Rational points on quartic hypersurfaces

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Introduction

Let $F \in \mathbb{Z}[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree $d \geq 3$, defining a hypersurface $X \subset \mathbb{P}^{n-1}_{\mathbb{Q}}$.

Goal: Establish the *Hasse principle:*

$$X(\mathbb{A}_{\mathbb{Q}}) \neq \varnothing \Rightarrow X(\mathbb{Q}) \neq \varnothing$$

Define the counting function

$$N(F, P) := \#\{\mathbf{x} \in \mathbb{Z}^n \cap [-P, P]^n \mid F(\mathbf{x}) = 0\}.$$

Heuristic: $N(F, P) \approx P^{n-d}$ (if $n > d$).

Birch's theorem

Theorem (Birch 1961). Suppose that

$$n - \dim \mathrm{Sing}(X) \geq 2^d(d-1) + 2$$

and $X_{ns}(\mathbb{A}_{\mathbb{Q}}) \neq \varnothing$. Then there exists $c_X > 0$ such that

$$N(F, P) = (c_X + o(1))P^{n-d}.$$

Number of variables needed in Birch's result for F non-singular ($\dim \mathrm{Sing}(X) = -1$):

d	2	3	4	5
$2^d(d-1)+1$	5	17	49	129

Browning, Prendiville 2017: $2^d(d-1) \rightsquigarrow 2^d\left(d - \frac{\sqrt{d}}{2}\right)$.

Cubic forms (d=3)

Note: $X(\mathbb{A}_{\mathbb{Q}}) \neq \varnothing$ as soon as $n \geq 10$ (Lewis 1952).

- Heath-Brown (1981): $X(\mathbb{Q}) \neq \varnothing$ if F is non-singular and $n \geq 10$.
- Hooley (1988): HP holds if F is non-singular and $n \geq 9$.
- Hooley (2013): HP holds if F has isolated double points as only singularities and $n \geq 9$.
- Hooley (2014): HP holds if F is non-singular and $n \geq 8$, if one assumes GRH for a class of Hasse-Weil L-functions.
- Heath-Brown (2007): $X(\mathbb{Q}) \neq \varnothing$ if $n \geq 14$.

Quartic forms (d=4)

- Here, one only knows $X(\mathbb{A}_{\mathbb{Q}}) \neq \varnothing$ for $n \geq 4221$ (Heath-Brown 2010).
- Theorem (Browning, Heath-Brown 2009) If $n - \dim \mathrm{Sing}(X) \geq 42$ and $X_{ns}(\mathbb{A}_{\mathbb{Q}}) \neq \varnothing$, then there exist $P_0 \geq 1$ and $c = c_X > 0$ such that

$$N(F, P) \geq cP^{n-4}$$

as soon as $P \geq P_0$.
- Hanselmann (2012): $n - \dim \mathrm{Sing}(X) \geq 41$ suffices.

Theorem A (M, Vishe 2019)

$n - \dim \mathrm{Sing}(X) \geq 31$ suffices.

In particular HP holds for non-singular quartic forms in at least 30 variables.

Quartic forms

Theorem A. If $n - \dim \mathrm{Sing}(X) \geq 31$ and $X_{ns}(\mathbb{A}_{\mathbb{Q}}) \neq \varnothing$, then there exist $P_0 \geq 1$ and $c = c_X > 0$ such that

$$N(F, P) \geq cP^{n-4}$$

as soon as $P \geq P_0$.

Smoothed counting function:

$$N_w(F, P) = \sum_{\substack{\mathbf{x} \in \mathbb{Z}^n \\ F(\mathbf{x})=0}} w\left(P^{-1}\mathbf{x}\right), \qquad (w \in C_c^\infty(\mathbb{R}^n \to \mathbb{R}_{\geq 0}).$$

Theorem A′. If $n - \dim \mathrm{Sing}(X) \geq 31$ and $X_{ns}(\mathbb{A}_{\mathbb{Q}}) \neq \varnothing$, then for a suitable smooth weight function[1] w, one has

$$N_w(F, P) = cP^{n-4} + O(P^{n-4-\delta}) \qquad (c > 0, \delta > 0).$$

Theorem A′ $\Rightarrow$ Theorem A since $N(F, P) \gg N_w(F, P)$.

