CRITICAL POINTS OF THE MOSER-TRUDINGER ENERGY IN THE SUPER-CRITICAL REGIME

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Let $\Omega$ be a bounded domain in $\mathbb{R}^2$, and let $H^1_0(\Omega)$ be the standard Sobolev space obtained as closure of the set $C^\infty_0(\Omega)$ of smooth, compactly supported functions on $\Omega$ in the norm $||u||_{H^1_0(\Omega)} = ||\nabla u||_{L^2(\Omega)}$. The energy functional

$$E(u) = \frac{1}{2} \int_{\Omega} (e^{u^2} - 1) \, dx, \quad u \in H^1_0(\Omega),$$

was first studied by Trudinger [13] and Moser [10] in the context of Nirenberg’s problem of finding conformal metrics on the standard sphere having a pre-assigned Gauss curvature. Besides its relevance in this and other geometric applications, the functional $E$ also is of independent interest.

As shown by Trudinger [13] and Moser [10], for any $\alpha \leq 4\pi$, the functional $E$ is bounded on the set $M_\alpha = \{ u \in H^1_0(\Omega); u \geq 0, ||\nabla u||_{L^2}^2 = \alpha \}$, and is unbounded for $\alpha > 4\pi$. Thus, the value $p = 4\pi$ may be regarded as a critical exponent for the map $M_1 \ni u \mapsto e^{u^2} \in L^p(\Omega)$. Surprisingly, and in sharp contrast with the limit case of Sobolev’s embedding in dimensions $n \geq 3$, Carleson-Chang [3] and Flucher [6] discovered that for any $\alpha \leq 4\pi$, including the critical case $\alpha = 4\pi$, the functional $E$ admits a maximizer in the space $M_\alpha$, corresponding to a solution $0 < u \in M_\alpha$ of the equation

$$-\Delta u = \lambda e^{u^2} \text{ in } \Omega$$

for some $\lambda > 0$; see [3] and [6]. Moreover, $E$ also admits a relative maximizer in $M_\alpha$ when $\alpha > 4\pi$ is sufficiently small. One therefore may expect to see also critical points of saddle-type for such $\alpha$, bifurcating from infinity at $\alpha = 4\pi$; indeed, when $\Omega$ is a ball this conjecture is strongly supported by numerical evidence [9]. However, standard variational techniques and standard tools from bifurcation theory fail in this “super-critical” range of energies.

In my first lecture I will recall the basic functional analytic properties characterizing the functional $E$. The second lecture will focus on variational approaches to the existence of saddle-points. In particular, I will recall the “entropy method” developed in [11], showing how the standard mountain-pass argument can be modified in the present context to yield the existence of saddle-points of $E$ on $M_\alpha$ for almost every $\alpha$ in a right neighborhood of $4\pi$. In my final lecture, I will present my recent joint work with Tobias Lamm and Frederic Robert [8] on the $L^2$-gradient flow for $E$ on $M_\alpha$, given by the equation

$$ue^{u^2} = \Delta u + \lambda e^{u^2} \text{ in } [0, \infty[ \times \Omega}$$

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with a function $\lambda = \lambda(t) > 0$ determined so that the Dirichlet integral of $u$ is preserved along the flow, and I will show how existence of saddle-points follows from a concentration-compactness alternative for this flow together with a precise quantization result in the case of concentration.

Note that our equation (1) is similar to the equation for scalar curvature flow, which in the case of 2 space dimensions is the Ricci flow studied by Hamilton [7] and Chow [4].

References


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