Einstein gravity amplitudes in twistor space

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Work with Tim Adamo, Freddy Cachazo & David Skinner. Adamo & M arxiv:1203.1026& arxiv:1207.3602, Cachazo & Skinner arxiv:1207.0741 Cachazo, M., Skinner arxiv:1207.????.

[Cf. also work by Bo Feng & Song He 1207.3220 and poised to go, & Mat Bullimore poised to go.]

Twistors strings $\rightsquigarrow$ remarkable progress for Yang-Mills.
What about gravity?

- Twistor-strings $\supset$ Conformal gravity [Berkovits, Witten 2004].
- Twistor-action for conformal gravity [M. 2005].

But Einstein gravity $\subset$ conformal gravity.
$\rightsquigarrow$ two strategies:

1. Try to compute Einstein gravity answer from Berkovits-Witten string ($N = 4$ SUSY) [Adamo M. 2012].
2. Guess full $N = 8$ formula, at least on momentum space generalizing Hodges' new MHV formula [Cachazo Skinner 2012].

In this talk I compare the two approaches and further developments.

From conformal gravity to Einstein gravity

Proposition (modified Maldacena after Anderson)

The conformal-gravity tree-level S-matrix evaluated on Einstein gravity wave functions with $\Lambda > 0$ gives $\Lambda\times$ Einstein S-matrix.

Proof: (Idea) Tree S-matrix = action evaluated on perturbatively constructed field g from data $g|_{\mathscr{I}} = \sum_{i-1}^{n} \epsilon_i g_i$

$$\mathcal{C}(1,\dots,n) = \text{coeff. of } \prod_i \epsilon_i \text{ in } S_{CG}[g] \qquad \text{similarly} \qquad \mathcal{M} \leftrightarrow S_{EG}$$

If g is Einstein $R_{ab} = \Lambda g_{ab}$ from Einstein data then:

$$S_{CG} = \int \mathrm{Weyl}^2 = \text{Euler class} + \int \Lambda^2 \mathrm{dvol}$$

whereas

$$S_{EG} = \int \Lambda \mathrm{dvol}$$

So also perturbatively $S_{CG} = \Lambda S_{EG} \quad \Rightarrow \quad \mathcal{C} = \Lambda \mathcal{M}$.
But need to take care of boundary terms. □

Twistor space is $\mathbb{PT} = \mathbb{CP}^{3|4}$, homogeneous coords:

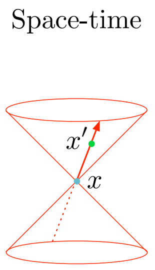
$$Z = Z_I = (\lambda_A, \mu^{A'}, \chi^a) \in \mathbb{T} := \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^{0|4}, \qquad Z \sim \zeta Z\,, \zeta \in \mathbb{C}^*\,.$$

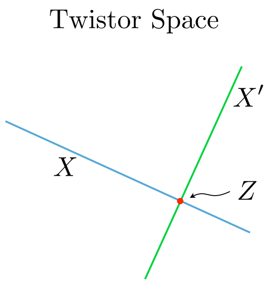
$\mathbb{T}$ = fund. repn of superconformal group $SU(2,2|4)$.

A point $x \in \mathbb{M}^{4|8} \leftrightarrow$ a line $X = \mathbb{CP}^1 \subset \mathbb{PT}$ via incidence relations

$$\mu^{A'} = -ix^{AA'}\lambda_A\,, \qquad \chi^a = \theta^{aA}\lambda_A\,.$$

Two points x, x' are null separated iff X and X' intersect.





Fields: $Z = Z(\sigma, \bar\sigma)$, $Y = Y_\sigma(\sigma,\bar\sigma)\mathrm{d}\sigma$, σ coord on worldsheet Σ:

$Z : \Sigma \to \mathbb{T}\,, \quad Y \in \Omega^{1,0}(\Sigma) \otimes T^*\mathbb{T}\,, \quad$ and $\quad a = a_{\bar\sigma}\mathrm{d}\bar\sigma \in \Omega^{0,1}(\Sigma)\,.$

The action is:

$$S[Z, Y, a] = \int_\Sigma Y_I \bar\partial Z^I + a\, Z^I\, Y_I\,, \qquad \bar\partial Z = \frac{\partial Z}{\partial \bar\sigma}\mathrm{d}\bar\sigma$$

Gauge freedom: $(Z, Y, a) \to (\mathrm{e}^\alpha Z, \mathrm{e}^{-\alpha} Y, a - \bar\partial\alpha)$, $\alpha \in \mathbb{C}$
reduces to target $\mathbb{PT}$.
Action can be peturbed by Vertex operators

$$V_F := V_f + V_g := \int_\Sigma f(Z)^I Y_I + g(Z)_I \mathrm{d}Z^I\,.$$

V_f deforms complex structure on $\mathbb{PT}$ and V_g is ‘B-field’.

