Circle Method and Analytic Theory of L-functions

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May 17, 2021

This mini course is about application of the Circle Method to the following:

Three Problems:

1. Subconvexity
2. Shifted convolution sum
3. Higher rank exponential sum

Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{for } s = \sigma + it,\ \sigma > 1.$$

- In this half-plane we have an integral representation

$$\pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2})\zeta(s) = \int_0^\infty x^{\frac{s}{2}-1}\psi(x)\mathrm{d}x, \quad \text{where } \psi(x) = \sum_{n=1}^{\infty} e^{-n^2\pi x}.$$

- The Poisson summation formula

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n), \quad \text{yields } 2\psi(x) + 1 = \frac{1}{\sqrt{x}}\left\{2\psi\left(\frac{1}{x}\right) + 1\right\}.$$

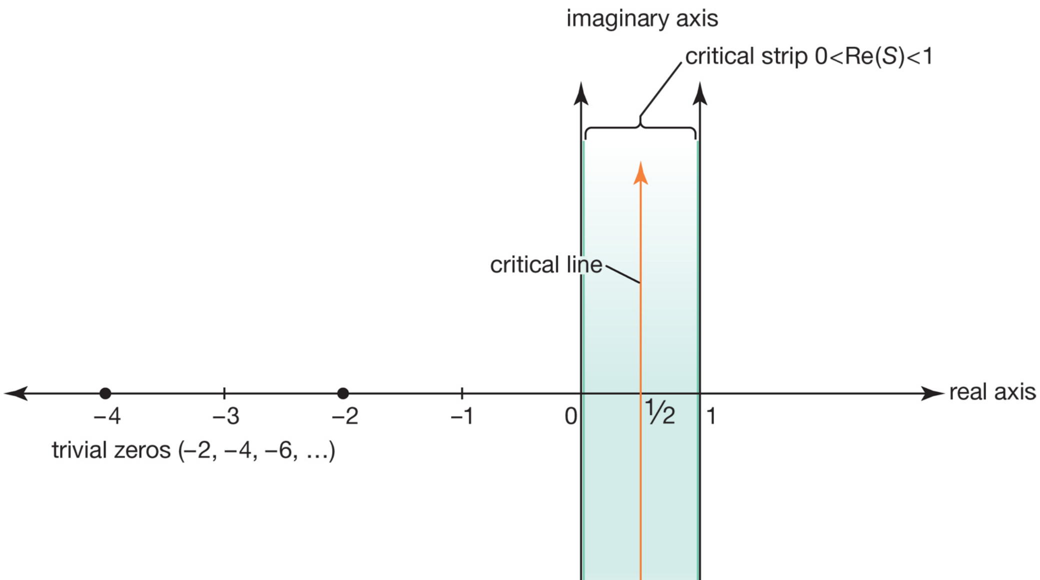
- Hence

$$\pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2})\zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty (x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}})\psi(x)\mathrm{d}x.$$

  We get analytic continuation and the functional equation

$$\pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2})\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma(\tfrac{1-s}{2})\zeta(1-s)$$

Critical strip



Subconvexity problem

- **Problem: To understand the size of the zeta function inside the critical strip.** (Because of Phragmen-Lindelöf principle, it's enough to study the size on the critical line $\zeta(1/2+it)$.)

- From functional equation and Phragmen-Lindelöf we get the trivial (convexity) bound
$$\zeta(1/2+it) \ll t^{1/4+\varepsilon}.$$
(Say $t > 2$.)

- But a much sharper bound is expected. The Riemann Hypothesis implies the Lindelöf Hypothesis
$$\zeta(1/2+it) \ll t^{\varepsilon}.$$

- The subconvexity problem is about improving the exponent $1/4$.

Thursday, February 10th, 1921.

Mr. H. W. RICHMOND, President, and later Mr. J. E. CAMPBELL, Vice-President, in the Chair.

Present thirty-seven members and twelve visitors.

Messrs. W. H. Glaser and R. F. Whitehead, and Prof. Olive C. Hazlett, were elected members of the Society.

Dr. H. Levy was nominated for membership.

Prof. H. S. Carslaw was admitted into the Society.

Prof. A. S. Eddington delivered a lecture "World Geometry (with particular reference to Weyl's electromagnetic theory)."

The following papers were communicated by title from the chair:—

*Note on the Electromagnetic Equations: J. Brill.

Researches in the Theory of the Riemann Zeta-Function: J. E. Littlewood.

A New Condition for Cauchy's Theorem: S. Pollard.

ABSTRACT.

Researches in the Theory of the Riemann ζ-Function

MR. J. E. LITTLEWOOD.

It would occupy too much space to give any detailed description of the methods used in these researches, or any full account of previous work in the same subjects, and I have confined myself in the main to a bare statement of results.

1. *Theorems on mean values.*

....

