Mathieu Moonshine

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based on work with
with S. Hohenegger, D. Persson, H. Ronellenfitsch
and R. Volpato

K3 sigma models

Consider CFT sigma model with target K3.

Spectrum:

$$\mathcal{H} = \bigoplus_{i,j} N_{ij}\, \mathcal{H}_i \otimes \bar{\mathcal{H}}_j \ .$$

repr. of N=4 superconformal algebra

Full spectrum complicated --- only known explicitly at special points in moduli space.

K3 Moduli Space

Moduli space of K3 sigma models described by

[Aspinwall, Morrison], [Aspinwall]
[Nahm,Wendland]

$$\mathcal{M} = \mathrm{O}(4,20;\mathbb{Z})\backslash \underbrace{\mathrm{O}(4,20;\mathbb{R})/\mathrm{O}(4,\mathbb{R}) \times \mathrm{O}(20,\mathbb{R})}$$

discrete autos
of fixed
`charge lattice'

Grassmannian, describing
pos. def. 4d subspace

$$\Pi \subset \mathbb{R}^{4,20}$$

Elliptic genus

Instead of full partition function consider
`partial index' = elliptic genus:

$$\phi_{\mathrm{K3}}(\tau, z) = \mathrm{Tr}_{\mathrm{RR}}\Big(q^{L_0 - \frac{c}{24}}\, y^{J_0}\, (-1)^F\, \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}\, (-1)^{\bar{F}} \Big) \equiv \phi_{0,1}(\tau, z) \ .$$

$$(q = e^{2\pi i\tau}, \ \ y = e^{2\pi i z})$$

- constant over moduli space

- defines weak Jacobi form of weight w=0
  and index m=1.

[Kawai et. al.]

Weak Jacobi form

A weak Jacobi form is a function

[Eichler,Zagier]

$$\phi \,:\, (\tau, z) \in \mathbb{H}_+ \times \mathbb{C} \;\mapsto\; \phi(\tau, z) \;,$$

whose expansion is of the form

$$\phi(\tau, z) = \sum_{n \geq 0, \ell \in \mathbb{Z}} c(n, \ell)\, q^n\, y^l \;.$$

Under a modular transformation it transforms as

$$\phi(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}) = (c\tau + d)^w e^{2\pi i m \frac{cz^2}{c\tau + d}} \phi(\tau, z) \;,$$

where w=0 and m=1 (for K3).

Weak Jacobi form

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[Eichler,Zagier]

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whose expansion is of the form

$$\phi(\tau, z) = \sum_{n\geq 0, \ell \in \mathbb{Z}} c(n,\ell)\, q^n\, y^l \ .$$

In addition it satisfies

$$\phi(\tau, z + \ell\tau + \ell') = e^{-2\pi i m(\ell^2\tau + 2\ell z)}\, \phi(\tau, z) \quad \ell, \ell' \in \mathbb{Z} \ .$$

(spectral flow)

BPS states

Define subspace of RR states:

BPS states = right-moving ground states

(These are the states that contribute to elliptic genus.)

Then, w.r.t left-moving N=4 have decomposition

$$\mathcal{H}_{\rm BPS} = 20 \cdot \mathcal{H}_{\frac{1}{4},j=0} \oplus 2 \cdot \mathcal{H}_{\frac{1}{4},j=\frac{1}{2}} \oplus \bigoplus_{n=1}^{\infty} D_n \otimes \mathcal{H}_{n+\frac{1}{4},j=\frac{1}{2}}$$

multiplicity spaces --- not constant over moduli space.

Indices

However, since elliptic genus is constant over moduli space, the indices

$$A_n = \mathrm{Tr}_{D_n}(-1)^{\bar{F}}$$

are constant. Explicitly:

$$\begin{aligned} A_1 &= 90 &&= \mathbf{45} + \overline{\mathbf{45}} \\ A_2 &= 461 &&= \mathbf{231} + \overline{\mathbf{231}} \\ A_3 &= 1540 &&= \mathbf{770} + \overline{\mathbf{770}} \end{aligned}$$

← dims of irreps of M24!