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Conjectured form for amplitudes in conformal gravity

Path integral reduces to integral over zero-modes

$$\mathcal{C}(1,\ldots,n) = \sum_{d=0}^{\infty} \int_{\mathscr{M}^g_{d,n}} \mathrm{d}\mu_d \langle V_{F_1} \ldots V_{F_n} \rangle_d \,, \qquad (1)$$

$g =$ genus of Σ and $d =$ degree of map $n =$ marked points.

- For tree-amplitudes, take $\Sigma \cong \mathbb{CP}^1$ $g = 0$.
- Coordinatize Σ with homogeneous coords $\sigma = (\sigma_0, \sigma_1)$
- So $\mathscr{M}_{d,n} =$ maps $Z : \mathbb{CP}^1 \to \mathbb{PT}$, degree-$d$ (weight d in σ)

$$Z(\sigma) = \sum_{r=0}^{d} U_r \sigma_0^r \sigma_1^{d-r} \,, \qquad \mathrm{d}\mu_d = \frac{\prod_r \mathrm{d}^{4|4} U_r}{\mathrm{Vol}\ GL(2)} \,.$$

- Correlator computed from

$$\langle Y_I(\sigma)\, Z^J_j(\sigma') \rangle_d = \frac{\delta_I^J}{(\sigma\,\sigma')} \frac{(\xi\,\sigma')^{d+1}}{(\xi\,\sigma)^{d+1}}$$

where $(\sigma\,\sigma') = \sigma_0\sigma_1' - \sigma_1\sigma_0'$ and $\xi \in \Sigma$ is gauge choice.

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Reduction to Einstein gravity

Introduce 'infinity twistors' $I_{\alpha\beta}$, $I^{\alpha\beta}$ (fermionic part $= 0$)

$$I_{\alpha\beta} = \begin{pmatrix} \varepsilon^{AB} & 0 \\ 0 & \Lambda\varepsilon_{A'B'} \end{pmatrix}, \qquad I^{\alpha\beta} = \begin{pmatrix} \Lambda\varepsilon_{AB} & 0 \\ 0 & \varepsilon^{A'B'} \end{pmatrix}.$$

$\Lambda =$ cosmological const., rank two when $\Lambda = 0$, and

$$I^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta} I_{\gamma\delta}\,, \quad I^{\alpha\beta} I_{\beta\gamma} = \Lambda\delta^{\alpha}_{\gamma}\,,$$

$\rightsquigarrow$ Poisson structure $\{,\}$, & 1-form τ, weights -2, 2:

$$\{h_1, h_2\} := I^{IJ}\partial_I h_1 \partial_J h_2\,, \qquad \tau = I_{IJ} Z^I \mathrm{d} Z^J\,,$$

Vertex operators for Einstein gravity are:

$$V_h = \int_\Sigma I^{IJ} Y_I \partial_J h =: \int_\Sigma Y \cdot \partial h\,, \quad V_{\tilde h} = \int_\Sigma \tilde h \wedge \tau\,. \qquad (2)$$

h, $\tilde h$ have weights 2, $-2 \leftrightarrow +$ve and $-$ve helicity multiplets.

Einstein amplitudes from twistor-strings

If Berkovits-Witten twistor-string correctly gives conformal gravity amplitudes, then $\mathcal{C} = \Lambda\mathcal{M}$ gives for Einstein N^{k}MHV

$$\begin{aligned}\Lambda\,\mathcal{M}_n^k &= \int_{\mathscr{M}_{k+1,n}} \mathrm{d}\mu_d \left\langle V_{\tilde{h}_1} \cdots V_{\tilde{h}_{k+2}} V_{h_{k+3}} \cdots V_{h_n} \right\rangle_d \\ &= \int_{\mathscr{M}_{k+1,n}} \mathrm{d}\mu_{k+1} \left\langle \tilde{h}_1\tau_1 \cdots \tilde{h}_{k+2}\tau_{k+2} Y_{k+3}\cdot\partial h_{k+3} \cdots Y_n\cdot\partial h_n \right\rangle_{k+1}\end{aligned}$$

we need degree of map $d = k+1$. Consequences:

- RHS is polynomial degree n in Λ.
- RHS $= 0$ when $\Lambda = 0$.
- $O(\Lambda)$ part gives Einstein at $\Lambda = 0$.

