A Taylor expansion theorem for an elliptic extension of the Askey–Wilson operator

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Outline

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2. Elliptic hypergeometric series
3. The Askey–Wilson operator
4. An elliptic Askey–Wilson operator
5. Well-poised expansions

A Taylor expansion theorem for an elliptic extension of the Askey–Wilson operator
Basic hypergeometric series

Let \(0 < |q| < 1\).

**q-Shifted factorials:**

\[
(a; q)^k := (1 - a)(1 - aq)(1 - aq^2) \cdots (1 - aq^{k-1})
\]

for \(k = 0, 1, 2, \ldots\).

**Compact notation:**

\[
(a_1, a_2, \ldots, a_m; q)^k := (a_1; q)^k(a_2; q)^k \cdots (a_m; q)^k.
\]

**Basic hypergeometric series:**

\[
s+1 \phi s[a_0, a_1, \ldots, a_s; b_1, b_2, \ldots, b_s; q, z] :=
\]

\[
\sum_{k=0}^{\infty} \frac{(a_0, a_1, \ldots, a_s; q)^k}{(q, b_1, \ldots, b_s; q)^k} z^k.
\]

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Basic hypergeometric series:

$$\phi_s^{s+1} \left[ \begin{array}{c} a_0, a_1, \ldots, a_s \\ b_1, b_2, \ldots, b_s \\ a, q, z \end{array} \right] := \sum_{k=0}^{\infty} \frac{(a_0, a_1, \ldots, a_s; q)_k}{(q, b_1, \ldots, b_s; q)_k} z^k.$$
q-Pfaff–Saalschütz summation:

\[
_3\phi_2 \left[ \begin{array}{c} a, b, q^{-n} \\ c, abq^{1-n}/c \end{array} ; q, q \right] = \frac{(c/a, c/b; q)_n}{(c, c/ab; q)_n}.
\]
**q-Pfaff–Saalschütz summation:**

\[ 3\phi_2 \left[ \begin{array}{c} a, b, q^{-n} \\ c, abq^{1-n}/c \\ q, q \end{array} \right] = \frac{(c/a, c/b; q)_n}{(c, c/ab; q)_n}. \]

**Nonterminating q-Pfaff–Saalschütz summation:**

\[ 3\phi_2 \left[ \begin{array}{c} a, b, c \\ e, f \\ q, q \end{array} \right] = \frac{(q/e, f/a, f/b, f/c; q)_\infty}{(aq/e, bq/e, cq/e, f; q)_\infty} + \frac{e (1/e, a, b, c, qf/e; q)_\infty}{(e, aq/e, bq/e, cq/e, f; q)_\infty} 3\phi_2 \left[ \begin{array}{c} aq/e, bq/e, cq/e \\ q^2/e, qf/e \\ q, q \end{array} \right], \]

where \( ef = abcq \).
Sears' balanced $4\phi_3$ transformation:

$$4\phi_3 \left[ a, b, c, q^{-n} \atop d, e, f \right; q, q] = \left( \frac{bc}{d} \right)^n \frac{(de/bc, df/bc; q)_n}{(e, f; q)_n} 4\phi_3 \left[ a, d/b, d/c, q^{-n} \atop d, de/bc, df/bc; q, q \right],$$

where $defq^{n-1} = abc$. 
Sears’ balanced $4\phi_3$ transformation:

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4\phi_3 \left[ \frac{a, b, c, q^{-n}}{d, e, f}; q, q \right] = \left( \frac{bc}{d} \right)^n \frac{(de/bc, df/bc; q)_n}{(e, f; q)_n} 4\phi_3 \left[ \frac{a, d/b, d/c, q^{-n}}{d, de/bc, df/bc; q, q} \right],
\]

where $defq^{n-1} = abc$.