[1] supported in a small neighbourhood of a given point $\mathbf{x}_0 \in X_{ns}(\mathbb{R})$

The δ-method (Duke/Friedlander/Iwaniec, Heath-Brown)

Let

$$\delta(n) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 1. Given $Q \geq 1$ and $\theta > 0$, we have

$$\delta(n) = \sum_{q=1}^{Q} \sum_{a=1}^{q}{}^{*} \int_{|z|<(qQ)^{-1+\theta}} p_q(z) e\left(\left(\frac{a}{q} + z\right) n\right) \, dz + O_{N,\theta}(Q^{-N\theta}),$$

where $p_q(z)$ is a smooth function satisfying

$$p_q(z) \ll 1, \qquad p_q(z) = 1 + O_N\left(\left(\frac{q}{Q}\right)^N\right) \text{ for } |z| < Q^{-2}.$$

The δ-method

Applying Lemma 1 to $N_w(F,P) = \sum_{\mathbf{x}} w(P^{-1}\mathbf{x})\delta(F(\mathbf{x}))$ gives:

Corollary 1. Given $P, Q \geq 1$ and $\theta > 0$, we have

$$N_w(F,P) = \sum_{q=1}^{Q} \int_{|z|<(qQ)^{-1+\theta}} p_q(z) S(q,z)\, dz + O_{N,\theta}(P^n Q^{-n\theta}), \qquad (1)$$

where

$$S(q,z) = \sum_{a=1}^{q}{}^{*} \sum_{\mathbf{x}\in\mathbb{Z}^n} w\left(P^{-1}\mathbf{x}\right) e\left(\left(\frac{a}{q} + z\right) F(\mathbf{x})\right).$$

This may be viewed as an exact Kloosterman refinement.

Note: In Corollary 1, F is allowed to be of any degree.

Major and minor arcs

Corollary 1. Given $P, Q \geq 1$ and $\theta > 0$, we have

$$N_w(F,P) = \sum_{q=1}^{Q} \int_{|z|<(qQ)^{-1+\theta}} p_q(z) S(q,z)\, dz + O_{N,\theta}(P^n Q^{-n\theta}) \qquad (1)$$

Split the range in (1) into $\mathfrak{M} \cup \mathfrak{m}$, where

$$\mathfrak{M} = \{(q,z) \mid 1 \leq q \leq P^{\Delta}, |z| \leq P^{-4+\Delta}\}.$$

By the properties of $p_q(z)$, we get

$$N_w(F,P) = \sum_{(q,z)\in\mathfrak{M}} \int S(q,z)\, dz + O\left(\sum_{(q,z)\in\mathfrak{m}} \int |S(q,z)|\, dz \right) + \text{error}.$$

Major and minor arcs

Assume for simplicity that F is non-singular.

The major arcs contribution

$$S_{\mathfrak{M}} = \sum_{1 \le q \le P^{\Delta}} \int_{|z| \le P^{-4+\Delta}} S(q,z)\, dz$$

can be shown to satisfy

$$S_{\mathfrak{M}} = cP^{n-4} + O(P^{n-4-\delta}),$$

where $c = \mathfrak{S}\mathfrak{I} > 0$, as soon as $n \ge 25$.

We focus on the minor arcs contribution

$$S_{\mathfrak{m}} = \sum_{1 \le q \le P^{\Delta}} \int_{P^{-4+\Delta} < |z| \le (qQ)^{-1+\theta}} |S(q,z)|\, dz + \sum_{P^{\Delta} \le q \le Q} \int_{|z| \le P^{-4+\Delta}} |S(q,z)|\, dz \overset{?}{\ll} P^{n-4-\delta}.$$

van der Corput differencing

Put

$$\mathcal{F}(\mathbf{x}) = \sum_{a=1}^{q}{}^{*} w\left(P^{-1}\mathbf{x}\right) e\left(\left(\frac{a}{q}+z\right) F(\mathbf{x})\right).$$

Fix $H \leq P$. We have

$$\sum_{1\leq h_i \leq H} \sum_{\mathbf{x}\in\mathbb{Z}^n} \mathcal{F}(\mathbf{x}+\mathbf{h}) = H^n S(q,z).$$

