Quiver moduli spaces and applications – Lecture 4

M. Reineke

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Overview

Aim

- *Derive motivic wall-crossing formula for quiver moduli spaces*
- *Sample application to Gromov-Witten invariants*

Part 1: Change of stability

Part 2: Harder-Narasimhan recursion

Part 3: Wall-crossing formula

Part 4: Sample application

Summary of previous talk

Q finite quiver, $\mathbf{d} \in \mathbb{N}^{Q_0}$ dimension vector, $\Theta : \mathbb{Z}^{Q_0} \to \mathbb{Z}$ stability, associated slope function $\mu = \mu_\Theta$.

Theorem

Exists complex algebraic variety $M_{\mathbf{d}}^{\Theta-\mathrm{sst}}(Q)$ parametrizing isomorphism classes of Θ-polystable representations of Q of dimension vector $\mathbf{d}$.
Irreducible normal variety of dimension $1 - \sum_{i \in Q_0} d_i^2 + \sum_{\alpha: i \to j} d_i d_j$ if exists Θ-stable of dimension vector $\mathbf{d}$.

$$M_{\mathbf{d}}^{\Theta-sst}(Q) \to \underbrace{M_{\mathbf{d}}^{\mathrm{ssimp}}(Q)}_{\text{affine}} \text{ projective.}$$

Coordinates given by appropriate determinants.

Wall-crossing I – change of stability

How does $M_{\mathbf{d}}^{\Theta-\mathrm{sst}}(Q)$ change *qualitatively* when changing the stability Θ?
Fix dimension vector $\mathbf{d}$. "Stability space" $(\mathbb{Z}^{Q_0})^*$ contains *walls* $W_{\mathbf{e}} = \{\Theta \in (\mathbb{Z}^{Q_0})^* \mid \mu(\mathbf{e}) = \mu(\mathbf{d})\}$ for $0 \neq \mathbf{e} \lneqq \mathbf{d}$. Induces decomposition of $\mathbb{Z}^{Q_0}$ into *chambers*.

Theorem

- $M_{\mathbf{d}}^{\Theta-\mathrm{sst}}(Q)$ *does not change for* Θ *in interior of chamber.*
- *If* $\mathbf{d}$ *is indivisible* $(\mathrm{gcd}(d_i) = 1)$ *and* Θ *in interior: moduli space* $M_{\mathbf{d}}^{\Theta-\mathrm{sst}}(Q)$ *is smooth.*
- $\mathbf{d}$ *indivisible,* Θ_0 *on a wall,* Θ *"deformation" of* Θ_0*:* $M_{\mathbf{d}}^{\Theta-\mathrm{sst}}(Q) \to M_{\mathbf{d}}^{\Theta_0-\mathrm{sst}}(Q)$ *resolution of singularities.*
- *Often a small resolution:* $\mathrm{H}_c^*(M_{\mathbf{d}}^{\Theta-\mathrm{sst}}(Q)) \simeq \mathrm{IH}_c^*(M_{\mathbf{d}}^{\Theta_0-\mathrm{sst}}(Q))$.
- *Canonical stability* $\Theta^{\mathrm{can}} = (\sum_{j\to i} d_j - \sum_{i \to j} d_j)_i$*:* $M_{\mathbf{d}}^{\Theta^{\mathrm{can}}-\mathrm{sst}}(Q)$ *often Fano.*

Wall-crossing II – Harder-Narasimhan recursion

Definition

Grothendieck ring of varieties $K_0(\mathrm{Var}_{\mathbb{C}})$:
spanned by isoclasses $[X]$ of $\mathbb{C}$-varieties modulo "cut-and-paste relation"

$$[X] = [U] + [X \setminus U] \text{ for } U \subset X \text{ open.}$$

Multiplication $[X] \cdot [Y] = [X \times Y]$.
Lefschetz motive $\mathbb{L} = [\mathbb{A}^1]$.
Consider localization
$R^{\mathrm{mot}} = K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{\pm 1/2}, (1-\mathbb{L}^n)^{-1} \, : \, n \geq 1]$.

Example: $\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{P}^{n-1}$, thus

$$[\mathbb{P}^n] = 1 + \mathbb{L} + \ldots + \mathbb{L}^n = \frac{1-\mathbb{L}^{n+1}}{1-\mathbb{L}}.$$

Wall-crossing II – Harder-Narasimhan recursion

Harder-Narasimhan filtration: Every representation V admits *unique* filtration $0 = V_0 \subset V_1 \subset \ldots \subset V_s = V$, all V_i/V_{i-1} Θ-semistable, $\mu(V_1/V_0) > \ldots > \mu(V_s/V_{s-1})$.