Jackson’s very-well-poised balanced $8\phi_7$ summation:

\[
8\phi_7 \left[ \frac{a, qa^{1/2}, -qa^{1/2}, b, c, d, a^2q^{1+n}/bcd, q^{-n}}{a^{1/2}, -a^{1/2}, aq/b, aq/c, aq/d, bcdq^{-n}/a, qa^{1+n}; q, q} \right] = \frac{(aq, aq/bc, aq/bd, aq/cd; q)_n}{(aq/b, aq/c, aq/d, aq/bcd; q)_n}.
\]
Nonterminating $8\phi_7$ summation:

$$8\phi_7 \left[ \frac{a}{a^2}, -qa^\frac{1}{2}, -qa^\frac{1}{2}, b, c, d, e, f \right]
= \left( \frac{aq, b}{c}, \frac{aq, c}{d}, \frac{aq, d}{e}, \frac{aq, e}{f}; q, q \right)_\infty$$

$$= \frac{(aq, e, f, \frac{aq}{a}, \frac{aq}{b}, \frac{aq}{d}, \frac{aq}{e}, \frac{aq}{f}; q)_\infty}{(aq, c, d, e, f, \frac{aq}{a}, \frac{aq}{b}, \frac{aq}{d}, \frac{aq}{e}, \frac{aq}{f}; q)_\infty}$$

$$+ \frac{b}{a} \left( \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{e}, \frac{aq}{f}, \frac{bc}{a}, \frac{bd}{a}, \frac{be}{a}, \frac{bf}{a}, \frac{b^2 q}{a}; q \right)_\infty$$

$$\times 8\phi_7 \left[ \frac{b^2}{a}, qba^{-\frac{1}{2}}, -qba^{-\frac{1}{2}}, b, \frac{bc}{a}, \frac{bd}{a}, \frac{be}{a}, \frac{bf}{a}; q, q \right],$$

where $qa^2 = bcdef$. 
Bailey’s very-well-poised balanced $10\phi_9$ transformation:

$$
10\phi_9\left[ \begin{array}{c}
a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d, e, f, \frac{\lambda aq^{n+1}}{ef}, q^{-n} \\
a^{\frac{1}{2}}, -a^{\frac{1}{2}}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{e}, \frac{aq}{f}, \frac{efq^{-n}}{\lambda}, aq^{n+1}; q, q \\
\end{array}\right] = \frac{(aq, aq/ef, \lambda q/e, \lambda q/f; q)_n}{(\lambda q, \lambda q/ef, aq/e, aq/f; q)_n} \times 10\phi_9\left[ \begin{array}{c}
\lambda, q\lambda^{\frac{1}{2}}, -q\lambda^{\frac{1}{2}}, \frac{\lambda b}{a}, \frac{\lambda c}{a}, \frac{\lambda d}{a}, e, f, \frac{\lambda aq^{n+1}}{ef}, q^{-n} \\
\lambda^{\frac{1}{2}}, -\lambda^{\frac{1}{2}}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{\lambda q}{e}, \frac{\lambda q}{f}, \frac{efq^{-n}}{a}, \lambda q^{n+1}; q, q \\
\end{array}\right],
$$

where $\lambda = qa^2/bcd$. 

A Taylor expansion theorem for an elliptic extension of the Askey–Wilson operator.
Elliptic hypergeometric series

Let $|p| < 1$.

(Modified Jacobi) theta functions:

$$\theta(x) = \theta(x; p) := \prod_{j=0}^{\infty} \left( 1 - p^j x \right) \left( 1 - p^j + 1/x \right).$$

Theta shifted factorials:

$$(a; q, p)_k := \theta(a) \theta(aq) \cdots \theta(aq^{k-1})$$ for $k = 0, 1, 2, \ldots$.

There holds $\theta(x; 0) = (1 - x)$ and $(a; q, 0)_k = (a; q)_k$.

Compact notations:

$$\theta(x_1, \ldots, x_m) := \theta(x_1) \cdots \theta(x_m),$$

$$(a_1, \ldots, a_m; q, p)_k := (a_1; q, p)_k \cdots (a_m; q, p)_k.$$
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Compact notations:

$$\theta(x_1, \ldots, x_m) := \theta(x_1)\cdots\theta(x_m),$$

$$(a_1, \ldots, a_m; q, p)_k := (a_1; q, p)_k \cdots (a_m; q, p)_k.$$
Inversion formula:

\[
\theta(1/x) = -\frac{1}{x} \theta(x).
\]

Quasi-periodicity:

\[
\theta(px) = -\frac{1}{x} \theta(x).
\]

