Quiver moduli spaces and applications – Lecture 5

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Overview

Aim

- *Introduce motivic Donaldson-Thomas invariants of quiver*
- *introduce Cohomological Hall algebras*

Part 1: Connection to Lecture 4

Part 2: Motivic DT invariants: definition, nature, examples

Part 3: Cohomological Hall algebras: definition, categorified WCF, structure, examples

Correction

Euler form $\langle \mathbf{d}, \mathbf{e} \rangle = \sum_{i \in Q_0} d_i e_i - \sum_{\alpha: i \to j} d_i d_j$. Antisymmetrization $\{\mathbf{d}, \mathbf{e}\} = \langle \mathbf{d}, \mathbf{e} \rangle - \langle \mathbf{e}, \mathbf{d} \rangle$. **"Motivic quantum torus"** $\mathbb{T}_Q = R^{\mathrm{mot}}[[t_i \,:\, i \in Q_0]]$ with multiplication $t^{\mathbf{d}} \cdot t^{\mathbf{e}} = (-\mathbb{L}^{1/2})^{\{\mathbf{d},\mathbf{e}\}} t^{\mathbf{d}+\mathbf{e}}$. Generating functions

$$A_Q = 1 + \sum_{\mathbf{d} \neq 0} (-\mathbb{L}^{1/2})^{\langle \mathbf{d},\mathbf{d} \rangle} \frac{[R_{\mathbf{d}}(Q)]}{[G_{\mathbf{d}}]} t^{\mathbf{d}} \in \mathbb{T}_Q,$$

$$A_Q^{\mu_\Theta} = 1 + \sum_{\mu_\Theta(\mathbf{d}) = \mu} (-\mathbb{L}^{1/2})^{\langle \mathbf{d},\mathbf{d} \rangle} \frac{[R_{\mathbf{d}}^{\Theta-\mathrm{sst}}(Q)]}{[G_{\mathbf{d}}]} t^{\mathbf{d}} \in \mathbb{T}_Q \text{ for } \mu \in \mathbb{Q}.$$

Theorem (Motivic wall-crossing formula)

$$A_Q = \prod_{\substack{\mu \in \mathbb{Q} \\ \text{decreasing}}} A_Q^{\mu_\Theta} \text{ in } \mathbb{T}_Q.$$

In particular, RHS is independent of Θ.

Two directions in studying quiver moduli spaces

"Horizontal" direction: Study $M_{\mathbf{d}}^{\Theta-\mathrm{sst}}(Q)$ for $\mathbf{d}$ indivisible, varying (generic) Θ: smooth (projective) rational varieties, known Betti numbers, can be studied via WCF, allows study of finer geometric properties.

"Vertical" direction: Study $M_{\mathbf{d}}^{\Theta-\mathrm{sst}}(Q)$ for fixed Θ, for *all* $\mathbf{d}$ of fixed slope $\mu \in \mathbb{Q}$: highly singular varieties, all known invariants "explode", encoded in motivic series A_Q^{μ}, which questions can be answered?

Example of motivic series

Compute A_Q for $Q = \bullet$:

$$A_Q = 1 + \sum_{\mathbf{d}\neq 0} (-\mathbb{L}^{1/2})^{\langle \mathbf{d},\mathbf{d}\rangle} \frac{[R_{\mathbf{d}}(Q)]}{[G_{\mathbf{d}}]} t^{\mathbf{d}} =$$

$$= 1 + \sum_{d\geq 1} (-\mathbb{L}^{1/2})^{d^2} \frac{[\mathrm{pt}]}{[\mathrm{GL}_d(\mathbb{C})]} t^d =$$

$$= 1 + \sum_{d\geq 1} (-\mathbb{L}^{1/2})^{d^2} \frac{1}{(\mathbb{L}^d - 1)\cdot \ldots \cdot (\mathbb{L}^d - \mathbb{L}^{d-1})} t^d =$$

$$= 1 + \sum_{d\geq 1} \frac{(\mathbb{L}^{1/2} t)^d}{(1-\mathbb{L})\cdot \ldots \cdot (1-\mathbb{L}^d)} =$$

$$= \prod_{k\geq 0} (1 - \mathbb{L}^{k+1/2} t)^{-1} \text{ by } q\text{-binomial theorem.}$$

Euler product factorization!

