Simultaneous diagonal and non-diagonal equations

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Systems of diagonal forms

The main question

For one or several polynomials $F_1, \ldots, F_r \in \mathbb{Z}[x_1, \ldots, x_s]$, how many integer solutions are there to the system of equations $F_1(\mathbf{x}) = \ldots = F_r(\mathbf{x}) = 0$?

In Yu-Ru's lectures, we have seen how to treat **diagonal forms**:

General philosophy

We get an asymptotic formula provided we have a good mean value estimate in the supercritical range.

- works when $s > s_{\text{crit}}$, usually $s \geqslant k^2 + O(k)$.
- shortcomings in the mean value can be compensated with Weyl-type inequality; costs extra variables ($s \geqslant s_{\text{crit}} + \#\text{Weyl}$).
- In the case of diagonal forms, we have Weyl-type inequalities that save a power of $\frac{1}{k(k-1)}$, rather than Weyl's original 2^{k-1}.
- Works well for systems with pairwise distinct degrees (VMVT), many copies of the same degree (Davenport–Lewis; Cook), or combinations thereof (B-Parsell).

Systems of general forms

Forms of general shape – here we have to rely on Weyl's inequality exclusively.

- works when $s > 2^d(d-1)$ for non-singular forms (Birch)
- works well for systems of many forms with the same degree (Birch; Myerson) – see Simon's lectures
- different ideas required for systems with differing degrees (Browning–Heath-Brown)

Question

Can we combine both approaches, so that the stronger methods for diagonal forms come to bear despite the presence of general forms?

Systems of diagonal and non-diagonal forms

Question

Can we combine both approaches, so that the stronger methods for diagonal forms come to bear despite the presence of general forms?

Consider one diagonal form F of degree k and a general form G of degree d (shamelessly exploiting the fact that there are different naming conventions in these two settings!)

- Suppose that $d \geqslant k$. In this case, Browning–Heath-Brown gives a result when
$$s > (2 + k)(d - 1)2^{d-1} + k2^{k-1}.$$
However, we need $s > 2^d(d - 1)$ even without the presence of the diagonal form – trying to exploit the fact that F is diagonal is not going to make much of a difference!
- If, however, $k \geqslant d$, then the B-HB bound is
$$s > (2 + d)(k - 1)2^{k-1} + d2^{d-1}.$$
In this setting, there is reason to hope that the 2^k-dependence on k can be replaced by a term growing more like k^2!

Systems of differing forms of general shape, I

How to approach the Browning–Heath-Brown result: For F of degree k and G of degree d, with $k > d$, set

$$T(\alpha, \beta) = \sum_{|\mathbf{x}| \leqslant X} e(\alpha F(\mathbf{x}) + \beta G(\mathbf{x})).$$

Recall the Weyl differencing operator $\Delta_{\mathbf{h}} H(\mathbf{x}) = H(\mathbf{x} + \mathbf{h}) - H(\mathbf{x})$, and write $\Delta_{\mathbf{g},\mathbf{h}} H(\mathbf{x}) = \Delta_{\mathbf{g}} \Delta_{\mathbf{h}} H(\mathbf{x})$. Each differencing step lowers the degree by one.

Naive approach:
Take $k-1$ Weyl differences. Then

- either $T(\alpha, \beta)$ is 'small',
- or we get a 'good' major arcs approximation for α.

Problem: This kills $G(\mathbf{x})$, so we lose all information on β.

