Thin tubes in mathematical physics, global analysis and spectral geometry

Daniel Grieser

February 14, 2008
Overview

1. Setup

Problems in MP, SG, GA

Methods: ΨDO or no?

Results: Fat graphs
Quantum graphs
Expectations

Results: Convex domains

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Thin tubes

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   - Fat graph theorem
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6. Results: Convex domains
Survey:


Details:


For global analysis: Hassell-Mazzeo-Melrose, Cappell-Lee-Miller, W.Müller, J.Müller-W.Müller, Park-Wojciechowski
Setup: Analysis

\[ M = \text{compact Riemannian manifold with boundary} \quad (\text{ex.: } M \subset \mathbb{R}^n \text{ open}) \]
Setup: Analysis

\( M = \text{compact Riemannian manifold with boundary} \)  
\( \text{(ex.: } M \subset \mathbb{R}^n \text{ open)} \)

Laplace-Beltrami operator \( \Delta \)

\( = -\sum_{i=1}^{n} \partial_{x_i}^2 \)

Eigenvalue problem

\[ \Delta u = \lambda u \]  
\( (\lambda \in \mathbb{R}, u: M \rightarrow \mathbb{C} \text{ satisfying boundary condition, e.g. Dirichlet, Neumann, mixed, Robin, ...}) \)

Eigenvalues:

\( \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \)

Eigenfunctions:

\( u_1, u_2, u_3, \ldots \)

Scaling

\( \lambda(cM) = c^{-2} \lambda(M) \)  
\( (c > 0) \)

\( cM = \text{all lengths are multiplied by } c \)
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($= - \sum_{i=1}^{n} \partial^2_{x_i}$)

Eigenvalue problem

$$\Delta u = \lambda u$$

($\lambda \in \mathbb{R}$, $u : M \rightarrow \mathbb{C}$ satisfying boundary condition, e.g. Dirichlet, Neumann, mixed, Robin, ...)

**Eigenvalues:** $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$

**Eigenfunctions:** $u_1 \quad u_2 \quad u_3 \quad \ldots$

$\lambda$ - eigenvalue

$u$ - eigenfunction
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\lambda^{(cM)} = c^{-2}\lambda^{(M)} \quad (c > 0)
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- Manifolds \( X^n_v \) for each \( v \in V \), \( Y^{n-1}_e \) for each \( e \in E \)
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- Finite graph: \( V = \text{vertices}, \ E = \text{edges} \)
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Fat graph \( M_\varepsilon \), thickness \( \varepsilon > 0 \)
Three problems

For a fat graph, determine asymptotics of $\lambda_k, \varepsilon, u_k, \varepsilon$ as $\varepsilon \to 0$.

Where is $\text{locmax} u_1, \{u_2 = 0\}$ for a convex planar domain?

Gluing formula for analytic torsion:

$$\tau(X_\ell \cup Y X_r) = \tau(X_\ell) + \tau(X_r) - ?(Y)$$

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(MP) (Mathematical Physics)
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Why care?
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Quantum mechanics of 'thin' structures
- Carbohydrate molecules
- Quantum wires, highly integrated circuits
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Optical fibers
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**MP** Quantum mechanics of 'thin' structures
- Carbohydrate molecules
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**SG** Qualitative properties of solutions of PDE
Why care?

(MP) Quantum mechanics of 'thin' structures
- Carbohydrate molecules
- Quantum wires, highly integrated circuits
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(SG) Qualitative properties of solutions of PDE

(GA) Decompose space into simple parts
A thin tube is...

an $n$-dimensional compact space $M_\varepsilon$ of size

$\approx \varepsilon$ in $n - 1$ directions and

$\approx 1$ in one direction.
Common theme: Thin tubes

A thin tube is...