Riemann relation:

\[
\theta(xy, x/y, uv, u/v) - \theta(xv, x/v, uy, u/y) = \frac{u}{y} \theta(yv, y/v, xu, x/u).
\]
Elliptic hypergeometric series:

\[ \sum_{k \geq 0} c_k, \]

where \( c_0 = 1 \) and \( g(k) = c_{k+1}/c_k \) is an elliptic (doubly periodic, meromorphic) function of \( k \) with \( k \) considered as a complex variable.
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Without loss of generality,

\[ g(x) = \frac{\theta(a_0 q^x, a_1 q^x, \ldots, a_s q^x)}{\theta(q^{1+x}, b_1 q^x, \ldots, b_s q^x)} z, \]

where

\[ a_0 a_1 \cdots a_s = q b_1 b_2 \cdots b_s \]

(elliptic balancing condition).
Elliptic hypergeometric series:

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g(x) = \frac{\theta(a_0 q^x, a_1 q^x, \ldots, a_s q^x)}{\theta(q^{1+x}, b_1 q^x, \ldots, b_s q^x)} z,
\]

where

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(elliptic balancing condition).

If we write \( q = e^{2\pi i \sigma} \), \( p = e^{2\pi i \tau} \), with complex \( \sigma, \tau \), then \( g(x) \) is periodic in \( x \) with periods \( \sigma^{-1} \) and \( \tau \sigma^{-1} \).
General solution:

\[ s+E_s \left[ a_0, a_1, \ldots, a_s; q, p; z \right] := \sum_{k=0}^{\infty} \frac{(a_0, a_1, \ldots, a_s; q, p)_k}{(q, b_1 \ldots, b_s; q, p)_k} z^k, \]

where \( a_0a_1 \cdots a_s = q b_1 b_2 \cdots b_s. \)
General solution:

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s_{s+1}E_s \left[ \begin{array}{c} a_0, a_1, \ldots, a_s \\ b_1, b_2, \ldots, b_s \end{array} \right] = \sum_{k=0}^{\infty} \frac{(a_0, a_1, \ldots, a_s; q, p)_k}{(q, b_1, \ldots, b_s; q, p)_k} z^k,
\]

where \( a_0 a_1 \cdots a_s = q b_1 b_2 \cdots b_s \).

For convergence, one usually requires \( a_s = q^{-n} \) (\( n \) being a nonnegative integer), so that the sum is finite.
Observe that
\[
\frac{\theta(aq^{2k})}{\theta(a)} = \frac{(qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, qa^{\frac{1}{2}}/p^{\frac{1}{2}}, -qa^{\frac{1}{2}}p^{\frac{1}{2}}; q, p)_k}{(a^{\frac{1}{2}}, -a^{\frac{1}{2}}, a^{\frac{1}{2}}p^{\frac{1}{2}}, -a^{\frac{1}{2}}/p^{\frac{1}{2}}; q, p)_k}(-q)^{-k}.
\]
Observe that
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\]

Very-well-poised elliptic hypergeometric series:

\[
s+1 V_s(a_0; a_5, \ldots, a_s, q, p; z) := \sum_{k=0}^{\infty} \frac{\theta(a_0q^{2k})}{\theta(a_0)} \frac{(a_0, a_5, \ldots, a_s; q, p)_k}{(q, a_0q/a_5, \ldots, a_0q/a_s; q, p)_k} (qz)^k,
\]
where
\[
q^2 a_5^2 a_6^2 \cdots a_s^2 = (a_0q)^{s-5}.
\]
Observe that
\[
\frac{\theta(aq^{2k})}{\theta(a)} = \frac{(qa^{k}, -qa^{k}, qa^{k}/p^{k}, -qa^{k}p^{k}; q, p)_k}{(a^{k}, -a^{k}, a^{k}p^{k}, -a^{k}/p^{k}; q, p)_k}(-q)^{-k}.
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Very-well-poised elliptic hypergeometric series:

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s_{+1} V_s(a_0; a_5, \ldots, a_s; q, p; z) := \\
\begin{aligned}
&\sum_{k=0}^{\infty} \frac{\theta(a_0 q^{2k})}{\theta(a_0)} \frac{(a_0, a_5, \ldots, a_s; q, p)_k}{(q, a_0 q/a_5, \ldots, a_0 q/a_s; q, p)_k} (qz)^k,
\end{aligned}
\]

where

\[
q^2 a_5^2 a_6^2 \cdots a_s^2 = (a_0q)^{s-5}.
\]