Apply Cauchy-Schwarz:

$$\begin{aligned} H^{2n}|S(q,z)|^2 &\ll P^n \sum_{\mathbf{x}} \sum_{\mathbf{h}_1,\mathbf{h}_2} \mathcal{F}(\mathbf{x}+\mathbf{h}_1)\overline{\mathcal{F}(\mathbf{x}+\mathbf{h}_2)} \\ &\ll H^n P^n \sum_{\mathbf{h}} \sum_{\mathbf{y}} \mathcal{F}(\mathbf{y}+\mathbf{h})\overline{\mathcal{F}(\mathbf{y})}; \end{aligned}$$

$$\implies |S(q,z)|^2 \ll \frac{P^n}{H^n} \sum_{\mathbf{h}\ll H} |\mathcal{T}_{\mathbf{h}}(q,z)|, \ldots$$

van der Corput differencing

$$\ldots \Longrightarrow |S(q,z)|^2 \ll \frac{P^n}{H^n} \sum_{\mathbf{h} \ll H} |\mathcal{T}_{\mathbf{h}}(q,z)|,$$

where

$$\mathcal{T}_{\mathbf{h}}(q,z) = \sum_{\mathbf{x} \in \mathbb{Z}^n} w_{\mathbf{h}}(P^{-1}\mathbf{x}) \sum_{a_1,a_2=1}^{q}{}^{*} e_q\left(a_1 F_{\mathbf{h}}(\mathbf{x}) + (a_1 - a_2)F(\mathbf{x})\right) e(zF_{\mathbf{h}}(\mathbf{x})),$$

with

$$w_{\mathbf{h}}(\mathbf{u}) = w\left(\mathbf{u} + P^{-1}\mathbf{h}\right) w(\mathbf{u}), \qquad F_{\mathbf{h}}(\mathbf{x}) = F(\mathbf{x}+\mathbf{h}) - F(\mathbf{x}).$$

"Semi-trivial" estimate in the case $\mathbf{h} = \mathbf{0}$ (and other bad cases): if $q \leq P^{2-4/(n+2)}$ is squarefree, then

$$\mathcal{T}_{\mathbf{h}}(q,z) \ll qP^{n+\varepsilon}. \tag{2}$$

Poisson summation

Suppose that $\mathbf{h} \neq \mathbf{0}$. Apply Poisson's summation formula:

$$\mathcal{T}_{\mathbf{h}}(q,z) = P^n q^{-n} \sum_{\mathbf{v} \in \mathbb{Z}^n} S(q,\mathbf{v}) J(z, q^{-1}\mathbf{v}),$$

where
$$S(q,\mathbf{v}) = \sideset{}{^*}\sum_{a_1,a_2=1}^{q} \sum_{\mathbf{x} \pmod{q}} e_q\left(a_1 F_{\mathbf{h}}(\mathbf{x}) + (a_1 - a_2)F(\mathbf{x}) + \mathbf{v}.\mathbf{x}\right),$$
$$J(z,\mathbf{u}) = \int_{\mathbb{R}^n} w_{\mathbf{h}}(\mathbf{x}) e(zF_{\mathbf{h}}(P\mathbf{x}) - P\mathbf{u}.\mathbf{x})\, d\mathbf{x}.$$

Lemma 2. One has

$$J(z,\mathbf{u}) \ll \int_{\substack{\mathbf{x} \ll 1 \\ |z\nabla F_{\mathbf{h}}(P\mathbf{x}) - \mathbf{u}| \ll \frac{P^{\varepsilon} Y}{P}}} d\mathbf{x} \quad + \quad (\text{small error}),$$

where $Y = \max\{1, \sqrt{|z|HP^3}\}$.

The lemma gives

$$\mathcal{T}_{\mathbf{h}}(q,z) \ll P^n q^{-n} \max_{\mathbf{v}_0} \sum_{|\mathbf{v} - \mathbf{v}_0| \leq \frac{qP^{\varepsilon} Y}{P}} |S(q,\mathbf{v})|.$$

Oscillatory integrals

Sketch of proof of Lemma 2. 'Smoothly partition' $J(z,\mathbf{u})$ into integrals

$$J_{\delta,\mathbf{y}}(z,\mathbf{u}) = \int_{\mathbb{R}^n} w_{\delta,\mathbf{y}}(\mathbf{x}) e(zF_{\mathbf{h}}(P\mathbf{x}) - P\mathbf{u}.\mathbf{x})\, d\mathbf{x},$$

where $w_{\delta,\mathbf{y}}$ is supported in a ball of radius δ around $\mathbf{y} \in [-1,1]^n$. Use an oscillatory integral bound of the type

$$\begin{cases} w \in C_c^\infty, f \in C^\infty \\ |\nabla f| \geq \lambda \text{ on } \operatorname{supp}(w) \end{cases} \implies \int w(\mathbf{z}) e(f(\mathbf{z}))\, d\mathbf{x} \ll_N \lambda^{-N} \qquad (3)$$

for any $N > 0$.