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(MP) Fat graph is thin tube
Common theme: Thin tubes

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(MP) Fat graph is thin tube

(SG) No chance for fixed domain $\Rightarrow$ should consider very eccentric domain

\[
\begin{align*}
\{ & \quad \approx \varepsilon \\
I & \quad \approx \varepsilon
\end{align*}
\]
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(MP) Fat graph is thin tube

(SG) No chance for fixed domain $\rightsquigarrow$ should consider very excentric domain

(GA) Scale $X_\ell, X_r$ to size $\varepsilon$, insert cylinder $\varepsilon Y \times I$, consider $\varepsilon \to 0$

(Fat graph for single edge)
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\[ \approx \varepsilon \]

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Methods

Type of problem: Singular perturbation/degeneration, multiscale

- Quadratic forms
  
  simple, robust, not sharp
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- Quadratic forms simple, robust, not sharp
- Matched asymptotic expansions rigid geometry, sharp
- Matching of scattering solutions simple, rigid geometry, very sharp
- Resolvent construction using adapted $\Psi$DO calculus NOT simple, less rigid, sharp
ΨDO kernel philosophy

Assume $P$ elliptic on $M$, $Q = P^{-1}$ (or parametrix)
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- Classical calculus: $p \approx p'$
- b (or cone) calculus ($p = (x, y)$, $x > 0$):
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- **Classical calculus**: $p \approx p'$
- **b (or cone) calculus** ($p = (x, y), x > 0$):
  - $x, x' > c, p \approx p'$
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- $b$-heat calculus, $Q = Q(x, y, x', y'; z)$: $x, x', p - p', z$
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- **b-heat calculus**, $Q = Q(x, y, x', y'; z)$: $x$, $x'$, $p - p'$, $z$
- **Surgery calculus**, $Q = Q(x, y, x', y'; z; \varepsilon)$: $x$, $x'$, $p - p'$, $z$, $\varepsilon$
Quantum graphs

Definition

Metric graph: (Finite) graph $(V, E)$ with edge lengths

Quantum graph: Metric graph $G$ with corner conditions for functions on edges

Special corner conditions:
For each $v \in V$ choose subspace $A_v \subset C(E(v))$, where $E(v) = \{e: e \sim v\}$.

\[ w = (w_e)_{e \in E} \text{ satisfies corner condition } (A_v) \iff (w_e(v))_{e \sim v} \in A_v \cap A_v^\perp. \]

Kirchhoff corner condition: $A_v = \text{span}\{(1, 1, \ldots, 1)\}$: $w$ continuous, sum of normal derivatives = 0 at $v$.

Decoupling: $A_v = \{0\}$: $w = 0$ at $v$, no relation between edges hitting $v$.
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\[
(w^e(v))_{e \sim v} \in A_v \\
(\partial_n w^e(v))_{e \sim v} \in A_v^\perp.
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\[(w^e(v))_{e \sim v} \in A_v\]

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**Kirchhoff corner condition**: \(A_v = \text{span}\{(1, 1, \ldots, 1)\} : w\) continuous, sum of normal derivatives \(= 0\) at \(v\).
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Expectations: Cylinder (with ends)

\[ Y = \text{cross section}, \; \nu = \text{smallest eigenvalue of } \Delta_Y, \; \varphi = \text{eigenfunction} \]
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Basic principle

Behavior of \( \lambda_{k, \epsilon} \) dominated by \( \nu \), for \( k \epsilon < C_0 \).
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**Cylinder** \( \text{Cyl} = \varepsilon Y \times I \) (Dirichlet BC)

\[
\Delta_{\text{Cyl}} = \varepsilon^{-2} \Delta_Y + (-\partial^2_x)_I
\]

\[
\lambda^\text{Cyl}_{k,\varepsilon} = \varepsilon^{-2} \nu + k^2 \pi^2 |l|^{-2}
\]

\[
u_{k,\varepsilon}(x, y) = \varphi(y/\varepsilon) \cdot \sin(k\pi|l|^{-1}x)
\]
Expectations: Cylinder (with ends)

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**Basic principle**

Behavior of \( \lambda_{k,\varepsilon} \) dominated by \( \nu \), for \( k\varepsilon < C_0 \).

**Cylinder with ends** (Dirichlet BC)

\[
\Delta_{M_{\varepsilon}} \equiv \varepsilon^{-2} \Delta_Y + (-\partial^2_x)_I \\
\lambda_{k,\varepsilon} \equiv \varepsilon^{-2} \nu + k^2 \pi^2 |I|^{-2} + O(\varepsilon) \\
u_{k,\varepsilon}(x, y) \equiv \varphi(y/\varepsilon) \cdot \sin(k\pi|I|^{-1}x) + O(\varepsilon)
\]

(Perturbation is of same order as \( |I| \sim |I| + \varepsilon \))
Along any edge $e$: $u_k, \varepsilon \approx \phi(y/\varepsilon) \cdot \varepsilon X_v$, $\varepsilon X_v$ = eigenfunction of $-\partial^2_x$ on edges, with corner conditions.