Write

\[
s_{+1} V_s(a_0; a_5, \ldots, a_s; q, p) := s_{+1} V_s(a_0; a_5, \ldots, a_s; q, p; 1).
\]
Elliptic hypergeometric series first appeared as elliptic solutions of the Yang–Baxter equation in work by Date, Jimbo, Kuniba, Miwa and Okado in 1987, and ten years later by I. B. Frenkel and V. Turaev.
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Their systematic study commenced at about the turn of the millenium, after further pioneering work of Spiridonov and Zhedanov, and of Warnaar.
Frenkel and Turaev’s $\mathbf{12 V_{11}}$ transformation:

\[
\mathbf{12 V_{11}}(a; b, c, d, e, f, \lambda aq^{n+1}/ef, q^{-n}; q, p) = \frac{(aq, aq/ef, \lambda q/e, \lambda q/f; q, p)_n}{(aq/e, aq/f, \lambda q/ef, \lambda q; q, p)_n} \times \mathbf{12 V_{11}}(\lambda; \lambda b/a, \lambda c/a, \lambda d/a, e, f, \lambda aq^{n+1}/ef, q^{-n}; q, p),
\]

where $\lambda = a^2 q/bcd$.
Frenkel and Turaev’s $_{12}V_{11}$ transformation:

$$\begin{align*}
_{12}V_{11}(a; b, c, d, e, f, \lambda aq^{n+1}/ef, q^{-n}; q, p)
&= \frac{(aq, aq/ef, \lambda q/e, \lambda q/f; q, p)_n}{(aq/e, aq/f, \lambda q/ef, \lambda q; q, p)_n} \\
&\times _{12}V_{11}(\lambda; \lambda b/a, \lambda c/a, \lambda d/a, e, f, \lambda aq^{n+1}/ef, q^{-n}; q, p),
\end{align*}$$

where $\lambda = a^2 q/bcd$.

Frenkel and Turaev’s $_{12}V_{11}$ transformation reduces to Bailey’s very-well-poised $_{10}\phi_9$ transformation when $p = 0$. 
A special case of the \( _{12}V_{11} \) transformation is the following summation, fundamental in the theory of elliptic hypergeometric series:

\[
\begin{align*}
\text{Frenkel and Turaev's } _{12}V_{11} \text{ summation:} \\
&= \frac{aq}{b}\frac{aq}{c}\frac{aq}{d}\frac{aq}{bcd} \\
&= \frac{aq}{bcde}.
\end{align*}
\]

Frenkel and Turaev's \( _{12}V_{11} \) summation reduces to Jackson's very-well-poised \( _{8}\phi_7 \) summation when \( p = 0 \).
A special case of the $\text{12}_V\text{11}$ transformation is the following summation, fundamental in the theory of elliptic hypergeometric series:

Frenkel and Turaev's $\text{10}_9$ summation:

$$\text{10}_9(a; b, c, d, e, q^{-n}; q, p) = \frac{(aq, aq/bc, aq/bd, aq/cd; q, p)_n}{(aq/b, aq/c, aq/d, aq/bcd; q, p)_n},$$

where $a^2 q^{n+1} = bcde$. 

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A special case of the $\binom{12}{11}$ transformation is the following summation, fundamental in the theory of elliptic hypergeometric series:

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where $a^2 q^{n+1} = bcde$.

Frenkel and Turaev's $\binom{10}{9}$ summation reduces to Jackson's very-well-poised $\binom{8}{7}$ summation when $p = 0$. 
The notations $s+1 E_s$ and $s+1 V_s$ and the observation that

$$\lim_{p \to 0} s+1 V_s(a_0; a_5, \ldots, a_s; q, p)$$

$$= s-1 W_{s-2}(a_0; a_5, \ldots, a_s; q, q)$$

$$:= s-1 \phi_{s-2} \left[ \begin{array}{c} a_0, qa_0^{1/2}, -qa_0^{1/2}, a_5, \ldots, a_s \\ a_0^{1/2}, -a_0^{1/2}, a_0 q/a_5, \ldots, a_0 q/a_s \end{array} \right] ; q, q$$

(assuming that $a_0, a_5, a_6, \ldots, a_s$ are independent of $p$) are due to Spiridonov [2002].
The Askey–Wilson operator

In the following we will be considering meromorphic functions $f(z)$ symmetric in $z$ and $1/z$.

