Quadratic differentials as stability conditions

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Our main result identifies spaces of meromorphic quadratic differentials on Riemann surfaces with spaces of stability conditions on certain CY_3 triangulated categories.

On the one hand this provides interesting examples of spaces of stability conditions on CY_3 categories. On the other, it opens the possibility of applying Donaldson-Thomas theory to problems involving the trajectory structure of quadratic differentials.

For technical details we rely heavily on results of Daniel Labardini-Fragoso.

Many of our constructions are inspired by

D. Gaiotto, G. Moore, A. Neitzke: 'Wall-crossing, Hitchin systems and the WKB approximation'.

The next step will be to try to understand the connections with Fock-Goncharov co-ordinates and cluster varieties described there.

Our categories are indexed by bordered surfaces with marked points $(\mathbb{S}, \mathbb{M})$. Here $\mathbb{S}$ is a smooth surface, with or without boundary, and $\mathbb{M} \subset \mathbb{S}$ is a non-empty set of marked points.

We always assume that each component of the boundary of $\mathbb{S}$ contains at least one point of $\mathbb{M}$. We also assume that $(\mathbb{S}, \mathbb{M})$ is not a sphere with $\leqslant 4$ marked points; or a disc with $\leqslant 3$ marked points which are all on the boundary.

Associated to $(\mathbb{S}, \mathbb{M})$ is a CY_3 triangulated category $\mathcal{D}(\mathbb{S}, \mathbb{M})$ which can be defined using quivers with potential. It also occurs (at least when $\mathbb{S}$ is closed) as the Fukaya category of a symplectic 6-manifold fibering over $\mathbb{S}$.

There is a distinguished group of autoequivalences

$$\mathsf{Aut}^* \mathcal{D}(\mathbb{S}, \mathbb{M}) \subset \mathsf{Aut}\, \mathcal{D}(\mathbb{S}, \mathbb{M})$$

and a distinguished connected component

$$\mathsf{Stab}^* \mathcal{D}(\mathbb{S}, \mathbb{M}) \subset \mathsf{Stab}\, \mathcal{D}(\mathbb{S}, \mathbb{M})$$

of the space of stability conditions on $\mathcal{D}(\mathbb{S}, \mathbb{M})$.

Theorem 1 *There is an isomorphism of complex orbifolds*

$$\mathsf{Quad}^{\pm}(\mathbb{S}, \mathbb{M}) \cong \frac{\mathsf{Stab}^* \mathcal{D}(\mathbb{S}, \mathbb{M})}{\mathsf{Aut}^* \mathcal{D}(\mathbb{S}, \mathbb{M})}.$$

The space on the left will be defined below: it is a moduli space of (signed) meromorphic quadratic differentials.

We can also precisely relate finite-length trajectories of a quadratic differential to stable objects in the corresponding stability condition.

Applying the machinery of Joyce, Kontsevich-Soibelman shows that quadratic differentials have associated BPS invariants, satisfying the wall-crossing formula. This is one of the starting points for Gaiotto-Moore-Neitzke.

For the form of our result stated above we unfortunately need to assume when $\mathbb{S}$ is closed that either $|\mathbb{M}| > 8$, or $g(\mathbb{S}) > 0$ and $|\mathbb{M}| > 5$. This is due to technical difficulties with self-folded triangles.

We don't need this assumption if we restrict to quadratic differentials with fixed non-zero residues.

Let $\mathcal{D}$ be a triangulated category, with Grothendieck group $K(\mathcal{D})$. Assume that $K(\mathcal{D}) = \mathbb{Z}^{\oplus n}$.

A stability condition $(Z, \mathcal{P})$ on $\mathcal{D}$ consists of a class of objects

$$\mathcal{P} = \bigcup_{\phi \in \mathbf{R}} \mathcal{P}(\phi) \subset \mathcal{D}$$

called the semistable objects, and a group homomorphism $Z\colon K(\mathcal{D}) \to \mathbb{C}$ called the central charge, together satisfying some axioms.

The set $\mathrm{Stab}(\mathcal{D})$ of all full, locally-finite stability conditions on $\mathcal{D}$ has a natural Hausdorff topology.

Theorem 2 *There is a local homeomorphism*

$$\mathrm{Stab}(\mathcal{D}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C}) \cong \mathbb{C}^n$$

sending a stability condition to its central charge.

In particular, the space $\mathrm{Stab}(\mathcal{D})$ is a complex manifold of dimension n.

The axioms for a stability condition $(Z, \mathcal{P})$ are as follows.