[Eguchi, Ooguri, Tachikawa]

Virtual representations

Note: similar decomposition for the lowest coefficients

$$\mathcal{H}_{\rm BPS} = 20 \cdot \mathcal{H}_{\frac{1}{4},j=0} \oplus 2 \cdot \mathcal{H}_{\frac{1}{4},j=\frac{1}{2}} \oplus \bigoplus_{n=1}^{\infty} D_n \times \mathcal{H}_{n+\frac{1}{4},j=\frac{1}{2}}$$

They involve virtual representations:

$$\begin{aligned} 20 &= \mathbf{23} - 3 \cdot \mathbf{1} \\ -2 &= -2 \cdot \mathbf{1} \ . \end{aligned}$$

Basics of M24

The Mathieu group M24 is one of the sporadic simple finite groups. It has order

$$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 = 244823040$$

26 irreducible representations: **1 .. 10395**

26 conjugacy classes: 1A .. 23A, 23B

Basics of M24

M24 is the subgroup of the permutation group of 24 points that leaves the binary Golay code invariant.

Equivalently, it is automorphism group of Niemeier lattice

$$\mathbb{M}_{24} \cong \mathrm{Aut}(A_1^{24}\ \mathrm{Niemeier})/(\mathrm{Weyl\ reflections})$$

Why Mathieu?

Mukai Theorem: any finite group of symplectic automorphisms ($\longrightarrow$ fixes all three complex structures) of a K3 surface is isomorphic to a subgroup of the Mathieu group M23.

[The Mathieu group M23 is the subgroup of M24 (as a subgroup of the permutation group) that leaves one fixed point invariant.]

However, the symplectic automorphisms of K3 have at least 5 orbits on the set of 24 points: form proper subgroups of M23.

Conjugacy classes

Because of this, useful to separate the conjugacy classes of M24 into two classes:

16 classes with representative in M23

1A, 2A, 3A, 4B, 5A, 6A, 7A, 7B, 8A, ← geometric

11A, 14A, 14B, 15A, 15B, 23A, 23B ← non-geometric

10 classes with no representative in M23

2B, 3B, 4A, 4C, 6B, 10A, 12A, 12B, 21A, 21B. ← non-geometric

Geometrical explanation?

This geometrical point of view therefore only explains small part of Mathieu Moonshine.

Natural generalisation: consider instead of geometrical symmetries, sigma-model symmetries

see later.....

Evidence

In order to determine `correct decomposition' consider analogue of McKay-Thompson series:

$$A_n = \dim(R_n) \;\rightarrow\; \mathrm{Tr}_{R_n}(g) \qquad \forall g \in \mathbb{M}_{24}$$

Basic idea: if space of BPS states carried rep of M24, then this would lead to `twining genus'

$$\phi_g(\tau, z) = \mathrm{Tr}_{\mathcal{H}_{\rm RR}}\Big(g\, q^{L_0 - \frac{c}{24}} e^{2\pi i z J_0} (-1)^F \, \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} (-1)^{\bar{F}} \Big) \,.$$