With $w(\mathbf{z}) = w_{\delta,\mathbf{y}}(\mathbf{y} + \delta\mathbf{z})$, $r(\mathbf{x}) = zF_{\mathbf{h}}(P\mathbf{x})$ (a cubic polynomial with coefficients of size $|z|HP^3$) and $f(\mathbf{z}) = r(\mathbf{y}+\delta\mathbf{z}) - P\mathbf{u}.(\mathbf{y}+\delta\mathbf{z})$, we have

$$\nabla f(\mathbf{z}) = \delta P(z\nabla F_{\mathbf{h}}(P(\mathbf{y}+\delta\mathbf{z})) - \mathbf{u}) + O(\delta^2|z|HP^3) = \nabla f(\mathbf{0}) + O(\delta^2|z|HP^3),$$

so choosing $\delta = \min\{1, (|z|HP^3)^{-1/2}\}$, we can use (3) with $\lambda = P^\varepsilon$ to get a negligible contribution from $J_{\delta,\mathbf{y}}(z,\mathbf{u})$ unless

$$P^\varepsilon \leq |\nabla f(\mathbf{0})| = \delta P\, |z\nabla F_{\mathbf{h}}(\mathbf{y}) - \mathbf{u}| + O(1),$$

which translates to the region defined in the Lemma.

Idealized minor arcs bound

Consider a typical minor arcs situation: $q \approx Q$ squarefree, so $|z| \ll Q^{-2}$. We have $\mathcal{T}_{\mathbf{0}}(q,z) \ll qP^n$ and

$$\mathcal{T}_{\mathbf{h}}(q,z) \ll P^n q^{-n} \max_{\mathbf{v}_0} \sum_{|\mathbf{v}-\mathbf{v}_0| \le \frac{qP^{\varepsilon}Y}{P}} |S(q,\mathbf{v})|, \quad \mathbf{h} \neq \mathbf{0}.$$

For generic $\mathbf{h}$, $\mathbf{v}$, we assume that we have

$$|S(q,\mathbf{v})| \ll q^{(n+2)/2} \quad \text{(full square-root cancellation).}$$

This gives

$$|S(q,z)|^2 \ll \frac{P^n}{H^n} \sum_{\mathbf{h} \ll H} |\mathcal{T}_{\mathbf{h}}(q,z)| \ll \frac{P^n}{H^n}\left(qP^n + H^n q^{1+n/2} Y^n + \dots\right)$$

$$\ll \frac{QP^n}{H^n}\left(P + HQ^{1/2} + \frac{H^{3/2}P^{3/2}}{Q^{1/2}} + \dots\right)^n \ll QP^{9n/5}.$$

Balance by putting $H = (Q/P)^{1/3}$ and $Q = P^{8/5}$.

$$\sum_{q \asymp Q \ (\square\text{-free})} \int_{|z| \le \frac{1}{qQ}} |S(q,z)|\, dz \ll Q^{-1/2} P^{9n/10} = P^{n-(n+8)/10}.$$

So typical contribution is OK if '$n \ge 32 + \varepsilon$'.

Bounding $S(p, \mathbf{v})$, p prime

Consider the varieties in $\mathbb{P}^{n-1}_{\mathbb{F}_p}$:

$$Z_{\mathbf{h}} = \{F(\mathbf{x}) = \mathbf{h}.\nabla F(\mathbf{x}) = 0\}, \qquad Z_{\mathbf{h},\mathbf{v}} = Z_{\mathbf{h}} \cap \{\mathbf{v}.\mathbf{x} = 0\}.$$

Let $s_p(\mathbf{h}) = \dim \mathrm{Sing}(Z_{\mathbf{h},\mathbf{v}})$ and $s_p(\mathbf{h}, \mathbf{v}) = \dim \mathrm{Sing}(Z_{\mathbf{h},\mathbf{v}})$.