Motivic Donaldson-Thomas invariants

Motivated by BPS state counts in string theory, Kontsevich and Soibelman defined:

Definition

Fix quiver Q, stability Θ, slope $\mu \in \mathbb{Q}$ such that $\langle \_, \_ \rangle_Q$ is symmetric on $\{\mu_\Theta(\mathbf{d}) = \mu\}$. Euler product factorization

$$A_Q^\mu = \prod_{\mu(\mathbf{d})=\mu} \prod_{i \in \mathbb{Z}} \prod_{k \geq 0} (1 - \mathbb{L}^{k+(i+1)/2} t^{\mathbf{d}})^{(-1)^{i+1} c_{\mathbf{d},i}} \in \mathbb{T}_Q$$

and define

$$\mathrm{DT}_{\mathbf{d}}^\Theta(Q) = \sum_i c_{\mathbf{d},i} (-\mathbb{L}^{1/2})^i \in \mathbb{Q}(\mathbb{L}^{1/2})$$

motivic Donaldson-Thomas invariants.

Nature of DT invariants

Theorem (Efimov)

If Q is symmetric and $\Theta = 0$, motivic DT invariant $\mathrm{DT}_{\mathbf{d}}^{0}(Q) \in \mathbb{N}[-\mathbb{L}^{\pm 1/2}]$.

Based on this theorem:

Theorem (Meinhardt-R.)

$$\mathrm{DT}_{\mathbf{d}}^{\Theta}(Q) = (-\mathbb{L}^{1/2})^{\langle \mathbf{d},\mathbf{d}\rangle - 1} \sum_i \dim IH_c^i(M_{\mathbf{d}}^{\Theta-\mathrm{sst}}(Q))(-\mathbb{L}^{1/2})^i$$

if exists Θ-stable representation for $\mathbf{d}$, and $\mathrm{DT}_{\mathbf{d}}^{\Theta}(Q) = 0$ otherwise.

Examples of DT invariants I

$Q = \bullet$:

$$A_Q = \prod_{k\geq 0}(1-\mathbb{L}^{k+1/2}t)^{-1},$$

so $\mathrm{DT}_1^0(Q) = 1$, $\mathrm{DT}_d^0(Q) = 0$ for $d \geq 2$, and
$M_d^{\mathrm{ssimp}}(Q) = \mathrm{pt}$ for all $d \geq 1$, but simples exist only for $d = 1$.

Q a single loop:

$$A_Q = 1 + \sum_{d\geq 1} \frac{\mathbb{L}^{d(d-1)/2}}{(1-\mathbb{L})\cdot\ldots\cdot(1-\mathbb{L}^d)}(-\mathbb{L}t)^d = \prod_{k\geq 0}(1-\mathbb{L}^{k+1}t),$$

so $\mathrm{DT}_1^0(Q) = -\mathbb{L}^{1/2}$, $\mathrm{DT}_d^0(Q) = 0$ for $d \geq 2$, and
$M_d^{\mathrm{ssimp}}(Q) = \mathbb{A}^d$, but simples exist only for $d = 1$.

Examples of DT invariants II

Q a double loop:

We recover the Rogers-Ramanujan type identity from the second lecture:

$$\sum_{d\geq 0} \frac{q^{d^2} t^d}{(1-q)\cdot \ldots \cdot (1-q^n)} =$$

$$= \prod_{d\geq 1} \prod_{i\geq 0} \prod_{k\geq 0} (1 - q^{i+k} t^d)^{(-1)^d \dim \mathrm{IH}_c^{2i+d(d+1)}(M_d^{\mathrm{ssimp}}(Q))}.$$

Cohomological Hall algebras I – definition

Categorify wall-crossing formula and DT invariants via *Cohomological Hall algebras*:

Definition

Q finite quiver, Θ stability, μ slope.
Sum of all equivariant cohomology of representation spaces

$$\mathrm{COHA}(Q) = \bigoplus_{\mathbf{d}} H^*_{G_{\mathbf{d}}}(R_{\mathbf{d}}(Q))$$

carries "parabolic induction" type multiplication turning it into $\mathbb{N}^{Q_0} \times \mathbb{N}$-graded associative algebra.
Similarly for

$$\mathrm{COHA}^{\mu-\mathrm{sst}}(Q) = \bigoplus_{\mu_\Theta(\mathbf{d})=\mu} H^*_{G_{\mathbf{d}}}(R_{\mathbf{d}}^{\Theta-\mathrm{sst}}(Q)).$$

Lemma

Poincaré series of $\mathrm{COHA}^{(\mu-\mathrm{sst})}(Q)$ *equals motivic generating series* $A_Q^{(\mu)}$.