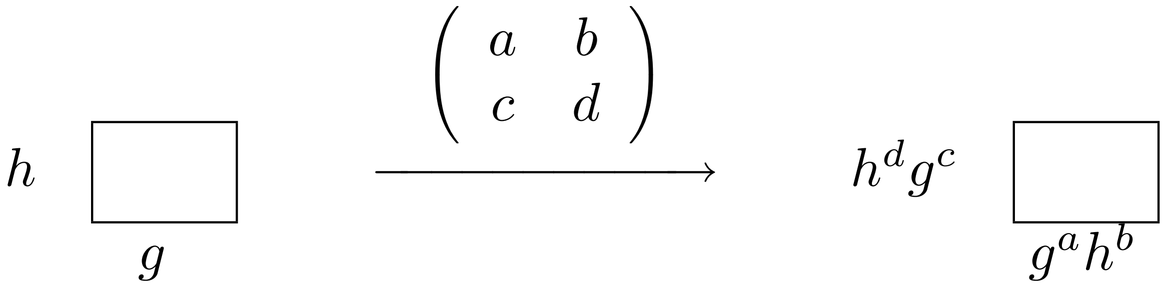
Constraints

These twining genera are constrained by

- Explicitly calculable for specific models and symmetries [Sen], [David, Jatkar, Sen], [Cheng]
- EOT ansatz fixes first few coefficients
- Modular properties

Modular properties

Indeed, standard orbifold arguments suggest that under a modular transformation



For twining genus: h=1. Thus to get same diagram need:

- c=0 mod N=o(g)
- a=1 mod N=o(g) [actually gcd(a,N)=1 sufficient]

Modular properties

Thus if group element g has order N, twining genus should be (up to a multiplier system) a Jacobi form of weight 0 and index 1 under the congruence subgroup

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \mid c \equiv 0 \mod N \right\} .$$

This is a strong constraint, provided one knows the multiplier phase...

Multiplier phase

Correct ansatz for multiplier phase:

[MRG,Hohenegger,Volpato]

$$\phi_g\Big(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\Big)=e^{\frac{2\pi i cd}{Nh}}\,e^{\frac{2\pi i\, cz^2}{c\tau+d}}\,\phi_g(\tau,z)\ ,\qquad \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma_0(N)\ ,$$

for specific h. [Later, h was identified with length of shortest cycle (when interpreted as permutation).]

[Duncan,Cheng]

With this insight explicit formulae for all twining genera were found.

[MRG,Hohenegger,Volpato]

[Also independently obtained by Eguchi & Hikami, using different methods.]

Proof of Moonshine

Twining genera give rise to explicit expression for

$$\mathrm{Tr}_{R_n}(g) \quad \forall\, n \in \mathbb{N},\;\; g \in \mathbb{M}_{24}$$

Using orthogonality of group characters, this then determines M24-representation of all multiplicity spaces!

We have worked this out for the first 500 multiplicity spaces, and all of them are indeed direct sums of M24 irreps with non-negative integer coefficients.

[Later checked by Tachikawa for first 1000 coefficients. According to Gannon, sufficient to prove for all.]

Partial Explanation

Natural way to `explain' Mathieu Moonshine:
generalise Mukai Theorem to sigma-models.

see also [Taormina, Wendland]

Moduli space:

$$\mathcal{M} = \mathrm{O}(4,20;\mathbb{Z})\backslash \underbrace{\mathrm{O}(4,20;\mathbb{R})/\mathrm{O}(4,\mathbb{R}) \times \mathrm{O}(20,\mathbb{R})}_{\Pi \subset \mathbb{R}^{4,20}}$$

N=(4,4) preserving symmetry of theory labelled by Π :

$$G_\Pi \subset \mathrm{O}(4,20;\mathbb{Z}) \ \text{ that fixes } \Pi \text{ pointwise}$$

Classification of symmetries

Following paradigm of Mukai-Kondo Theorem, can classify the possible symmetry groups to be either of

[MRG,Hohenegger,Volpato]

$$\text{(i)}\quad G = G'.G'',\ \text{where } G' \text{ is a subgroup of } \mathbb{Z}_2^{11},\ \text{and } G'' \text{ is a subgroup of } \mathbb{M}_{24}$$

$$\text{(ii)}\quad G = 5^{1+2} : \mathbb{Z}_4$$

$$\text{(iii)}\quad G = \mathbb{Z}_3^4 : A_6$$

$$\text{(iv)}\quad G = 3^{1+4} : \mathbb{Z}_2.G'',\ \text{where } G'' \text{ is either trivial, } \mathbb{Z}_2 \text{ or } \mathbb{Z}_2^2.$$

extra-special group semi-direct product [. normal subgroup]

Observations

(1) None of the K3 sigma-models has M24 as symmetry group.

