0.

Cubic 4-folds and K3 surfaces.

Plan of talk.

Describe analogy   Beauville-Donagi, Voisin, ...

cubic 4-folds $\longleftrightarrow$ K3 surfaces

X   S

1. Hassett : H^*

Codim-1 locus of cubics which look like K3 to H^*.

HH conj

Rational cubics

K conj

2. Kuznetsov: $D^b(\mathrm{coh})$

Locus of cubics which look like K3 to $D^b(\mathrm{coh})$

3. Joint work with Nick Addington

(Almost) Same loci

1. $X \subset \mathbb{P}^5$ smooth cubic 4-fold

moduli space $H^0(\mathcal{O}(3))/GL(6,\mathbb{C})$ 20-dimensional

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Hodge diamond $H^*(X) =$

$$\begin{array}{ccccc} & & 1 & & \\ & & 1 & & \\ 0 & 1 & 21 & 1 & 0 \\ & & 1 & & \\ & & 1 & & \end{array}$$

$H^{2,2}(X)$

$H^{3,1} \subset H^4 \rightsquigarrow$ period pt

Torelli (Voisin)

Remove uninteresting bit coming from $\mathbb{P}^5$ and Lefschetz theorems to leave

$$H^4_{prim}(X) = 0 \quad 1 \quad 20 \quad 1 \quad 0$$

cf. K3 surface S $\quad H^2(S) = \ 1 \quad 20 \quad 1$

$H^{2,0}$ $\quad H^{1,1}(S)$

<u>But</u> intersection forms have signatures

$H^4_{prim}(X) \quad (20, 2)$

$H^2(S) \quad (3, 19)$

2.

<u>Polarised</u> K3s (S,L) have $H^2_{prim}(S) = \langle L \rangle^\perp$ of signature $(2,19)$. Can often be found in $H^4_{prim}(X)$, as orthog to some integral $(2,2)$ class T.

So pass to Noether-Lefschetz loci on both sides.

Data on each side classified by an integer $d > 0$, up to $\mathrm{Aut}(H^*_{\mathbb{Z}})$.

- degree $L = \int_S L^2$
- discriminant $\langle h^2, T \rangle$

$$= \det \begin{pmatrix} h^2 . h^2 & h^2 . T \\ T . h^2 & T^2 \end{pmatrix}$$

3.

Generally $H^{1,1}(S,\mathbb{Z}) := H^{1,1}(S) \cap H^2(S,\mathbb{Z})$ is zero; NL_d is locus where $\beta \in H^2(S,\mathbb{Z})$ ($\deg\beta = d$) has type $(1,1)$ so $H^{1,1}_{\mathbb{Z}}$ has rank $(\geq)$ 1.

divisor: one equation $\int_\beta \sigma^{2,0} = 0$

Similarly, for general X, $\mathrm{rk}\, H^{2,2}_{\mathbb{Z}} = 1$ generated by h^2

ie $H^{2,2}_{prim}(X,\mathbb{Z}) = 0$

h hyperplane class on $\mathbb{P}^5$

For X in a divisor NL_d in moduli space, $H^{2,2}_{\mathbb{Z}} = \langle h^2, T \rangle$ has rank $(\geq)$ 2

irreducible!

one equation $\int_T \sigma^{3,1} = 0$

Now can find that $H^2_{prim}(S) = \langle L \rangle^{\perp}$ can be isomorphic to $\langle h^2, T\rangle^{\perp} \subset H^4_{prim}(X)$ for some X, S.

as a Hodge structure with a Tate twist. discriminant d degree d

<u>Thm</u> (Hassett)
$\langle h^2, T\rangle^{\perp} \subset H^4(X)$ is isomorphic to the primitive Hodge structure of a polarised K3 surface (S, L) iff

