SYMMETRY IN AMPLITUDES

[cf. talks by Volovich, Mason,Arkani-Hamed]

Tree-level planar n-particle color-ordered amplitudes

$\mathcal{A}_n(\lambda_k, \tilde{\lambda}_k, \eta_k)$ $\quad k = 1, \ldots, n$ $\qquad$ Null mom. $p_k = \lambda_k \tilde{\lambda}_k$

$\lambda_k, \tilde{\lambda}_k \in C^2$ c. c. spinors $\qquad$ $\eta_k \in C^{0|4}$ Grassmann

Yangian of $\mathfrak{psu}(2,2|4)$ [Drummond-Henn-Plefka '09]

$$\mathfrak{J}^A = \sum_{i=1}^{n} \mathfrak{J}_i^A \qquad \widehat{\mathfrak{J}}^A = f^A_{BC} \sum_{j<k=1}^{n} \mathfrak{J}_j^B \mathfrak{J}_k^C$$

$$\widehat{\mathfrak{B}} = \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} \left(\mathfrak{Q}_k^{\alpha b} \mathfrak{S}_{j,\alpha b} - \bar{\mathfrak{Q}}^{\dot{\alpha}}_{k,b} \bar{\mathfrak{S}}^b_{j,\dot{\alpha}} - \mathfrak{Q}_j^{\alpha b} \mathfrak{S}_{k,\alpha b} + \bar{\mathfrak{Q}}^{\dot{\alpha}}_{j,b} \bar{\mathfrak{S}}^b_{k,\dot{\alpha}} \right)$$

[Beisert-Schwab '11]

SECRET (BONUS) SYMMETRY IN PURE SPINORS

Group variable $g \in PSU(2,2|4)$

$$J = -dg\, g^{-1}$$

Lax connection

$$[\partial_+ + J_+(z)\,,\, \partial_- + J_-(z)] = 0$$

Non-local conserved charges generated by transfer matrix

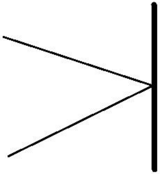
$$T(z) = g(+\infty)^{-1} \left[P\ \exp \int_C \left(-J_+(z) d\tau^+ - J_-(z) d\tau^- \right) \right] g(-\infty)$$

$$I_\xi = \text{Str}\ \xi \left(\int \int_{\sigma_1 > \sigma_2} [j(\sigma_1), j(\sigma_2)] - \int k \right) \qquad \xi \in \mathfrak{psu}(2,2|4)$$

[Berkovits-Mikhailov '11]

SECRET SYMMETRY IN BOUNDARY PROBLEMS

[MacKay-Regelskis '10; Regelskis '11]



$\mathfrak{g}$ $\qquad$ $\mathfrak{h}$ $\qquad$ $K \;:\; V_1(u) \otimes V_b \longrightarrow V_1(-u) \otimes V_b$

$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h} \qquad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m} \qquad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h} \qquad \mathfrak{m} = \mathfrak{g}/\mathfrak{h}$

Twisted Yangian $\qquad \Delta(\mathfrak{J}) \in Y(\mathfrak{g}) \otimes Y(\mathfrak{g}, \mathfrak{h})$

$Y(\mathfrak{g}, \mathfrak{h})$ gen. by $\mathfrak{J}^i$ and $\tilde{\mathfrak{J}}^p = \widehat{\mathfrak{J}}^p + f^p_{qj}\, \mathfrak{J}^q \mathfrak{J}^j \qquad i \sim \mathfrak{h} \quad p \sim \mathfrak{m}$

QUANTUM AFFINE SECRET SYMMETRY

[Beisert-Koroteev '08; De Leeuw-Matsumoto-Regelskis '11] [de Leeuw-Regelskis-AT '12]

$$DA = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix} \qquad D = \mathrm{diag}(1,-1,-1,-1)$$

$$K_i E_j = q^{DA_{ij}} E_j K_i \ , \qquad K_i F_j = q^{-DA_{ij}} F_j K_i \ ,$$

$$[E_j, F_j\} = D_{jj} \frac{K_j - K_j^{-1}}{q - q^{-1}} \ , \qquad [E_i, F_j\} = 0$$

$$\{[E_1, E_k], [E_3, E_k]\} - (q - 2 + q^{-1}) E_k E_1 E_3 E_k = g\alpha_k (1 - V_k^2 U_k^2) \ ,$$

$$\{[F_1, F_k], [F_3, F_k]\} - (q - 2 + q^{-1}) F_k F_1 F_3 F_k = g\alpha_k^{-1} (V_k^{-2} - U_k^{-2})$$

$$B_{E,F} = b_{E,F}\, diag(1, \dots, -1 \dots)$$

$$\begin{aligned} \Delta B_E =& B_E \otimes 1 + 1 \otimes B_E \\ &+ (U_2^{-1} \otimes 1)(K_{123}^{-1} E_4 \otimes \tilde{E}_{123} + K_{23}^{-1} \tilde{E}_{14} \otimes \tilde{E}_{23} + K_{12}^{-1} \tilde{E}_{34} \otimes \tilde{E}_{12} + K_2^{-1} \tilde{E}_{134} \otimes E_2) \\ &+ K_{124}^{-1} E_3 \otimes \tilde{E}_{124} + K_3^{-1} \tilde{E}_{124} \otimes E_3 \end{aligned}$$

CONCLUSIONS

• A deep mathematical structure is there, in some aspects almost reducible to standard, in some others much harder

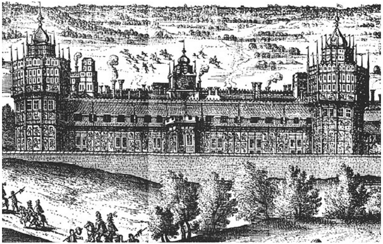
• Nevertheless, blooming of developments allowed to unveil some of the most useful bits of it

• Role of secret symmetry, quantum double, maybe one day universal R-matrix ? Relation to Q-operator [Matthias' talk]

• Exciting links to condensed matter theory (Hubbard model, recent holographic CMT) and Pohlmeyer reduction

[Grigoriev, Tseytlin, Hoare, Hollowood, Miramontes]

• Fascinating connections with Yangian and dual superconformal symmetries of scattering amplitudes await to be fully investigated



…Thank you