(a) if $E \in \mathcal{P}(\phi)$ then $Z(E) \in \mathbf{R}_{>0} \exp(i\pi\phi)$;

(b) $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$ for all $\phi \in \mathbf{R}$;

(c) if $\phi_1 > \phi_2$ and $A_j \in \mathcal{P}(\phi_j)$ then

$$\mathrm{Hom}_{\mathcal{D}}(A_1, A_2) = 0;$$

(d) for each $0 \neq E \in \mathcal{D}$ there is a collection of triangles

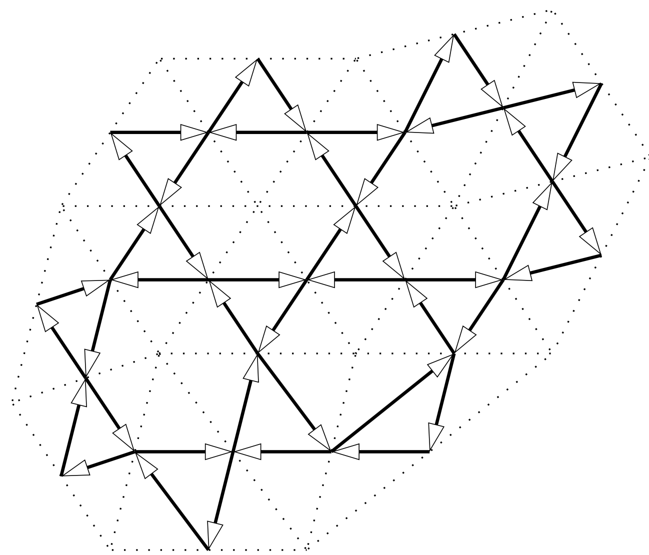
$$\begin{array}{ccccccc} 0 = E_0 & \longrightarrow & E_1 & \longrightarrow \cdots \longrightarrow & E_{n-1} & \longrightarrow & E_n = E \\ & \nwarrow \quad \swarrow & & & & \nwarrow \quad \swarrow & \\ & A_1 & & & & A_n & \end{array}$$

with $A_j \in \mathcal{P}(\phi_j)$ for all j, and

$$\phi_1 > \phi_2 > \cdots > \phi_n.$$

The other axioms imply that the filtrations appearing in (d) are unique up to isomorphism.

A non-degenerate ideal triangulation T of the surface $(\mathbb{S}, \mathbb{M})$ has an associated quiver with potential (Q, W)



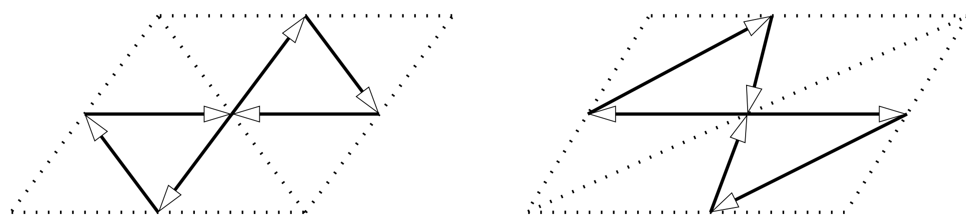
$$W = \sum_f T(f) - \sum_p C(p).$$

The associated CY_3 triangulated category

$$\mathcal{D}(T) = \mathcal{D}(Q, W) = \left\langle S_e : e \in E(T) \right\rangle$$

is defined as the subcategory of the derived category of the complete Ginzburg algebra consisting of objects with finite-dimensional cohomology.

Flipping an edge of T induces a mutation of the quiver with potential (Q, W).



A general result of Keller and Yang gives distinguished equivalences

$$\Phi_{\pm} \colon \mathcal{D}(T_1) \cong \mathcal{D}(T_2).$$

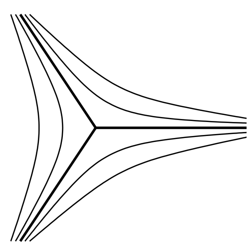
All ideal triangulations of $(\mathbb{S}, \mathbb{M})$ can be related by flips, hence $\mathcal{D}(\mathbb{S}, \mathbb{M})$ is well-defined up to equivalence.

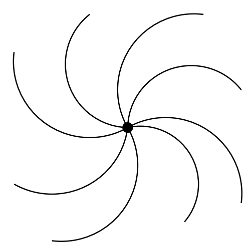
The group $\text{Aut}^* \mathcal{D}(\mathbb{S}, \mathbb{M})$ is defined by considering all composites of the equivalences $\Phi_{\pm}$ for chains of flips beginning and ending at the same triangulation, considered up to diffeomorphism.