To check we must insert momentum eigenstates

$$\tilde{h}_j = \int_{\mathbb{C}} s_j \,\mathrm{d}s_j\, \bar{\delta}^2(s_j\lambda_j - p_j)\, \mathrm{e}^{is_j[[\mu_j\tilde{\lambda}_j]]}\,, \quad h_j = \int_{\mathbb{C}} \frac{\mathrm{d}s_j}{s_j^3}\, \bar{\delta}^2(s_j\lambda_j - p_j)\, \mathrm{e}^{is_j[[\mu_j\tilde{\lambda}_j]]}\,,$$

construct correlator and integrate to compare to known results.

Lemma

Each $\langle Y_i \cdot \partial h_i \, \tau_j \rangle$ contraction leads to a power of Λ in the answer.

Proof: Each such contraction leads to $I^{\alpha\beta} I_{\beta\gamma} = \Lambda \delta^{\alpha}_{\gamma}$.$\Box$

Degree $= 0$**:** maps are just points $\rightsquigarrow$ easy integration over $\mathbb{PT}$, gives $\Lambda\times$ standard $\overline{\text{MHV}}$-3 point amplitude $+\Lambda^2\times$ new term.

Degree $= 1$**:** Maps are now lines $\leftrightarrow$ points in $M^{4|8}$

$$\mu^{A'} = -ix^{AA'}\lambda_A \,, \qquad \chi^a = \theta^{aA}\lambda_A \,, \qquad \lambda_A = \sigma_A \,.$$

and this formula fixes vol $GL(2)$ freedom.

Three point:

- requires one Y-contraction $\langle \tilde{h}_1 \tau_1 \tilde{h}_2 \tau_2 Y_3 \cdot \partial h_3 \rangle$.
- Standard rules above lead to an error term $O(\Lambda)$!
- Correct answer is obtained by geometric construction:

$$\mathcal{C}(1,2,3) = \Lambda \, \mathcal{M}(1,2,3) = \Lambda \, \frac{\langle 1\,2 \rangle^2}{\langle 1\,3 \rangle^2 \langle 2\,3 \rangle^2} (1 + \Lambda \Box_p) \delta^{4|8}\left(\sum_i P_i\right)$$

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Degree = 1, general n, $\Lambda = 0$

- The $\langle Y \cdot \partial h_i \, h_j \rangle$ or $\langle Y \cdot \partial h_i \, \tilde{h}_j \rangle$ contractions $\rightsquigarrow$ the factor

$$\tilde{\phi}^i_j := \frac{[ij]\langle \xi \, j\rangle^2}{\langle ij\rangle\langle \xi i\rangle^2}, \qquad i \neq j$$

- At $\Lambda = 0$, $\langle Y_i \cdot \partial h_i \, \tau_j \rangle = 0$, so $Y_i \cdot \partial h_i$ contracted with all h_j, $\tilde{h}_j$s gives factor

$$-\tilde{\phi}^i_i := \sum_{j \neq i} \frac{[ij]\langle \xi \, j\rangle^2}{\langle ij\rangle\langle \xi i\rangle^2}, \qquad \text{defines } \tilde{\phi}^i_j \text{ for } i = j$$

- Gauge invariant by momentum conservation.
- For the MHV amplitude at $\Lambda = 0$ this gives

$$\prod_{i=1}^{n-2} (-\tilde{\phi}^i_i) \, .$$

This is generically non-zero!

Resolution: only allow Feynman diagrams for the correlator that are connected trees.

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Proposition

If only connected trees are allowed for the contractions, the conformal gravity amplitude vanishes at $\Lambda = 0$.

Matrix tree theorem: $\sum$ Feynman tree graphs $= \det \mathcal{L}_{n-1\,n}$ where:

- draw master graph G of all possible $\langle Y_i\, h_j\rangle$, $\langle Y_i\, \tilde{h}_j\rangle$ contractions orienting line from i to j.
  1. $n-2$ white vertices for $Y_i \cdot \partial h_i$,
  2. 2 black vertices for $\tilde{h}_i \tau_i$.
  3. white vertices have $n-1$ outgoing edges, $n-3$ incoming.
  4. black vertices $n-2$ incoming.
- The weighted Laplacian matrix $\mathcal{L}$ of G has (i,j)-entries

$$\mathcal{L} = \begin{cases} \tilde{\phi}^i_j & i \neq j, n-1, n \\ 0 & i = n-1, n \\ \tilde{\phi}^i_i := -\sum_{j\neq i} \tilde{\phi}^i_j & i = j \end{cases}$$

- Let $\mathcal{L}_{n-1\,n} := \{\mathcal{L}-$ rows & columns $n-1$ and $n\}$.