[In case (i) only those elements of M24 appear which have at least 4 orbits when acting as a permutation. Thus 12B, 21A, 21B, 23A, 23B never arise.]

(2) Some K3 sigma-models have symmetries that are not contained in M24.

[In particular, this is the case for (ii), (iii) and (iv), as well as (i) with non-trivial G'.]

(3) All symmetry groups fit inside the Conway group

[but there is no evidence for `Conway Moonshine' in the elliptic genus.]

Exceptional cases

Call K3 sigma model exceptional if symmetry group is not contained in M24.

Recent observation: [MRG,Volpato]

- If sigma model is cyclic torus orbifold, then it it is always exceptional.
- Every sigma model corresponding to case (ii) is equal to $\mathbb{T}^4/\mathbb{Z}_5$
- Every sigma model corresponding to case (iii) & (iv) is equal to $\mathbb{T}^4/\mathbb{Z}_3$

Quantum Symmetry

Basic idea of argument: K3 sigma model is cyclic torus orbifold if and only if it has `quantum symmetry' whose orbifold is a torus.

$$\mathrm{K3} = \mathbb{T}^4/\mathbb{Z}_n \;\leftrightarrow\; \mathbb{T}^4 \cong \mathrm{K3}/\tilde{\mathbb{Z}}_n$$

Suppose that K3 has a symmetry g of order n, which satisfies level matching (= trivial multiplier phase) so that orbifold is consistent.

Orbifold K3

Suppose that K3 has a symmetry g of order n, which satisfies level matching (= trivial multiplier phase) so that orbifold is consistent.

By usual orbifold rules, elliptic genus of orbifold equals

$$\tilde{\phi}(\tau, z) = \frac{1}{n} \sum_{i,j=1}^{n} \phi_{g^i, g^j}(\tau, z)$$

twisted twining genus -- can be obtained from twining genus of g^d where d=gcd(i,j,n) by suitable modular transformation.

Orbifold theory

The orbifold is a torus theory if and only if its elliptic genus vanishes for z=0.

But for z=0 we have simply

$$\phi_{g^d}(\tau, 0) = \mathrm{Tr}_{24}(g^d) = \text{constant}$$

and hence

$$\phi_{g^i, g^j}(\tau, 0) = \mathrm{Tr}_{24}(g^{\gcd(i,j,n)})$$

Orbifold theory

Thus

$$\tilde{\phi}(\tau, 0) = \frac{1}{n} \sum_{i,j=1}^{n} \mathrm{Tr}_{24}(g^{\gcd(i,j,n)}) \ .$$

coincides with trace in standard 24d rep of Conway group.

For each conjugacy class of Conway can hence decide whether orbifold (if consistent) will lead to torus or not.

Conway classes

Conway group has 167 conjugacy classes, but only 42 contain symmetries that are realised by some K3.

Of these, 31 have necessarily trivial multiplier phase (since trace over 24d rep non-trivial):

- 21 lead to K3, i.e. $\tilde{\phi}(\tau, 0) = 24$

- 10 lead to T4, i.e. $\tilde{\phi}(\tau, 0) = 0$

↖ none of them has representative in M24

Quantum symmetry of T4-orbifold is always exceptional!

Other classes

The remaining 11 classes all lead to inconsistent orbifolds (=level matching not satisfied), i.e.

$$\tilde{\phi}(\tau, 0) \neq 0, 24$$

except for one possible torus orbifold. (It also does not have representative in M24.)

- If sigma model is cyclic torus orbifold, then it it is always exceptional.

✓

Other cases

To prove:

- Every sigma model corresponding to case (ii) is equal to $\mathbb{T}^4/\mathbb{Z}_5$

note that case (ii) contains 5C class of Conway (not in M24) whose orbifold is a torus.