(*) d even, not divisible by $4, 9$, nor any prime of the form $6n+5$

ie $d =$ "6", $14, 26, 38, \ldots$

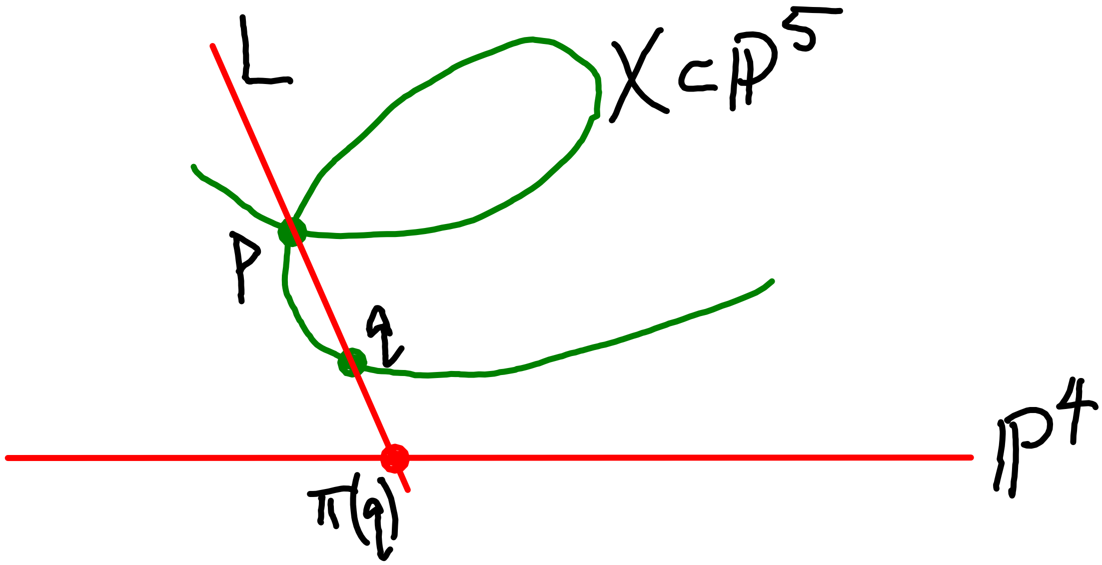
Abstract K3 produced by Torelli.
But $\exists$ geometric constructions for small d.

5. "$d=6$" Cubic 4-folds X with an odp $p\in X$.

$\exists$ a $\mathbb{P}^4$ of lines L in $\mathbb{P}^5$ through p.

Generic L hits X in $3-2=1$ more point $q\in X$

$\Rightarrow$ Birational map $X \overset{\pi}{\dashrightarrow} \mathbb{P}^4$, $q \mapsto L$.



In fact $\pi : Bl_p X \longrightarrow \mathbb{P}^4$ blows down the universal line ($\mathbb{P}^1$ bundle) over $S \subset \mathbb{P}^4$

$S = \{\text{lines in } X \text{ through } p\} = \text{cubic} \cap \text{quadric}$

$= K3$ surface!

6.

ie $Bl_p X \cong Bl_S \mathbb{P}^4$

Gives a correspondence $\subset X \times S$

and so maps $H^4(X) \longleftrightarrow H^2(S)$ etc.

<u>$d=14$</u> Beauville-Donagi.

Pfaffian cubics

2-forms of $rk \le 4$ $\qquad$ $\mathbb{P}(\Lambda^2 \mathbb{C}^6)$

$$
\begin{tikzcd}
Pf(4,6) \subset \mathbb{P}^{14} \arrow[rr, leftrightarrow, "\text{proj.}", "\text{dual}"'] \arrow[d, "\substack{\text{Intersect with} \\ \mathbb{P}^5 \subset \mathbb{P}^{14}}"'] & & Gr(2,6) \subset (\mathbb{P}^{14})^* \arrow[d, "\substack{\text{Intersect with} \\ \text{dual } \mathbb{P}^8 \subset (\mathbb{P}^{14})^*}"] \\
X & & S
\end{tikzcd}
$$

Gives correspondence in $X \times S$, $H^4(X) \leftrightarrow H^2(S)$.

All cubics $X \in NL_{14}$ rational

7.