Labardini-Fragoso shows how to deal with degenerate triangulations.

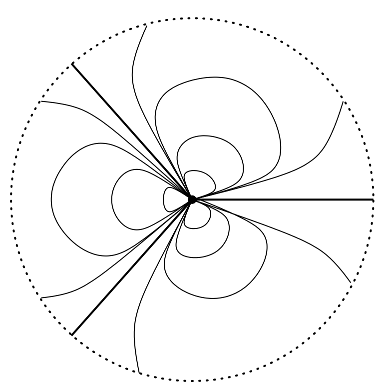
A meromorphic quadratic differential on a compact Riemann surface S is a meromorphic section of the line bundle $\omega_S^{\otimes 2}$.

The condition $\langle\sqrt{\phi}, v\rangle \in \mathbf{R}$ defines a foliation on S with singularities at the zeroes and poles of ϕ. At a simple zero and a generic double pole it looks as follows.





At a pole of order $m \geqslant 3$ there are $m-2$ distinguished tangent directions along which trajectories enter.



Define a surface with boundary $\mathbb{S}$ by taking the real oriented blow-up of S at each pole of ϕ of order $\geqslant 3$. Define $\mathbb{M} \subset \mathbb{S}$ to be the poles of ϕ order $\leqslant 2$ together with the distinguished tangent directions at the poles of order $\geqslant 3$.

There is a complex orbifold $\mathrm{Quad}(\mathbb{S}, \mathbb{M})$ parameterizing equivalence classes of pairs (S, ϕ) for which the associated surface is diffeomorphic to $(\mathbb{S}, \mathbb{M})$, and such that ϕ has simple zeroes.

The residue of a differential ϕ at a double pole $p \in S$ is

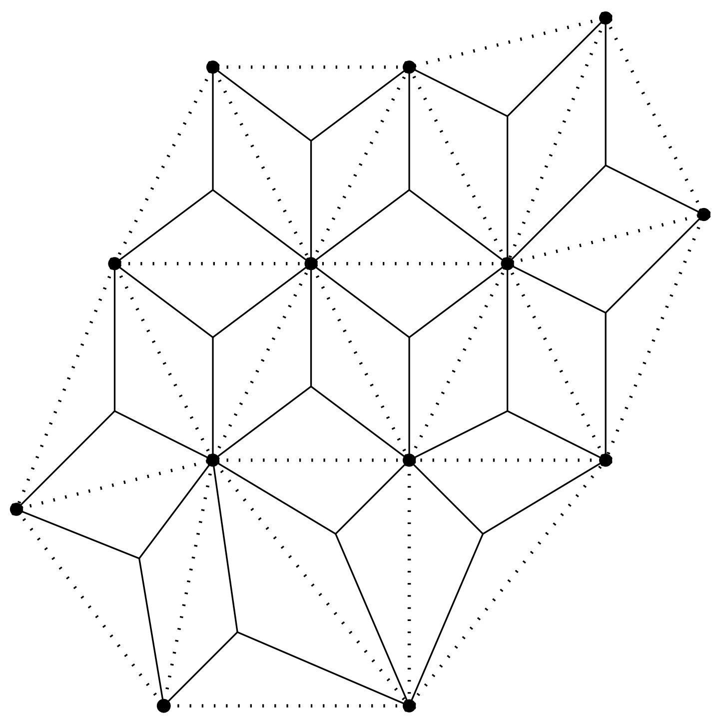
$$\mathrm{Res}_p(\phi) = \pm \int_{\beta_p} \sqrt{\phi} \in \mathbb{C},$$

where the path of integration is a small loop around p. There is a branched cover

$$\mathrm{Quad}^{\pm}(\mathbb{S}, \mathbb{M}) \longrightarrow \mathrm{Quad}(\mathbb{S}, \mathbb{M})$$

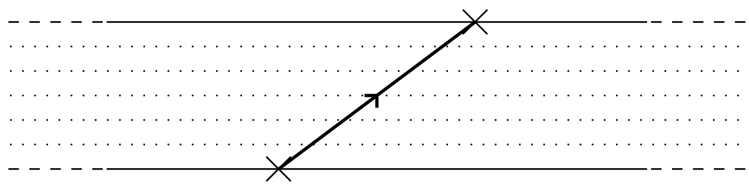
consisting of differentials with a fixed choice of sign of each such residue.

For a generic differential in $\mathrm{Quad}(\mathbb{S},\mathbb{M})$ the surface $\mathbb{S}$ is decomposed into horizontal strips and half-planes.