Questions
Is this true?
If yes, which corner conditions?
Along any edge $e$:

$$u_{k,\varepsilon} \approx \varphi(y/\varepsilon) \cdot w_k^e(x),$$

$w_k^e$ = eigenfunction of $-\partial^2_x$ of $e$. 

$\lambda_{k,\varepsilon} = \varepsilon^{\nu}$ + $\mu_k + O(\varepsilon)$ (*).
Along any edge $e$:
\[ u_{k,\varepsilon} \approx \varphi(y/\varepsilon) \cdot w_k^e(x), \]
\[ w_k^e = \text{eigenfunction of } -\partial^2_x \text{ of } e. \]

Therefore:
\[ \lambda_{k,\varepsilon} \overset{?}{=} \varepsilon^{-2} \nu + \mu_k + O(\varepsilon) \quad (*) \]

$\mu_k = \text{eigenvalues of } -\partial^2_x \text{ on edges, with corner conditions.}$
Along any edge $e$:

$$u_{k,\varepsilon} \approx \varphi\left(\frac{y}{\varepsilon}\right) \cdot w_{k}^{e}(x),$$

$$w_{k}^{e} = \text{eigenfunction of } -\partial_{x}^{2} \text{ of } e.$$ 

Therefore:

$$\lambda_{k,\varepsilon} \approx \varepsilon^{-2}\nu + \mu_{k} + O(\varepsilon)$$

$$\mu_{k} = \text{eigenvalues of } -\partial_{x}^{2} \text{ on edges, with corner conditions.}$$

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Therefore:
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Rescaling: Long cylinders

\[ M_\varepsilon \]

\[ \varepsilon X_v \]

\[ \varepsilon Y_v \]
Rescaling: Long cylinders

\[ \varepsilon \rightarrow 0 \]

Eigenvalues: \[ \lambda_k, \varepsilon \]

\[ \varepsilon X_v \]

\[ \varepsilon Y_e \]

\[ M_\varepsilon \]

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Rescaling: Long cylinders

\[ M_\varepsilon \]

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\[ \downarrow \]

\[ G \]
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\[ \varepsilon X_v \]

\[ \varepsilon Y_v \]

\[ G \]

\[ \varepsilon^{-1} M_\varepsilon \approx \varepsilon^{-1} \]

\[ v \]

\[ X_v \]
Rescaling: Long cylinders

\[ M_\varepsilon \quad \rightarrow \quad \varepsilon X_v \]

\[ \varepsilon Y_e \]

\[ \varepsilon^{-1} M_\varepsilon \quad \approx \quad \varepsilon^{-1} X_v \]

Eigenvalues: \( \lambda_k, \varepsilon\)
Rescaling: Long cylinders

\[ M_\varepsilon \]

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\[ \varepsilon Y_e \]

\[ \downarrow \varepsilon \rightarrow 0 \]

\[ \varepsilon^{-1} M_\varepsilon \]

\[ \varepsilon^{-1} \]

\[ \approx \]

\[ \varepsilon^{-1} \]

\[ \downarrow \varepsilon \rightarrow 0 \]

\[ X^\infty = \bigcup_v X_v^\infty \]

\[ G \]

\[ X_v^\infty \]
Rescaling: Long cylinders

\[ M_\varepsilon \xrightarrow{\varepsilon \to 0} \varepsilon Y_e \]

\[ \varepsilon X_v \]

\[ \varepsilon^{-1} M_\varepsilon \xrightarrow{\varepsilon \to 0} \approx \varepsilon^{-1} Y_e \]

\[ Y_e \]

\[ \varepsilon X_v \]

\[ G \]

Eigenvalues:

\[ \lambda_{k,\varepsilon} \]

\[ \varepsilon^2 \lambda_{k,\varepsilon} \]

\[ X^\infty = \bigcup_v X_v^\infty \]

\[ X_v^\infty \]
Scattering theory and Theorem

\[ \sigma_{\text{cont}}(\Delta \chi^\infty) = [\nu, \infty) \]
Scattering theory and Theorem

- $\sigma_{\text{cont}}(\Delta X^\infty) = [\nu, \infty)$
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Scattering theory and Theorem