Non-example

<u>$d=8$</u> Cubics containing a plane.

$$P \subset X.$$ $\operatorname{disc}\langle h^2, P\rangle = 8$

Different $\mathbb{P}^2 := \{3\text{-planes } P \subset \mathbb{P}^3 \subset \mathbb{P}^5\}$

Such a $\mathbb{P}^3$ intersects X in

$$P \cup Q$$ ← quadric surface

$\Rightarrow \mathbb{P}^2$-family of quadric surfaces Q.

In fact $\mathrm{Bl}_P X \xrightarrow{\pi} \mathbb{P}^2$ quadric surface fibration

Generic fibre $\mathbb{P}^1 \times \mathbb{P}^1$

Singular fibres (cone on conic) over discriminant (sextic curve $C_6 \subset \mathbb{P}^2$)

8.

So $M = \left\{ \begin{array}{l} \text{lines in quadric surface} \\ \text{fibres of } Bl_p X \to \mathbb{P}^2 \end{array} \right\}$

is a $\mathbb{P}^1$ bundle over

$S =$ double cover of $\mathbb{P}^2$ branched over sextic C_6.

Parametrises choices of ruling on fibres

$\mathbb{P}^1$ bdl $M \to S$ has a Brauer class $\alpha \in H^2(S, \mathcal{O}_S^*)$

- Obs to finding line bdl on M restricting to $\mathcal{O}(1)$ on each $\mathbb{P}^1$
- Obs to finding vector bundle whose projectivisation is $\mathbb{P}^1 \to M \to S$
- Obs to finding a rational section of $M \to S$.

9.

In general $H^2_{prim}(S) \cancel{\Leftrightarrow} H^4_{prim}(X)$

unless work with $\mathbb{Z}[\frac{1}{2}]$ or $\mathbb{Q}$.

But when $\exists\, T' \in H^{2,2}_{\mathbb{Z}}$ (as well as P) such that $\int_{\substack{\text{Quadric}\\ \text{surface fibre}}} T'$ is odd

ie $\int_X T' \wedge (h^2 - P)$ is odd

Since $\int_X h^2 = 2$, can take $\int_X T' \wedge (h^2 - P) = 1$ wlog

By pushing and pulling from X to M get divisor on M with degree 1 on $\mathbb{P}^1$ fibres.

$\Leftrightarrow$ $Br = 0$ and we can find a <u>section</u> of the quadric surface bundle $Bl_P X \to \mathbb{P}^2$.

10.

Stereographic projection from this section $\Rightarrow X$ birat. to $\mathbb{P}^2$ bdl$/\mathbb{P}^2$

$\Rightarrow X$ rational.

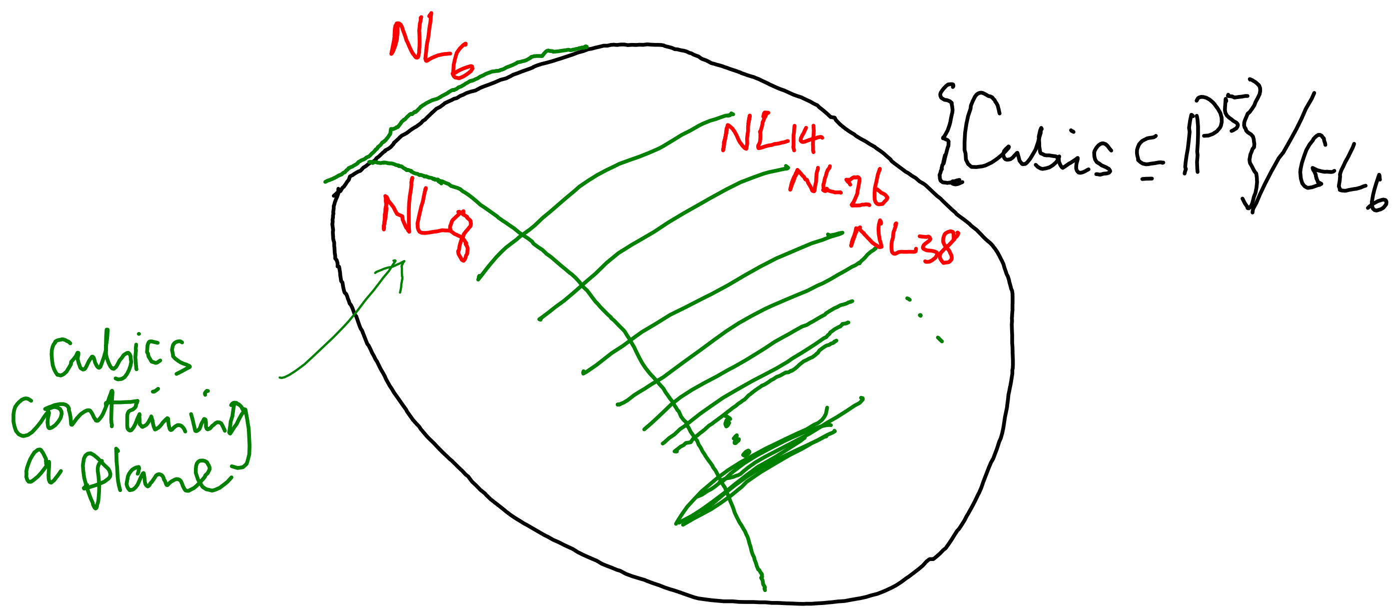
Also we show (~~Hassett probably knew~~)