Definition

The Hodges matrix $\tilde{\Phi} := D^{-1}\tilde{\phi}D$ *where* $D = \mathrm{diag}\{\langle\xi\, i\rangle^2\}$

- $\tilde{\Phi}$ has co-rank three

$$\sum_j \tilde{\Phi}^i_j \lambda_{jA}\lambda_{jB} = 0\,.$$

  following from momentum conservation.
- $\mathcal{L}_{n-1\,n} = \tilde{\phi}_{n-1\,n}$ (with $n-1$ and nth row & column removed) as they only differ in the $n-1$ and nth row and column.
- So $\det \mathcal{L}_{n-1\,n} = \det \tilde{\phi}_{n-1\,n} = 0$ as $\tilde{\Phi}$ and $\tilde{\phi}$ have 3-d kernel.

$\mathcal{C} = 0$ at $\Lambda = 0$ now follows.□

MHV at order Λ, the Hodges formula

- At $O(\Lambda)$ one Y_i must contract with τ_{n-1} or τ_n.
- The other $n-3$ contractions must connect remaining white vertices with one outgoing edge connecting to i, $n-1$ or n.
- Matrix tree theorem gives sum of contributions as factor

$$\det \mathcal{L}_{i\,n-1\,n} = \det \tilde{\Phi}_{i\,n-1\,n}$$

multiplied by 3pt amplitude for $\mathcal{M}(i, n-1, n)$.
- This is a version of Hodges' formula.

Note: $\tilde{\Phi}$ and Hodges formula have manifest permutation symmetry and polynomial complexity.

- Contractions $\langle Y_i \cdot \partial h_i\, h_j \rangle_d\ \langle Y_i \cdot \partial h_i\, \tilde{h}_j \rangle_d$ now give generalized Hodges matrices

$$\tilde{\phi}_i^j = \begin{cases} \frac{[i\,j]}{(i\,j)} \frac{(\xi\,j)^{d+1}}{(\xi\,i)^{d+1}} & i \neq j \\ -\sum_{k\neq i} \tilde{\phi}_i^k & i = j \end{cases}$$

- This has co-rank $k+3$ because relations

$$\sum_j \tilde{\phi}_i^j \sigma_{jA_1} \dots \sigma_{jA_{k+2}} \frac{(\xi\,i)^{d+1}}{(\xi\,j)^{d+1}} = 0$$

  follow from $\sum_j \tilde{\lambda}_j \sigma_{jA_1} \dots \sigma_{jA_{k+1}} = 0$.
- At $\Lambda = 0$, Matrix-tree theorem gives as determinant of a $n-k-2$ minor of $\tilde{\phi}$ that vanishes by co-rank $k+3$.
- At $O(\Lambda)$ with one $\langle Y_i\, \tau_j \rangle$ contraction, Matrix-tree theorem yields answer as determinant of $n-k-3$ minor of $\tilde{\phi}$.

The Cachazo-Skinner formula

Theorem (Cachazo, M, Skinner)

The tree-level S-matrix for $N = 8$ supergravity is given by

$$\mathcal{M}_n^{d-1}(1,\ldots,n) = \int_{\mathscr{M}_{d,n}} \mathrm{d}\mu_d \ \mathrm{det}'(\tilde{\Phi}^d)\mathrm{det}'(\Phi^d)\prod_{i=1}^{n} \mathrm{D}\sigma_i \ h_i\left(Z(\sigma_i)\right),$$

where $\tilde{\Phi}$ is conjugate to $\tilde{\phi}$ as above and

$$\Phi_i^j = \frac{\langle\lambda(\sigma_i)\,\lambda(\sigma_j)\rangle}{(i\,j)} \qquad\qquad i \neq j$$

etc., has rank d and det' is det of a minor of maximal rank divided by Vandermonde factors.

- Parity invariance follows by formal Fourier transform.
- Factorization follows by careful analysis following Gukov-Motl-Neitzke/Vergu strategy for Yang-Mills.
- $1/z^2$ fall-off under BCFW shift follows from link repn.

Conclusions: Twistor-strings vs Cachazo-Skinner

For the Berkovits-Witten twistor-string we have

- confirmed that twistor-string gives zero at $\Lambda = 0$ as required by Maldacena argument.
- obtained Hodges formula for MHV (with subtlety over 3pt)
- obtained minor of $\tilde{\Phi}$ in Cachazo-Skinner formula.

But

- We do not know how to obtain $\det' \Phi$ or other Vandermonde factors.
- We can only see $N = 8$ permutation symmetry as an emergent phenomenom from $N = 4$ perspective.

So

- more work needed to understand Berkovits-Witten twistor-string.
- Is there an $N = 8$ SUGRA twistor-string as well?

Thank You!