Complete the sum:

$$\begin{aligned} S^{\mathrm{comp}}(p, \mathbf{v}) &:= \sum_{a_1, a_2 = 1}^{p} \sum_{\mathbf{x} \pmod{p}} e_p\left(a_1 F_{\mathbf{h}}(\mathbf{x}) + (a_1 - a_2) F(\mathbf{x}) + \mathbf{v}.\mathbf{x}\right) \\ &= p^2 \sum_{F(\mathbf{x}) \equiv F_{\mathbf{h}}(\mathbf{x}) \equiv 0 \pmod{p}} e_p(\mathbf{v}.\mathbf{x}) =: p^2 \Sigma(p, \mathbf{v}). \end{aligned}$$

Lemma 3. We have

$$\Sigma(p, \mathbf{v}) \ll \begin{cases} (\sqrt{p})^{n-1+s_p(\mathbf{h})} & \text{if } s_p(\mathbf{h}, \mathbf{v}) \leq s_p(\mathbf{h}), \\ (\sqrt{p})^{n+s_p(\mathbf{h})} & \text{otherwise} \end{cases}$$

Proof. Use (Katz, *Estimates for singular exponential sums*, 1999). □

Bounding $S(p, \mathbf{v})$, p prime

Using Lemma 3 one obtains

$$S^{\text{comp}}(p, \mathbf{v}) \ll p^{(n+3+s_p(\mathbf{h}))/2}(p, \Phi(\mathbf{v}))^{1/2}$$

for a certain homogeneous polynomial $\Phi = \Phi_{\mathbf{h}} \in \mathbb{Z}[\mathbf{x}]$.

Important for later: Φ can be chosen to have no linear factors over $\mathbb{Z}$!

Multiplicativity gives

$$|S(q, \mathbf{v})| \ll q^{(n+2)/2+\varepsilon}$$

for q squarefree and $\mathbf{h}, \mathbf{v}$ generic .

Averaged van der Corput differencing

Idea from (Heath-Brown, *Cubic forms in 14 variables*, 2007):

For $\mathcal{H} \subseteq [-P, P]^n$ we have

$$|S(q,z)|^2 \ll P^n \#\mathcal{H}^{-1} \sum_{\mathbf{h} \in \mathcal{H}} |\mathcal{T}_{\mathbf{h}}(q,z)| \qquad (\text{earlier } \mathcal{H} = [-H, H]^n),$$

where

$$\mathcal{T}_{\mathbf{h}}(q,z) = \sum_{\mathbf{x} \in \mathbb{Z}^n} w_{\mathbf{h}}(P^{-1}\mathbf{x}) \sideset{}{^*}\sum_{a_1, a_2 = 1}^{q} e_q\left(a_1 F_{\mathbf{h}}(\mathbf{x}) + (a_1 - a_2) F(\mathbf{x})\right) e(z F_{\mathbf{h}}(\mathbf{x})).$$

Average smoothly over an interval of size $Z \gg HP^3$:

$$M(q,Z) = \int \omega(Z^{-1}z) |S(q,z)|^2 \, dz \qquad (\omega \text{ smooth}).$$

The contribution to $M(q,Z)$ from $\mathbf{x}$ with

$$|F_{\mathbf{h}}(\mathbf{x})| \geq HP^{3+\varepsilon} \tag{4}$$

is small, since ...

Averaged van der Corput differencing

... The contribution to $M(q,Z)$ from $\mathbf{x}$ with

$$|F_{\mathbf{h}}(\mathbf{x})| \geq HP^{3+\varepsilon} \tag{4}$$

is small, since

$$\left| \int \omega(Z^{-1}z) e(zF_{\mathbf{h}}(\mathbf{x}))\, dz \right| = Z\, |\hat{\omega}(ZF_{\mathbf{h}}(\mathbf{x}))| \ll_N Z|ZF_{\mathbf{h}}(\mathbf{x})|^{-N} \ll P^{-\varepsilon N}.$$

Choose w such that[2] $\partial_{x_1} F(\mathbf{x}) \gg P^3$ for all $\mathbf{x} \in P\,\mathrm{supp}(w)$. Then we can enlarge $\mathcal{H}$ to

$$\mathcal{H}' = [-P, P] \times [-H, H]^{n-1},$$

since the range $HP^{\varepsilon} \leq |h_1| \leq P$ gives a negligible contribution by (4). Thus:

Lemma 4. We have

$$M(q,Z) \ll P^{-N} + \left(\frac{P}{H}\right)^{n-1} \sum_{\mathbf{h} \ll HP^{\varepsilon}} \int \omega(Z^{-1}z) |\mathcal{T}_{\mathbf{h}}(q,z)|\, dz.$$