that in this case can find $T \in \langle h^2, T, P\rangle$

with $\mathrm{disc}\langle h^2, T\rangle = d$ such that

$X \in NL_d$, and all NL_d (d satisfying ⊛)

intersect NL_8.



So $H^2_{prim}(S,\mathbb{Z}) \hookrightarrow H^4_{prim}(X,\mathbb{Z})$ (already known - see Kuznetsov result next)

Rationality.

Harris? Hassett? Conjecture that X is rational iff $\exists\ H^2_{prim}(S,\mathbb{Z}) \hookrightarrow H^4_{prim}(X,\mathbb{Z})$

ie $H^2_{prim}(S,\mathbb{Z}) \cong \langle h^2, T\rangle^{\perp}$ for some $T \in H^{2,2}_{prim}(X,\mathbb{Z})$ with discriminant d satisfying $(*)$.

Rough idea:
birational map should blow up a surface S somewhere, and that will give a correspondence to a K3 surface S.

One thing better than correspondences:
Fourier-Mukai kernels.

12. Kuznetsov's derived category approach.

Categorifies the Hodge theory approach, in some sense

$\{E \in D(X) : \forall i = 0,1,2,\ \mathrm{Ext}^*(\mathcal{O}(i), E) = 0\}$

$$D(X) = \langle \mathcal{A}_X, \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2) \rangle$$

exceptional collection:

$$\mathrm{Ext}^* = \begin{tikzcd} \bullet \arrow[loop above, "id"] \arrow[r, "\times 6"'] & \bullet \arrow[loop above, "id"] \arrow[r, "\times 6"'] & \bullet \arrow[loop above, "id"] \end{tikzcd}$$

$\Rightarrow$ Can use Gram-Schmidt to project onto $\langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2) \rangle^{\perp} = \mathcal{A}_X$

(Replace E by $\mathrm{Cone}\left(R\mathcal{H}om(\mathcal{O}, E) \otimes \mathcal{O} \to E\right)$ etc.)

$$\begin{tikzcd} \mathcal{A}_X \arrow[r, shift left] & D(X) \arrow[l, shift left, "\pi_A"] \end{tikzcd}$$

13.

$\mathcal{A}_X$ is a 2-dimensional CY category

$R\mathrm{Hom}(A,B) \cong R\mathrm{Hom}(B,A)^*[-2]$

ie Serre functor $\cong [2]$.

with same sized Hochschild homology as $D(K3)$, etc. "Noncommutative K3"

Mirror LG-model $Y^4 \to \mathbb{C}$ has 4 singular fibres: 3 odps and one with singular set a K3.

"Explains" Beauville-Donagi holomorphic symplectic structure on Fano of lines $F(X)$ in X..

$F(X)$ is a moduli space of objects $\pi_{\mathcal{A}}(\mathcal{I}_L)$ of $\mathcal{A}$, so inherits Mukai's holomorphic symplectic structure coming the trivialisation of the Serre functor

holo 2-form

Call $\mathcal{A}_X$ <u>geometric</u> iff $\mathcal{A}_X \cong D(S)$ for some K3 surface S.