Writing $z = e^{i\theta}$ (note that $\theta$ need not be real), we may consider $f$ to be a function in $x = \cos \theta = (z + 1/z)/2$ and write

$$f\left[ x \right] := f(z).$$

The Askey–Wilson operator $D_q$, acting on $x = \cos \theta$, is defined as follows:

$$D_q f\left[ x \right] = f\left( q^{1/2} z \right) - f\left( q^{-1/2} z \right) \left( q^{1/2} - q^{-1/2} \right) \sin \theta,$$

where $\iota\left[ x \right] = x$ (i.e., $\iota(z) = (z + 1/z)/2$).

Equivalently,

$$D_q f\left[ x \right] = f\left( q^{1/2} z \right) - f\left( q^{-1/2} z \right) i \left( q^{1/2} - q^{-1/2} \right) \sin \theta.$$
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D_q f[x] = \frac{f(q^{1/2}z) - f(q^{-1/2}z)}{\iota(q^{1/2}z) - \iota(q^{-1/2}z)},
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Equivalently,

$$D_q f[x] = \frac{f(q^{1/2} z) - f(q^{-1/2} z)}{i(q^{1/2} - q^{-1/2}) \sin \theta}.$$
The operator $\mathcal{D}_q$ is a \textit{q}-analogue of the differentiation operator. In particular, since

$$\mathcal{D}_q T_n[x] = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} U_{n-1}[x],$$

where

$$T_n[\cos \theta] = \cos n\theta$$

and

$$U_n[\cos \theta] = \sin(n+1)\theta / \sin \theta$$

are the Chebyshev polynomials of the first and second kind, respectively, one readily sees that $\mathcal{D}_q$ maps polynomials to polynomials, lowering the degree by one.
In the calculus of the Askey–Wilson operator the so-called “Askey–Wilson monomials”

\[ \phi_n(x; a) = (az, a/z; q)_n \]

form a natural basis for polynomials or power series in \( x \).
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\[ \phi_n(x; a) = (az, a/z; q)_n \]

form a natural basis for polynomials or power series in \( x \).

One computes

\[ D_q(az, a/z; q)_n = -\frac{2a(1 - q^n)}{(1 - q)}(aq^{1/2}z, aq^{1/2}/z; q)_{n-1}. \]
We recall the following Taylor theorem for polynomials $f[x]$, proved by Ismail [1995]:

**Theorem.** If $f[x]$ is a polynomial in $x$ of degree $n$, then

$$f[x] = \sum_{k=0}^{n} f_k \phi_k(x; a),$$

where

$$f_k = \frac{(q - 1)^k}{(2a)^k (q; q)_k} q^{-k(k-1)/4} (D_q^k f)[x_k],$$

with

$$x_k := \frac{1}{2} (aq^k + q^{-k}/a).$$
The application of this Theorem to $f(z) = (b z, b/z; q)_n$ immediately gives the $q$-Pfaff–Saalschütz summation, in the form

$$\frac{(b z, b/z; q)_n}{(b a, b/a; q)_n} = 3\phi_2 \left[ az, a/z, q^{-n} \right. \left. ab, q^{1-n} a/b; q, q \right],$$

A Taylor expansion theorem for an elliptic extension of the Askey–Wilson operator
The application of this Theorem to $f(z) = (bz, b/z; q)_n$ immediately gives the $q$-Pfaff–Saalschütz summation, in the form

$$\frac{(bz, b/z; q)_n}{(ba, b/a; q)_n} = 3\phi_2 \left[ \begin{array}{c} az, a/z, q^{-n} \\ ab, q^{1-n} a/b \\ q, q \end{array} \right],$$

while its application to the Askey–Wilson polynomials,

$$\omega_n(x; a, b, c, d; q) := 4\phi_3 \left[ \begin{array}{c} az, a/z, abcdq^{n-1}, q^{-n} \\ ab, ac, ad \\ q, q \end{array} \right],$$