We have also constructed the asymmetric $\mathbb{Z}_5$ orbifold explicitly and checked that it has the symmetry in case (ii).

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Similarly for:

- Every sigma model corresponding to case (iii) & (iv) is equal to $\mathbb{T}^4/\mathbb{Z}_3$ ✓

Exceptions

However, there are also exceptional K3 sigma-models in case (i). Some of them are cyclic torus orbifolds, but some of them are not

--- maybe non-abelian orbifolds?

More fundamentally, it would be important to understand what the significance of these (cyclic) torus orbifolds is....

Twisted twining genera

Another avenue: analyse analogue of Norton's generalised Moonshine, i.e. twisted twining genera

[MRG,Persson,Ronellenfitsch,Volpato, in progress]

$$Z_{g,h} = g\ \underset{h}{\square}$$

(g,h) commuting M24 elements

Main observation: twisted twining genera behave exactly as twisted twining characters of a holomorphic orbifold VOA.

[cf. Gannon]

Holomorphic orbifold

[Dijkgraaf,Vafa,Verlinde,Verlinde]
[Dijkgraaf,Witten]
[Dijkgraaf,Pasquier,Roche]
[Bantay]
[Coste,Gannon,Ruelle]

Twisted twining characters of holomorphic orbifold obey

$$Z_{g,h}(\tau+1) = c_g(g,h)\, Z_{g,gh}(\tau) \ ,$$

$$Z_{g,h}(-1/\tau) = \overline{c_h(g,g^{-1})}\, Z_{h,g^{-1}}(\tau) \ ,$$

where

$$c_h(g_1,g_2) = \frac{\alpha(h,g_1,g_2)\alpha(g_1,g_2,(g_1g_2)^{-1}h(g_1g_2))}{\alpha(g_1,h,h^{-1}g_2h)} \ .$$

3 cocycle

[Dijkgraaf,Witten]
[Dijkgraaf,Pasquier,Roche]
[Bantay]
[Coste,Gannon,Ruelle]

Here

$$c_h(g_1, g_2) = \frac{\alpha(h, g_1, g_2)\alpha(g_1, g_2, (g_1g_2)^{-1}h(g_1g_2))}{\alpha(g_1, h, h^{-1}g_2h)} \; .$$

determined by 3-cocycle

$$\alpha \in H^3(G, U(1))$$

characterises holomorphic orbifold

Multiplier phases

For the case at hand, the relevant cohomology group is

$$H^3(\mathbb{M}_{24}, U(1)) \cong \mathbb{Z}_{12}$$

[Dutour Sikiric, Ellis]

We have found all twisted twining genera, such that their multiplier phases are precisely as above, with α corresponding to primitive cocycle.

Consistency check

As before, knowledge of all twisted twining genera allows one to find decomposition of g-twisted sector with respect to stabiliser of g.

Actually, representation projective with 2-cocycle

$$c_g(\cdot,\cdot) \in H^2(C_g, U(1))$$

Decomposition leads to positive integer multiplicities!

Obstructions

Twisted twining genera related to one another by conjugation

[Dijkgraaf,Witten]
[Coste,Gannon,Ruelle]

$$Z_{g,h} = \frac{c_g(h,k)}{c_g(k,k^{-1}hk)} Z_{k^{-1}gk,k^{-1}hk}$$

If k commutes with g and h, twisted twining genus vanishes unless

$$\frac{c_g(h,k)}{c_g(k,h)} = 1$$ --- interesting obstructions!

Conclusions & open problems

Mathieu Moonshine has been `established' on the level of the

(twisted) twining genera

However, underlying `microscopic' understanding is still missing, in particular, not clear why M24 should arise, what is special about torus orbifolds, etc

Conclusions & open problems

Structure of twisted twining genera suggests underlying VOA explanation, but not clear how to construct this algebra directly --- should be some sort of BPS algebra...

[Harvey, Moore], ...