<u>Conjecture</u> (Kuznetsov)

X rational $\iff$ $\mathcal{A}_X$ geometric.

Intuition again that birational map will blow up a surface S, introducing D(S) into D(X)

15.

<u>Eg</u> $d=8$ again: $P \subset X$.

$$\begin{tikzcd} \text{Quadric surface} \arrow[r, hook] & Bl_P X \arrow[d] \\ & \mathbb{P}^2 \end{tikzcd} \qquad \begin{tikzcd} S = \{\text{rulings of fibres}\} \arrow[d, two heads] \\ \mathbb{P}^2 \end{tikzcd}$$

S parametrises certain sheaves on X

$$\begin{array}{c} \Downarrow \\ x \longmapsto l_* f_L \end{array}$$

where L is a line in the fibre $Q \xhookrightarrow{l} Bl_P X$ in the family x.

In fact S = moduli space of objects in $\mathcal{A}_X$.

$\pi_{\mathcal{A}} l_* f_L$

Universal object on $S \times X$ <u>twisted</u> by Brauer class $\alpha \in H^2(\mathcal{O}_S^*)$.

Use as FM transform: $D(S,\alpha) \longrightarrow D(X)$

<u>Thm</u> (Kuznetsov)

Factors through an equivalence

$$D(S,\alpha) \cong \mathcal{A}_X .$$

So X geometric if $\alpha = 0$.

What we do: <u>try</u> to show that the two conjectures of HH and K are the same.

ie that $X \in NL_d$ (d satisfying $(*)$)

$$\Longleftrightarrow \mathcal{A}_X \text{ geometric.}$$

17.

Think of $a = [\mathcal{O}_{point}]$, $b = [\mathcal{O}_S]$ in $K(D(S))$

1. d satisfies ⊛ $\Leftrightarrow$ $\exists\, a, b \in K(\mathcal{A}_X)$ such that a is pointlike $\langle a, a\rangle = 0$ and b is linelike $\langle a, b\rangle = 1$, $\langle b, b\rangle = 2$

(Hint that) $\mathcal{A}_X$ contains points! Geometric!
The K3 we want is moduli space of these points!

2. Show each NL_d (d satisfies ⊛) intersects NL_8.

3. Here $\alpha = 0$; Kuznetsov gives us $D(S) \cong \mathcal{A}_X$.

But this is the "wrong" equivalence for NL_d; it's only the "right" one for NL_8.

18.

4. Mukai-Orlov

another K3

$\Rightarrow \exists$ 2dim moduli space $\mathcal{M}$ of sheaves on S of class a

(fine moduli space since $\exists b$ with $\langle a,b\rangle = 1$)

$$\begin{array}{ccc} \mathcal{O}_m & \longmapsto & m \\ \Uparrow & & \Uparrow \end{array}$$

$$\text{Given} \quad D(\mathcal{M}) \cong D(S) \cong \mathcal{A}_X$$

$\cong$

The "right" equivalence for NL_d.

5. Now deform off NL_8 back into NL_d.

Intuitively: have classes a, b on all of NL_8 so should be able to form moduli space of objects of type a to get our K3 and equivalence $D(K3) \cong D(\mathcal{A}_X)$

But don't have stability conditions etc. to make this moduli space.

(On $NL_g \cap NL_d$ could use Kuznetsov then Mukai–Orlov to form this moduli space)

Instead use Torelli/Hodge theory to deform M with $X \in NL_d$

Show the universal object / FM kernel $\in D(M \times X)$

deforms with them, to all orders ($\Rightarrow$ over a Zariski open).

cf. Toda, Huybrechts–Macrì–Stellari.

Eg. to 1st order:

$Obs \in Ext^2_{M\times M}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \cong H^2(M, \mathbb{C})$

can be identified with

$(\text{def. of } M \in H^{1,1}(M))$

$- (\text{def of } X \in H^{2,2}(X))$

thought of as lying in $H^{1,1}(M) \subset H^{2,2}(X)$

By construction of the deformation of M, this is zero.