gives a connection coefficient identity which, by specialization, can be reduced to the Sears transformation, in the form

$$\omega_n(x; a, b, c, d; q) = \frac{a^n(bc, bd; q)_n}{b^n(ac, ad; q)_n} \omega_n(x; b, a, c, d; q).$$
Ismail and Stanton [2003] extended the above polynomial Taylor Theorem to hold for entire functions of exponential growth, resulting in infinite Taylor expansions.
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Marco and Parcet [2006] extended this yet further to hold for arbitrary $q$-differentiable functions, resulting in infinite Taylor expansions with explicit remainder term.

Among other results they were able to recover the nonterminating $q$-Pfaff–Saalschütz summation.
An elliptic Askey–Wilson operator

Since

\[ D_{q}(az, a/z; q)^n (cz, c/z; q)^n = (1 - c/a)(1 - acq^{n-1}) \]
\[ \times (1 - czq^{-1/2})(1 - czq^{1/2})(1 - cq^{-1/2}z)(1 - cq^{1/2}/z) \]
\[ \times (-1)^n a(1 - q^n)(1 - q) \]
\[ \times (aq^{1/2}z, aq^{1/2}/z; q)^{n-1} \]

it makes sense to define a \( c \)-generalized Askey–Wilson operator acting on \( x \) (or \( z \)) by

\[ D_{c, q} \]

A Taylor expansion theorem for an elliptic extension of the Askey–Wilson operator.
Since

\[ D_q \left( \frac{(az, a/z; q)_n}{(cz, c/z; q)_n} \right) = \frac{(1 - c/a)(1 - acq^{n-1})}{(1 - czq^{-1/2})(1 - czq^{1/2})(1 - cq^{-1/2}/z)(1 - cq^{1/2}/z)} \times \frac{(-1)^2a(1 - q^n)}{(1 - q)} \left( \frac{aq^{1/2}z, aq^{1/2}/z; q}_{(cq^{3/2}z, cq^{3/2}/z; q)} \right)^{n-1}, \]

it makes sense to define a \( c \)-generalized Askey–Wilson operator acting on \( x \) (or \( z \)) by

\[ D_{c, q} = \frac{(1 - czq^{-1/2})(1 - czq^{1/2})(1 - cq^{-1/2}/z)(1 - cq^{1/2}/z)}{(1 - c/a)(1 - acq^{n-1})} \times \frac{(-1)^2a(1 - q^n)}{(1 - q)} \left( \frac{aq^{1/2}z, aq^{1/2}/z; q}_{(cq^{3/2}z, cq^{3/2}/z; q)} \right)^{n-1}. \]
Since
\[
D_q \frac{(az, a/z; q)_n}{(cz, c/z; q)_n} = \frac{(1 - c/a)(1 - acq^{n-1})}{(1 - czq^{-\frac{1}{2}})(1 - czq^{\frac{1}{2}})(1 - cq^{-\frac{1}{2}}/z)(1 - cq^{\frac{1}{2}}/z)} \times \frac{(-1)^2a(1 - q^n)}{(1 - q)} \frac{(aq^{\frac{1}{2}}z, aq^{\frac{1}{2}}/z; q)_{n-1}}{(cq^{\frac{3}{2}}z, cq^{\frac{3}{2}}/z; q)_{n-1}},
\]
it makes sense to define a \textit{c-generalized Askey–Wilson operator} acting on \(x\) (or \(z\)) by
\[
D_{c, q} = (1 - czq^{-\frac{1}{2}})(1 - czq^{\frac{1}{2}})(1 - cq^{-\frac{1}{2}}/z)(1 - cq^{\frac{1}{2}}/z) D_q.
\]
This operator acts “degree-lowering” on the “rational monomials”

\[
\frac{(az, a/z; q)_n}{(cz, c/z; q)_n}
\]

in the form

\[
\mathcal{D}_{c,q} \frac{(az, a/z; q)_n}{(cz, c/z; q)_n} = (-1)2a(1 - c/a)(1 - acq^{n-1})(1 - q^n) \frac{(aq^{1/2}z, aq^{1/2}/z; q)_{n-1}}{(1 - q) (cq^{3/2}z, cq^{3/2}/z; q)_{n-1}}.
\]
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\]