Systems of differing forms of general shape, II

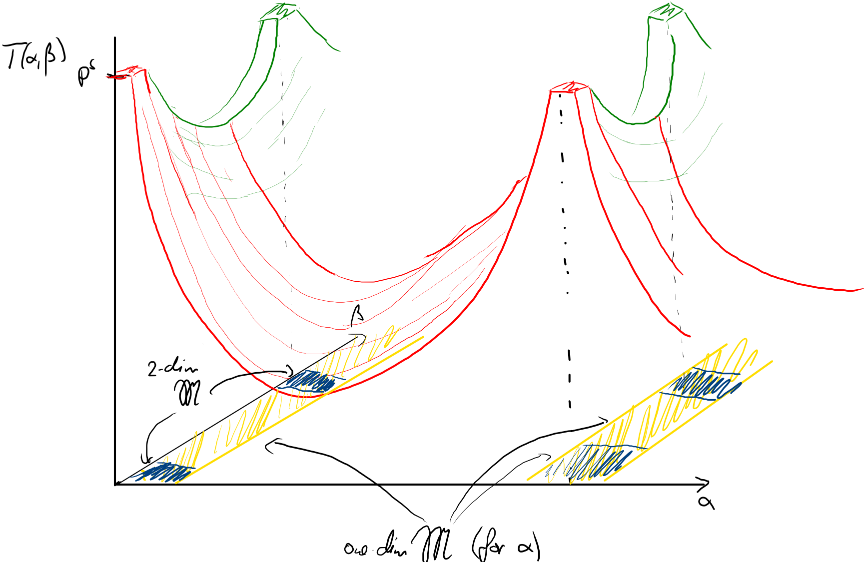
Strategy of Browning–Heath-Brown

- difference $k-1$ times $\to$ G disappears, and we get a bound when $\alpha \in \mathfrak{m}$.
- Now suppose that $\alpha \approx a/q \in \mathfrak{M}$, $q < Q$. Then $\|q\alpha\| \ll QX^{-k}$.
- Assuming that $\alpha \in \mathfrak{M}$, take $d-1$ differences, but arrange that the last $\mathbf{h}$ has a factor q. Then

$$\|\alpha \Delta_{\mathbf{h}_1,\dots,\mathbf{h}_{d-2},q\mathbf{g}_{d-1}} F(\mathbf{x})\| \ll \|q\alpha\| \left|\Delta_{\mathbf{h}_1,\dots,\mathbf{h}_{d-2},\mathbf{g}_{d-1}} F(\mathbf{x})\right| \approx 0.$$

- Consequently, $e(\alpha F(\mathbf{x})) \approx 1$, and we can proceed to get a major/minor arcs dissection for β.

Sketch of the major arcs



The underlying idea when $F(\mathbf{x}) = x_1^k + \ldots + x_s^k$

If we want to use the diagonal structure, we need to "get rid of" the general form.

Idea

Taking d Weyl differences does exactly that!

So let's take d Weyl steps. This gives

$$|T(\alpha,\beta)|^{2^d} \ll X^{(2^d-d-1)s} \sum_{\mathbf{h}_1,\ldots,\mathbf{h}_d} \left| \sum_{|\mathbf{x}|\leqslant X} e(\alpha \Delta_{\mathbf{h}_1\ldots,\mathbf{h}_d} F(\mathbf{x}) + \beta \Delta_{\mathbf{h}_1\ldots,\mathbf{h}_d} G(\mathbf{x})) \right|.$$

Crucial input

- Since G is of degree d, the expression $\Delta_{\mathbf{h}_1\ldots,\mathbf{h}_d} G(\mathbf{x})$ is a constant in $\mathbf{x}$ (depending only on $\mathbf{h}_1, \ldots, \mathbf{h}_d$) and disappears in the absolute value!
- The discrete differencing operator is additive, and its action on a pure power is

$$\Delta_{h_1,\ldots h_d} x^k = h_1 \cdots h_d p_{\mathbf{h}}(x),$$

where $p_{\mathbf{h}}$ is an (essentially) monic polynomial of degree $k-d$.

The effect of Weyl differencing

Recall $F(\mathbf{x}) = x_1^k + \ldots + x_s^k$, and $\Delta_{h_1,\ldots h_d} x^k = h_1 \cdots h_d p_{\mathbf{h}}(x)$.
Thus, we find

$$|T(\alpha,\beta)|^{2^d} \ll X^{(2^d-d-1)s} \sum_{\mathbf{h}_1,\ldots,\mathbf{h}_d} \left| \sum_{|\mathbf{x}|\leqslant X} e(\alpha \Delta_{\mathbf{h}_1\ldots,\mathbf{h}_d} F(\mathbf{x}) + \beta \Delta_{\mathbf{h}_1\ldots,\mathbf{h}_d} G(\mathbf{x})) \right|$$
$$\ll X^{(2^d-d-1)s} \left(\sum_{h_1,\ldots,h_d} \left| \sum_{|x|\leqslant X} e(\alpha h_1 \cdots h_d p_{\mathbf{h}}(x)) \right| \right)^s .$$

Suppose that $s = 2^d t$, and write

$$f_{\mathbf{h}}(\alpha) = \sum_{|x|\leqslant X} e(\alpha h_1 \cdots h_d p_{\mathbf{h}}(x)).$$