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u(x, y) = e^{-ix\sqrt{\lambda - \nu}} \varphi(y) \tilde{a} + e^{ix\sqrt{\lambda - \nu}} \varphi(y) S_v(\lambda) \tilde{a} + O(e^{-cx}),
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**Theorem**

Let \( A_v \) be the \((+1)\)-eigenspace of \( S_v(\nu) \). Let \( \mu_k \) be the eigenvalues of the quantum graph \( G \) with corner conditions \((A_v)_{v \in V} \). Then the eigenvalues on \( M_{\varepsilon} \) are

\[
\begin{align*}
\lambda_{k, \varepsilon} &= \varepsilon^{-2} \tau_k + O(e^{-c/\varepsilon}), & k = 1, \ldots, D \\
\lambda_{k, \varepsilon} &= \varepsilon^{-2} \nu + \mu_{k-D} + O(\varepsilon), & k > D
\end{align*}
\]
Remarks

**Theorem**

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- More precisely: full convergent asymptotics, also for eigenfunctions
Remarks

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- \( \exists \) eigenvalues \( \ll \varepsilon^{-2}\nu \) (compare expectation)
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- $\exists$ eigenvalues $<< \varepsilon^{-2}\nu$ (compare expectation)
- Neumann boundary condition $\Leftrightarrow \nu = 0$, Kirchhoff
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Theorem

For \( \nu > 0 \) and generic \( Y_e, X_v \) one has \( A_v = 0 \) for all \( \nu \), i.e. **decoupling.**
Sketch of proof

- **Step 1**: Show that such eigenvalues exist:
  
  Use scattering solution/eigenfunction on $\mathbf{X}_\infty$ to construct approximate eigenfunctions on $\mathbf{M}_{\varepsilon}$. For $\lambda > \nu$, this gives coupling condition
  
  $$\det(I - e^{i\alpha\varepsilon} - 1 L_S(\lambda)) = 0 \quad (\alpha = \sqrt{\lambda - \nu}) \quad (*)$$

  Spectral approximation lemma $\Rightarrow$ $\exists$ eigenvalues close to solutions.

- **Step 2**: Show that all eigenvalues are obtained this way:
  
  - $\lambda < \nu$: easy with exponential damping
  - $\lambda \geq \nu$: A priori estimates $\Rightarrow$ eigenfunctions on $\mathbf{M}_{\varepsilon}$ are close to scattering solutions.

  Stability analysis of $(*)$. 

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    \[
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Expectations: Variable thickness

\[ Y = Y(x) \text{ cross section at } x \in I \]

\[ \nu(x) = \text{smallest eigenvalue of } \Delta Y(x) \]
Y = Y(x) cross section at \( x \in I \)
\( \nu(x) \) = smallest eigenvalue of \( \Delta Y(x) \)

By basic principle:

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\Delta_{M_\epsilon} \approx P_\epsilon := \epsilon^{-2} \nu(x) + (-\partial_x^2)
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\( P_\epsilon \): semiclassical Schrödinger operator
Expectations: Variable thickness

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$\nu(x) =$ smallest eigenvalue of $\Delta Y(x)$

By basic principle:

$$\Delta_{\mathcal{M}_\varepsilon} \approx P_\varepsilon := \varepsilon^{-2} \nu(x) + (-\partial_x^2)$$

$P_\varepsilon$: semiclassical Schrödinger operator

$$\lambda_{k,\varepsilon} \approx k\text{th eigenvalue of } P_\varepsilon$$

$u_{k,\varepsilon} \approx \text{product structure}$
Results: Spectral geometry

\[ M \subset \mathbb{R}^2 \text{ convex.} \]

\[ u_1: \text{ unique maximum at locmax} \]

\[ u_2: \text{ nodal line} \]

\[ u_1 - u_2(0) \]

Let \( \text{diam } M = 1, \text{ inr } M = \varepsilon. \)


The location of \( \text{locmax } u_1 \) and of \( u_1 - u_2(0) \) is determined geometrically up to an error \( C \) by solution of ODE up to an error \( C \varepsilon \).

This is optimal in order of magnitude.

Remark: For optimality need third term (\( O(\varepsilon) \)) in asymptotics.
$M \subset \mathbb{R}^2$ convex.

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M \subset \mathbb{R}^2 \text{ convex.}

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Thin tube problems arise in many contexts. Analysis of thin tube problems:

- (singular) ΨDO method
- 'direct' methods

Fat graphs decouple generically (non-Neumann BC)

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