Taking one trajectory from each horizontal strip gives an ideal triangulation of $(\mathbb{S},\mathbb{M})$. The dual graph to the horizontal strip decomposition is the quiver Q considered before.

Each horizontal strip is equivalent to a strip in $\mathbb{C}$ equipped with the differential $dz^{\otimes 2}$.



Quadratic differentials with a given horizontal strip decomposition are completely determined by the complex numbers

$$z_i = \int_{\gamma_i} \sqrt{\phi} \in \mathfrak{h}$$

where γ_i connects the two zeroes in the boundary of the given horizontal strip.

The corresponding stability condition has heart $\mathcal{A} = \mathcal{P}((0,1])$ given by the standard heart

$$\mathcal{A}(Q,W) \subset \mathcal{D}(Q,W) \cong \mathcal{D}(\mathbb{S},\mathbb{M}).$$

Its central charge defined by $Z(S_i) = z_i$, where $S_i \in \mathcal{A}$ is the simple module corresponding to the vertex i of Q.

(a) Considering quadratic differentials (S, ϕ) for which S and the residues of ϕ are fixed defines a complex submanifold

$$B_0 \subset \mathrm{Quad}(\mathbb{S}, \mathbb{M}).$$

How to see this in terms of stability conditions?

(b) How can we obtain quadratic differentials with non-simple zeroes? The obvious answer is to consider quivers associated to more general polygonal decompositions of surfaces. More interestingly, we should glue these different strata together. This means making a space of degenerate stability conditions $\overline{\mathrm{Stab}}(\mathcal{D})$ where the masses of stable objects are allowed to become zero.

(c) How to obtain spaces of stability conditions on more general dimer models, for example the quiver for local $\mathbb{P}^2$? Maybe consider Riemann surfaces with orbifold points?

We saw above that a large open subset

$$\mathsf{Stab}_0(\mathcal{D}) \subset \mathsf{Stab}(\mathcal{D})$$

is a disjoint union of cells indexed by ideal triangulations

$$\bigsqcup_T \mathfrak{h}^n = \mathsf{Stab}_0(\mathcal{D}),$$

with adjacent cells related by mutations.

The same combinatorics can also be used to define the cluster variety $\mathcal{X}$, which is a union of algebraic tori glued by birational equivalences

$$\bigcup_T (\mathbb{C}^*)^n = \mathcal{X}.$$

These two spaces $\mathsf{Stab}_0(\mathcal{D})$ and $\mathcal{X}$ can be defined in the same way for any quiver Q.

When the quiver Q arises from a triangulation of a surface $(\mathbb{S}, \mathbb{M})$ these two spaces are related.

Fock and Goncharov showed that

$$\mathcal{X} \subset \mathsf{Loc}_{\mathsf{PGL}_2(\mathbb{C})}(\mathbb{S}, \mathbb{M})$$

is a Zariski open subset of the moduli space of framed $\mathsf{PGL}_2(\mathbb{C})$ local systems on $(\mathbb{S}, \mathbb{M})$.

In a different complex structure the space of local systems becomes the space of parabolic Higgs bundles. Thus we expect a map

$$H \colon \mathcal{X} \longrightarrow B \subset \overline{\mathsf{Stab}}(\mathcal{D})$$

whose generic fibres are compact tori.

Is this sort of thing true for more general quivers Q?

In the case $\mathbb{S}$ is closed we can interpret $\mathcal{D}(\mathbb{S},\mathbb{M})$ as a Fukaya category of a symplectic 6-manifold underlying certain CY_3 quasi-projective varieties $Y(S,\phi)$ fibering over $\mathbb{S}$.

Let $V \to S$ be a rank two holomorphic bundle with $\det(V) = \omega_S(D)$, where $D = \sum_{p\in\mathbb{M}} p$. Now form a short exact sequence

$$0 \longrightarrow W \longrightarrow \mathrm{Sym}^2(V) \longrightarrow \mathcal{O}_D \longrightarrow 0.$$

The determinant map

$$\det\colon \mathrm{Sym}^2(V) \longrightarrow \omega_S(D)^{\otimes 2}$$

restricts to a quadratic map $\det_W$ on the bundle W which is rank one at points of $\mathbb{M}$. The threefold

$$Y(S,\phi) = \Big\{\det_W = \phi\Big\} \subset W$$

is an affine conic fibration over S, with nodal fibres over the zeroes of ϕ and fibres singular at infinity over points of $\mathbb{M}$.