Clearly,

\[
D_{0,q} = D_q.
\]
More generally, for parameters $c, q, p$ with $|q|, |p| < 1$, we define an elliptic extension of the Askey–Wilson operator, acting on functions symmetric in $z^{\pm 1}$, by

$$D_{c,q,p} f(z) = 2q^{\frac{1}{2}}z \frac{\theta(czq^{-\frac{1}{2}}, czq^{\frac{1}{2}}, cq^{-\frac{1}{2}}/z, cq^{\frac{1}{2}}/z; p)}{\theta(q, z^2; p)} \times (f(q^{\frac{1}{2}}z) - f(q^{-\frac{1}{2}}z)).$$
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Note that

$$D_{c, q, 0} = D_{c, q}.$$
More generally, for parameters $c, q, p$ with $|q|, |p| < 1$, we define an elliptic extension of the Askey–Wilson operator, acting on functions symmetric in $z^{\pm 1}$, by

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Note that

$$D_{c,q,0} = D_{c,q}.$$

In particular, we have

$$D_{c,q,p} \frac{(az, a/z; q, p)_n}{(cz, c/z; q, p)_n} = \frac{(-1)2a}{\theta(c/a, acq^{n-1}, q^n; p)} \frac{\theta(aq^{\frac{1}{2}} z, aq^{\frac{1}{2}}/z; q, p)_{n-1}}{\theta(cq^{\frac{3}{2}} z, cq^{\frac{3}{2}}/z; q, p)_{n-1}}.$$

A Taylor expansion theorem for an elliptic extension of the Askey–Wilson operator
Rains considered a multivariable generalization of the operator $D_{c,q,p}$ which played a crucial role in his construction of $BC_n$-symmetric biorthogonal abelian functions generalizing Koornwinder’s orthogonal polynomials.

He further used his operator to derive $BC_n$-symmetric extensions of Frenkel and Turaev’s $10V_9$ summation and $12V_{11}$ transformation.
Which space of functions are we working in?
Which space of functions are we working in?

Let \( c \) be fixed. We define

\[
W^n_c := \text{span} \left\{ \frac{g_k(z)}{(cz, c/z; q, p)_k} \right\}_{0 \leq k \leq n},
\]

where \( g_k(z) \) runs over all functions being holomorphic for \( z \neq 0 \) with \( g_k(z) = g_k(1/z) \) and

\[
g_k(pz) = \frac{1}{p^k z^{2k}} g_k(z).
\]
Which space of functions are we working in?

Let $c$ be fixed. We define

$$W^n_c := \text{span} \left\{ g_k(z) (cz, c/z; q, p)_k, 0 \leq k \leq n \right\},$$

where $g_k(z)$ runs over all functions being holomorphic for $z \neq 0$ with $g_k(z) = g_k(1/z)$ and

$$g_k(pz) = \frac{1}{p^k z^{2k}} g_k(z).$$

Note that $W^n_c$ consists of certain abelian functions. (For $p \to 0$ these degenerate to certain rational functions we may call “well-poised”.)
Lemma. For any arbitrary but fixed $a$, the set

$$\left\{ \frac{(az, a/z; q, p)_k}{(cz, c/z; q, p)_k}, 0 \leq k \leq n \right\}$$

forms a basis for $W^*_c$. 
Lemma. For any arbitrary but fixed $a$, the set
\[
\left\{ \frac{(az, a/z; q, p)_k}{(cz, c/z; q, p)_k}, 0 \leq k \leq n \right\}
\]
forms a basis for $W_c^n$.