Theorem

$$\frac{[R_{\mathbf{d}}(Q)]}{[G_{\mathbf{d}}]} = \sum_{\substack{\mathbf{d}=\mathbf{e}^1+\ldots+\mathbf{e}^s \\ \mu(\mathbf{e}^1)>\ldots>\mu(\mathbf{e}^s)}} \mathbb{L}^{\textit{something}} \cdot \prod_{k=1}^{s} \frac{[R_{\mathbf{e}^k}^{\Theta-\mathrm{sst}}(Q)]}{[G_{\mathbf{e}^k}]} \in R^{\mathrm{mot}}.$$

If $\mathbf{d}$ *indivisible and* Θ *generic:*

$$(\mathbb{L}-1)\cdot\frac{[R_{\mathbf{d}}^{\Theta-\mathrm{sst}}(Q)]}{[G_{\mathbf{d}}]} = [M_{\mathbf{d}}^{\Theta-\mathrm{sst}}(Q)] \overset{!}{=} \sum_i \dim H_c^{2i}(M_{\mathbf{d}}^{\Theta-\mathrm{sst}}(Q))\mathbb{L}^i.$$

so motives and Betti numbers of these moduli spaces can be determined recursively.

Wall-crossing III – Wall-crossing formula

Euler form $\langle \mathbf{d}, \mathbf{e} \rangle = \sum_{i \in Q_0} d_i e_i - \sum_{\alpha: i \to j} d_i d_j$. Antisymmetrization $\{\mathbf{d}, \mathbf{e}\} = \langle \mathbf{d}, \mathbf{e} \rangle - \langle \mathbf{e}, \mathbf{d} \rangle$. **"Motivic quantum torus"** $\mathbb{T}_Q = R^{\mathrm{mot}}[[t_i \,:\, i \in Q_0]]$ with multiplication $t^{\mathbf{d}} \cdot t^{\mathbf{e}} = (-\mathbb{L}^{1/2})^{\{\mathbf{d},\mathbf{e}\}} t^{\mathbf{d}+\mathbf{e}}$. Generating functions

$$A_Q = 1 + \sum_{\mathbf{d} \neq 0} (-\mathbb{L}^{1/2})^{\langle \mathbf{d}, \mathbf{d} \rangle} \frac{[R_{\mathbf{d}}(Q)]}{[G_{\mathbf{d}}]} t^{\mathbf{d}} \in \mathbb{T}_Q,$$

$$A_Q^{\mu_\Theta} = 1 + \sum_{\mu_\Theta(\mathbf{d}) = \mu} (-\mathbb{L}^{1/2})^{\langle \mathbf{d}, \mathbf{d} \rangle} \frac{[R_{\mathbf{d}}^{\Theta-\mathrm{sst}}(Q)]}{[G_{\mathbf{d}}]} t^{\mathbf{d}} \in \mathbb{T}_Q \text{ for } \mu \in \mathbb{Q}.$$

Theorem (Motivic wall-crossing formula)

$$A_Q = \prod_{\substack{\mu \in \mathbb{Q} \\ \text{decreasing}}} A_Q^{\mu_\Theta} \quad \text{in } \mathbb{T}_Q.$$

In particular, RHS is independent of Θ.

Similar factorization formulas in (variants of) quantum tori appear e.g. in quantum dilogarithm identities, Gross-Pandharipande-Siebert tropical vertex in Gromov-Witten theory, Gross-Hacking-Keel-Kontsevich scattering diagrams....

If these can be matched to above quiver WCF (!!), quiver moduli can compute interesting invariants.

Wall-crossing IV – Sample application

Consider m-subspace quiver S_m

$$\begin{array}{ccccc} & & j & & \\ & \nearrow & \ldots & \nwarrow & \\ i_1 & & \ldots & & i_m \end{array}$$

dimension vector $\mathbf{d} = (1, \ldots, 1, d)$, stability $\Theta = (d, \ldots, d, -m)$.

$$M_{\mathbf{d}}^{\Theta-\mathrm{sst}}(S_m) \simeq (\mathbb{P}^{d-1})^m_{\mathrm{st}} // \mathrm{PGL}_d(\mathbb{C}),$$

stable configurations of m points in $\mathbb{P}^{d-1}$ modulo projective similarities.

If $\gcd(d, m) = 1$: it is smooth projective rational Fano of dimension $(d-1)(m-d-1)$.

Gromov-Witten invariant N_d:
number of irreducible rational degree d curves in $\mathbb{P}^2$ passing through a given generic configuration of $2d-1$ points and having order d tangency to a given line at a given point (well-defined!).

d	1	2	3	4	5	6	7
N_d	1	1	7	138	5477	367640	37541883

Theorem

$$N_d = \chi((\mathbb{P}^{d-1})^{2d-1}_{\mathrm{st}} // \mathrm{PGL}_d(\mathbb{C})).$$

Final slide

Thank you!