Then

$$|T(\alpha,\beta)| \ll X^{s-(d+1)t} \left(\sum_{h_1,\ldots,h_d} |f_{\mathbf{h}}(\alpha)| \right)^t .$$

Weyl differencing: Conclusion

Suppose that $s = 2^d t$, and write

$$f_{\mathbf{h}}(\alpha) = \sum_{|x| \leqslant X} e(\alpha h_1 \cdots h_d p_{\mathbf{h}}(x)).$$

Then

$$|T(\alpha, \beta)| \ll X^{s-(d+1)t} \left(\sum_{h_1, \dots, h_d} |f_{\mathbf{h}}(\alpha)| \right)^t .$$

When $h_1 \cdots h_d \neq 0$, then $f_{\mathbf{h}}(\alpha)$ is a one-dimensional exponential sum of degree $d - k$. However, the contribution with $h_1 \cdots h_d = 0$ is of a smaller order of magnitude.

We want to use mean values. Unfortunately, from the mean value we can get at most $X^{s-(k-d)}$. The main term will be of order $X^{s-(k+d)}$, so we need to save another X^{2d} from Weyl inequalities.

The minor arcs for α: the mean value

Consider the minor arcs for α,

$$\mathfrak{m}_k(Q) = \{\alpha : \|\alpha q\| \leqslant QX^{-k} \Rightarrow q > Q\}.$$

Set $t = 2v_0 + v_1$, then for any $\beta \in [0,1]$ we have

$$\int_{\mathfrak{m}_k(Q)} |T(\alpha,\beta)| \, \mathrm{d}\alpha \ll X^{s-(d+1)t} \sup_{\alpha \in \mathfrak{m}_k(Q)} \Big(\sum_{\mathbf{h}} |f_{\mathbf{h}}(\alpha)| \Big)^{v_1} \int_0^1 \Big(\sum_{\mathbf{h}} |f_{\mathbf{h}}(\alpha)| \Big)^{2v_0} \, \mathrm{d}\alpha.$$

In the mean value, we treat the **h** as constants. Thus,

$$\int_0^1 \Big(\sum_{h_1 \cdots h_d \neq 0} |f_{\mathbf{h}}(\alpha)| \Big)^{2v_0} \, \mathrm{d}\alpha \ll X^{2v_0 d} \sup_{\mathbf{h}} \int_0^1 |f_{\mathbf{h}}(\alpha)|^{2v_0} \, \mathrm{d}\alpha.$$

Finally, for any $h_1 \cdots h_d \neq 0$ and any $v_0 \geqslant s_0 :=$ [critical point for degree $k - d$] we have

$$\int_0^1 |f_{\mathbf{h}}(\alpha)|^{2v_0} \, \mathrm{d}\alpha \ll X^{2v_0 - (k-d) + \varepsilon}.$$

the minor arcs for α: the Weyl inequality

The minor arcs of degree j are given by $\mathfrak{m}_j(Q) = \{\alpha : \|\alpha q\| \leqslant QX^{-j} \Rightarrow q > Q\}$. We need to understand

$$\sup_{\alpha \in \mathfrak{m}_k(Q)} \sum_{\mathbf{h}} |f_{\mathbf{h}}(\alpha)| .$$

Since the exponential sum $f_{\mathbf{h}}(\alpha)$ is of degree $k-d$, Weyl's inequality gives

$$|f_{\mathbf{h}}(\alpha)| \ll X^{1+\varepsilon} Q^{-1/\sigma} \qquad \text{for } \alpha \in \mathfrak{m}_{k-d}(Q),$$

where $1/\sigma$ is the Weyl exponent for degree $k-d$.
Note: This is a different set of minor arcs!!
Fortunately, this is how it should be, because the **h**'s play a role:

Lemma 1

Suppose that $\alpha \in \mathfrak{m}_k(Q)$. Then $\displaystyle\sum_{0<H\leqslant X^d} |f(H\alpha)| \ll X^{d+1+\varepsilon} Q^{-1/\sigma}$.

Applying this with $H = h_1 \cdots h_d$ yields the bound.