We now define
\[
\mathcal{D}_{c,q,p}^{(k)} = \mathcal{D}_{cq^{3/2},q,p}^{(k-1)} \mathcal{D}_{c,q,p},
\]
with $\mathcal{D}_{c,q,p}^{(0)} = \varepsilon$, the identity operator.
Expansion Theorem. Let $f$ be in $W^n_c$, then

$$f(z) = \sum_{k=0}^{n} f_k \frac{(az, a/z; q, p)_k}{(cz, c/z; q, p)_k},$$

where

$$f_k = \frac{(q - 1)^k q^{-k(k-1)/4}}{(2a)^k (q, c/a, acq^{k-1}; q, p)_k} \left[ D^{(k)}_{c,q,p} f \right]_{z=aq^k}.$$
The application of this Theorem to

\[ f(z) = \frac{(bz, b/z; q, p)_n}{(cz, c/z; q, p)_n} \]

immediately gives Frenkel and Turaev's \(10\ V_9\) summation, in the form

\[ \frac{(ac, c/a, bz, b/z; q, p)_n}{(ab, b/a, cz, c/z; q, p)_n} = 10\ V_9(acq^{-1}; az, a/z, c/b, bcq^{n-1}, q^{-n}; q, p). \]
The application of this Theorem to

\[ f(z) = \frac{(bz, b/z; q, p)_n}{(cz, c/z; q, p)_n} \]

immediately gives Frenkel and Turaev’s $10\,V_9$ summation, in the form

\[ \frac{(ac, c/a, bz, b/z; q, p)_n}{(ab, b/a, cz, c/z; q, p)_n} = 10\,V_9(acq^{-1}; az, a/z, c/b, bcq^{n-1}, q^{-n}; q, p). \]

Similarly, the application of this Theorem to Spiridonov’s elliptic extension of Rahman’s family of biorthogonal rational functions gives a connection coefficient identity which, by specialization, can be reduced to Frenkel and Turaev’s $12\,V_{11}$ transformation.
For the remaining talk, we consider the space $W_n^c$ with $p = 0$ and $n \to \infty$. We thus consider infinite convergent expansions, of which we merely sketch very few details. For instance, using an extension of the Taylor Expansion Theorem and a symmetry argument, we obtain the expansion
\[
\left( \frac{cz}{d}, \frac{c}{dz}, \frac{cz}{e}, \frac{c}{ez} ; q \right)_\infty
= \left( \frac{c^2 z}{bde}, \frac{c^2}{bdez} ; q \right)_\infty
\sum_{k \geq 0} f_k \left( \frac{bz}{de}, \frac{b}{z} ; q \right)_k
\left( \frac{cz}{de}, \frac{c}{dez} ; q \right)_k
\sum_{k \geq 0} g_k \left( \frac{c^2 z}{bde}, \frac{c^2}{bdez} ; q \right)_k,
\]
where after the explicit computation of the coefficients $f_k$ and $g_k$ one recovers the nonterminating $8\phi_7$ summation.
For the remaining talk, we consider the space $W_n$ with $p = 0$ and $n \to \infty$. We thus consider infinite convergent expansions, of which we merely sketch very few details.
Well-poised expansions

For the remaining talk, we consider the space $W^n_c$ with $p = 0$ and $n \to \infty$. We thus consider infinite convergent expansions, of which we merely sketch very few details.

For instance, using an extension of the Taylor Expansion Theorem and a symmetry argument, we obtain the expansion

$$\frac{(cz/d, c/dz, cz/e, c/ez; q)_\infty}{(cz, c/z, c^2z/bde, c^2/bdez; q)_\infty} = \frac{(cz/de, c/dez; q)_\infty}{(c^2z/bde, c^2/bdez; q)_\infty} \sum_{k \geq 0} f_k \frac{(bz, b/z; q)_k}{(cz, c/z; q)_k}$$

$$+ \frac{(bz, b/z; q)_\infty}{(cz, c/z; q)_\infty} \sum_{k \geq 0} g_k \frac{(cz/de, c/dez; q)_k}{(c^2z/bde, c^2/bdez; q)_k},$$

where after the explicit computation of the coefficients $f_k$ and $g_k$ one recovers the nonterminating $8\phi_7$ summation.
To give another example, by expanding the “quadratic” infinite product

$$f(z) = \frac{(azq, aq/z, b^2 z/a, b^2/az; q^2)_\infty}{(bz, b/z; q)_\infty}$$

in terms of the “well-poised monomials”

$$\frac{(az, a/z; q)_k}{(bz, b/z; q)_k},$$

we recover a particular nonterminating $8\phi_7$ summation, namely Bailey’s $q$-analogue of Watson’s $3F_2$ summation.