Theorem (Categorified WCF; Franzen-R.)

$$\mathrm{COHA}(Q) \simeq \bigotimes_{\substack{\mu\in\mathbb{Q} \\ \text{decreasing}}} \mathrm{COHA}^{\mu-\mathrm{sst}}(Q)$$

as $\mathbb{N}^{Q_0} \times \mathbb{N}$*-graded vector spaces.*

Cohomological Hall algebras III - relation to DT invariants

Theorem (Categorified DT invariants; Efimov)

Q symmetric: $\mathrm{COHA}(Q) \simeq \mathrm{Sym}(V^{**}[z])$,

and this implies $\mathrm{DT}^0_{\mathbf{d}}(Q) = \sum_i \dim V^{\mathbf{d},i}(-\mathbb{L}^{1/2})^i$.

Theorem (Categorified DT invariant; Davison-Meinhardt)

$\langle\_,\_\rangle_Q$ *symmetric on* $\{\mu(\mathbf{d}) = \mu\}$*: exists filtration* $\mathcal{F}$ *on* $\mathrm{COHA}^{\mu-\mathrm{sst}}(Q)$ *such that*

$$\mathrm{gr}_{\mathcal{F}}(\mathrm{COHA}^{\mu-\mathrm{sst}}(Q)) \simeq \mathrm{Sym}(V^{**}[z]), \text{ and thus}$$

$$\mathrm{DT}^{\Theta}_{\mathbf{d}}(Q) = \sum_i \dim V^{\mathbf{d},i}(-\mathbb{L}^{1/2})^i.$$

$Q=\bullet$: $\mathrm{COHA}(Q)\simeq\mathrm{Sym}(\mathbb{Q}^{1,1}[z])$ exterior algebra in countably many generators.

Q a loop: $\mathrm{COHA}(Q)\simeq\mathrm{Sym}(\mathbb{Q}^{1,0}[z])$ symmetric algebra in countably many generators.

Q a two-cycle:

$$\mathrm{COHA}(Q)\simeq\mathrm{Sym}((\mathbb{Q}^{(1,0),1}\oplus\mathbb{Q}^{(0,1),1}\oplus\mathbb{Q}^{(1,1),0})[z]),$$

since

$$M^{\mathrm{ssimp}}_{(1,0)}(Q)=M^{\mathrm{ssimp}}_{(0,1)}(Q)=\mathrm{pt}\text{ and }M^{\mathrm{ssimp}}_{(1,1)}(Q)=\mathbb{A}^1,$$

and no simples in other dimensions.

Cohomological Hall algebra V – examples

Q Kronecker $\bullet \Rightarrow \bullet$, $\Theta = (1, -1)$:

$\mathrm{COHA}^{\mu-\mathrm{sst}}(Q) = 0$ if $\mu \notin \{-1, -\frac{1}{3}, -\frac{1}{5}, \ldots, 0, \ldots, \frac{1}{5}, \frac{1}{3}, 1\}$,

$\mathrm{COHA}^{\mu-\mathrm{sst}}(Q) \simeq \mathrm{Sym}(\mathbb{Q}[z])$ if $\mu = \pm\frac{1}{2d+1}$,

$\mathrm{COHA}^{0-\mathrm{sst}}(Q)$ has generators $e_0, e_1, e_2, \ldots, f_1, f_2, \ldots$ subject to

$$[E(X), E(Y)] = 2(Y - X)\frac{YE(Y) - XE(X)}{Y - X}\frac{YF(Y) - XF(X)}{Y - X},$$

$$[E(X), F(Y)] = (Y - X)\left(\frac{YF(Y) - XF(X)}{Y - X}\right)^2, \; [F(X), F(Y)] = 0$$

for $E(X) = \sum_{n\geq 0} e_n X^n$, $F(X) = \sum_{n\geq 0} f_{n+1} X^n$.

Final slide

Thank you!