[2] after permuting variables

Averaged van der Corput differencing

Lemma 4. We have

$$M(q,Z) \ll P^{-N} + \left(\frac{P}{H}\right)^{n-1} \sum_{\mathbf{h} \ll HP^{\varepsilon}} \int \omega(Z^{-1}z) |\mathcal{T}_{\mathbf{h}}(q,z)| \, dz,$$

where

$$\mathcal{T}_{\mathbf{h}}(q,z) = \sum_{\mathbf{x} \in \mathbb{Z}^n} w_{\mathbf{h}}(P^{-1}\mathbf{x}) \sideset{}{^*}\sum_{a_1,a_2=1}^{q} e_q\left(a_1 F_{\mathbf{h}}(\mathbf{x}) + (a_1 - a_2) F(\mathbf{x})\right) e(zF_{\mathbf{h}}(\mathbf{x})).$$

We gain a factor P/H, so in the end we gain $(P/H)^{1/2} = P^{2/5}$ in the idealized minor arcs bound:

$$P^{n-(n+8)/10} \rightsquigarrow P^{n-(n+12)/10} \qquad \text{Now OK for } n \geq 28 + \varepsilon!$$

Complementary bounds

For squarefull numbers q, we can only satisfactorily bound the average

$$\sum_{|\mathbf{v}-\mathbf{v}_0|\leq V} |S(q,\mathbf{v})|.$$

If $V \approx qY/P$ is so small that nothing can be gained from this, then we employ either

- Birch's bounds for $|S(q,z)|$ ("4 Weyl differencings"), or
- Bounds from (Browning-Prendiville, 2017) ("1 van der Corput differencing + 3 Weyl differencings")

Through a delicate optimization of all available bounds, we are (only) able to conclude that

$$S_{\mathfrak{m}} \ll P^{n-4-\delta}$$

as soon as $n \geq 30$.

Crucial idea to go from 30 to 29

Recall

$$\mathcal{T}_{\mathbf{h}}(q,z) = P^n q^{-n} \sum_{\mathbf{v}\in\mathbb{Z}^n} S(q,\mathbf{v}) J(z, q^{-1}\mathbf{v}),$$

where

$$J(z,q^{-1}\mathbf{v}) = \int_{\mathbb{R}^n} w_{\mathbf{h}}(\mathbf{x}) e(zF_{\mathbf{h}}(P\mathbf{x}) - q^{-1}P\mathbf{v}.\mathbf{x})\, d\mathbf{x}$$

$$\overset{\text{Lemma 2}}{\ll} \int_{\substack{\mathbf{x}\ll 1 \\ |qz\nabla F_{\mathbf{h}}(P\mathbf{x}) - \mathbf{v}| \ll \frac{P^{\varepsilon}Y}{P}}} d\mathbf{x} \quad + \quad \text{(small error)}.$$

Capitalize average over $|z| \ll Z$:

$$\int \omega(Z^{-1}z) J(z,q^{-1}\mathbf{v})\, dz \ll \int_{\substack{z\ll Z, \mathbf{x}\ll 1 \\ |F_{\mathbf{h}}(P\mathbf{x})| \leq Z^{-1}YP^{\varepsilon} \\ |qz\nabla F_{\mathbf{h}}(\mathbf{x}) - \mathbf{v}| \ll \frac{qP^{\varepsilon}Y}{P}}} d\mathbf{x} \quad + \quad \text{(small error)},$$

where now $Y = \max\{1, \sqrt{ZHP^3}\}$.

Crucial idea to go from 30 to 29

...

$$\int \omega(Z^{-1}z) J(z, q^{-1}\mathbf{v})\, dz \ll \int_{\substack{z \ll Z, \mathbf{x} \ll 1 \\ |F_{\mathbf{h}}(P\mathbf{x})| \le Z^{-1} Y P^{\varepsilon} \\ |qz \nabla F_{\mathbf{h}}(\mathbf{x}) - \mathbf{v}| \ll \frac{qP^{\varepsilon} Y}{P}}} d\mathbf{x} \quad + \quad \text{(small error)},$$

where $Y = \max\{1, \sqrt{ZHP^3}\}$.

Idea: consider a small region R where both z and $\mathbf{x}$ vary by at most $\delta \approx Y^{-1}$. Then the integral

$$\iint_R \omega(z) \cdots e(zZF_{\mathbf{h}}(P\mathbf{x})) \cdots \, dz\, d\mathbf{x}$$

oscillates in the z-direction unless

$$|F_{\mathbf{h}}(P\mathbf{x})| \le Z^{-1} Y P^{\varepsilon}. \tag{5}$$

For large q, the upper bound (5) is trivial, but for small q, one can hope to bound the measure of the set of $\mathbf{x} \ll 1$ satisfying (5).