The minor arcs for α: Summary

Set $s_0 =$ [crit pt for degree $k-d$] and $1/\sigma =$ [Weyl exp for degree $k-d$].
Suppose that $v_0 \geqslant s_0$, then the mean value is

$$\int_0^1 \left(\sum_{h_1 \cdots h_d \neq 0} |f_{\mathbf{h}}(\alpha)| \right)^{2v_0} \mathrm{d}\alpha \ll X^{2v_0 d} \sup_{\mathbf{h}} \int_0^1 |f_{\mathbf{h}}(\alpha)|^{2v_0} \,\mathrm{d}\alpha \ll X^{2v_0 d} X^{2v_0 - (k-d) + \varepsilon}.$$

Moreover, we have the Weyl-type inequality

$$\sup_{\alpha \in \mathfrak{m}_k(Q)} \left(\sum_{\mathbf{h}} |f_{\mathbf{h}}(\alpha)| \right)^{v_1} \ll X^{v_1(d+1+\varepsilon)} Q^{-v_1/\sigma}.$$

Thus, with $t = 2v_0 + v_1$ we have altogether

$$\int_{\mathfrak{m}_k(Q)} |T(\alpha,\beta)| \,\mathrm{d}\alpha \ll X^{s-(d+1)t} \sup_{\alpha \in \mathfrak{m}_k(Q)} \left(\sum_{\mathbf{h}} |f_{\mathbf{h}}(\alpha)| \right)^{v_1} \int_0^1 \left(\sum_{\mathbf{h}} |f_{\mathbf{h}}(\alpha)| \right)^{2v_0} \mathrm{d}\alpha$$
$$\ll X^{s-(k-d)+\varepsilon} Q^{-v_1/\sigma}.$$

This is ok (for $Q = X$) when $v_1 > 2d\sigma$.
Recalling that $s = 2^d t$, this means that we need $s > 2^d(2s_0 + 2d\sigma)$.

Combining our argument with Browning–Heath-Brown

We now use this bound inside the B-HB technology. Set $Q = X^\theta$. Our input is as follows:

Step 1 By similar arguments to above, we have that

$$\sup_{\alpha \in \mathfrak{m}_k(X^\theta)} |T(\alpha, \beta)| \ll X^{s+\varepsilon}(X^\theta)^{-t/\sigma}.$$

Conversely, if the exponential sum is larger, then $\alpha \in \mathfrak{M}_k(X^\theta)$.

Step 2 Suppose that $\alpha \in \mathfrak{M}_k(X^\theta)$ with denominator q. Then the diagonal form is essentially invisible, and we can perform $d-1$ Weyl steps, and see that when $T(\alpha, \beta)$ is large, then β also has a good rational approximation. Thus, define major arcs β. Since these lie inside the major arcs for α, they are joint major arcs for α and β.

Step 3 Use pruning arguments to make the major arcs as small as possible. There are two stages for this, first we prune the major arcs for α only, and after some changeover point we continue with the joint major arcs for α and β.

Step 4 Major arcs: work (just like in B–HB).

The theorem

Theorem 2 (B-Parsell 2020)

Suppose that $F(\mathbf{x}) = c_1 x_1^k + \ldots + c_n x_n^k$ *with* $c_1 \cdots c_n \neq 0$*, and let* $G \in \mathbb{Z}[x_1, \ldots, x_n]$ *be a non-singular homogeneous polynomial of degree* $d \geqslant 2$*.*
Let s_0 *be the critical point for the mean value of an exponential sum of degree* $k - d$*, and let* σ *be the corresponding Weyl exponent.*
Then

$$N_{F,G}(X) = CX^{s-k-d} + O(X^{s-k-d-\delta}),$$

provided that

$$s > 2^{d+1}(s_0 + d\sigma.)$$

In particular, the bound is

$$s > \begin{cases} 2^k(d+1) & \text{Hua \& Weyl bounds} \\ 2^d[(2d+1)k^2 - O_d(k)] & \text{Vinogradov bounds.} \end{cases}$$

Final comments

- Solving a diagonal form of high degree and a general form of low degree corresponds to counting solutions to a diagonal form that has been restricted to some low-degree hypersurface. In this sense, we generalise work by Brüdern and Robert on linear slices of diagonal forms.
- Similar results: For $d \geqslant 2$, only when G is also diagonal. For $d = 1$ (the Brüdern–Robert case), the G is automatically diagonal, and ultimately VMVT bounds prevail.
- The methods generalise to systems of forms G of same degree, and also to forms F that are sums of n-ary forms. In the latter setting, the bounds get a bit weaker, though.