4. In a paper written in collaboration with Prof. G. H. Hardy, which we hope will be published shortly, it is shown that $\zeta(\frac{1}{2}+it) = O(t^{\frac{1}{6}+\epsilon})$, that intermediate upper bounds exist for σ's between $\frac{1}{2}$ and 1, and that (with special reference to the neighbourhood of $\sigma = 1$) *there is a constant A such*

(Hardy-Littlewood-) Weyl bound

Theorem

For $t > 2$ one has

$$\zeta(\tfrac{1}{2} + it) \ll t^{\frac{1}{6} + \varepsilon}.$$

- Hardy-Littlewood paper on subconvexity was never published. Landau later published the first proof.
- The bound of Hardy-Littlewood-Weyl is 1/3rd way down towards LH.
- Current record: Bourgain (2017) using decoupling

$$\zeta(\tfrac{1}{2} + it) \ll t^{\frac{1}{6} - \frac{1}{84} + \varepsilon}$$

Proof of Weyl bound

- Functional eqn + Cauchy Theorem $\implies$ Approximate functional eqn

$$\zeta(\tfrac{1}{2}+it) = \sum_{n\le\sqrt{\frac{t}{2\pi}}} \frac{1}{n^{\frac{1}{2}+it}} + \chi(\tfrac{1}{2}+it) \sum_{n\le\sqrt{\frac{t}{2\pi}}} \frac{1}{n^{\frac{1}{2}-it}} + O\left(\frac{\log t}{t^{1/4}}\right)$$

  where $\chi(s) = 2^{s-1}\pi^s(\sec\frac{s\pi}{2})/\Gamma(s)$.
- Trivial estimation yields convexity bound. For subconvexity we need to show cancellation in the sum

$$\sum_{n\ll\sqrt{t}} n^{it} = \sum_{n\ll\sqrt{t}} e\left(\frac{t}{2\pi}\log n\right)$$

  where $e(z) = e^{2\pi i z}$.
- This is an example of an exponential sum. A central theme in analytic number theory.

Proof of Weyl bound

- Step 1: Reduce to polynomial phase function: Using short shifts in the phase function $\log(n+a)$, and Taylor expansion.
- Step 2: Exponential sums with polynomial phase (Weyl sums).
- For Weyl bound we only need quadratic Weyl sum estimates:

$$\left|\sum_{m=1}^{M} e(\alpha m^2 + \beta m)\right|^2 \leq M + 2\sum_{r=1}^{M-1} \min\{M, |\mathrm{cosec}\, 2\pi\alpha r|\}$$

Examples of L-functions

The Riemann zeta function is just the tip of the iceberg.

Dirichlet L-function

Given a primitive character $\chi \in \mathrm{Hom}\left((\mathbb{Z}/M\mathbb{Z})^\star, \mathbb{C}^\star\right)$, we define

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

(Note $\chi(p) = 0$ for $p|M$.)

- Conv. abs. for $\sigma > 1$, and extends to an entire function.
- For $a = \chi(-1)$, $\omega_\chi = i^{-a} G(\chi)/\sqrt{M}$

$$\Lambda(s,\chi) := M^{\frac{s}{2}} \Gamma_{\mathbb{R}}(s+a) L(s,\chi) = \omega_\chi \Lambda(1-s,\bar{\chi}).$$

- The local factor $L_p(s,\chi)^{-1} = 1 - \chi(p)p^{-s}$ is of degree one in p^{-s}. (Like zeta this is a degree one L-function.)

$L(s, \chi)$ continued...

- GRH $\implies$ GLH, i.e. (for $t > 2$)

$$L(\tfrac{1}{2} + it, \chi) \ll (tM)^{\varepsilon}.$$

From the functional equation and the Phragmen-Lindelöf one gets the convexity bound $\ll (tM)^{\frac{1}{4}+\varepsilon}$.

- There are two components - t t-aspect and M level aspect.
- Functional eqn $\implies$ Approximate functional eqn

$$L(\tfrac{1}{2} + it, \chi) \approx \sum_{n \ll \sqrt{Mt}} \frac{\chi(n)}{n^{1/2+it}} + \varepsilon_{\chi}(t) \sum_{n \ll \sqrt{Mt}} \frac{\overline{\chi(n)}}{n^{1/2-it}}.$$

- Mt is the conductor of $L(s, \chi)$ (roughly it measures the complexity of the L-value).
- Note: The convexity bound is given by $(\text{conductor})^{1/4}$ and the length of the approx functional eqn is $(\text{conductor})^{1/2}$.

Discriminant modular form

Set $q = e(z) = e^{2\pi i z}$, $z \in \mathbb{H}$
$\Delta(z) = q \prod_{n=1}^{\infty}(1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n$.

Modularity

Δ is a modular form (cuspform) of weight 12, level 1:

$$\Delta\left(\frac{az+b}{cz+d}\right) = (cz+d)^{12}\Delta(z), \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

In general: $\Gamma_0(N)$ congruence group of level N, ψ Dirichlet char mod N: $M_k(N, \psi)$

1. f holomorphic on $\mathbb{H}$.
2. $f(\gamma z) = \psi(d)(cz+d)^k f(z)$ for all $\gamma \in \Gamma_0(N)$.
3. f holomorphic at the cusps.

Example: $(\Delta \otimes \chi)(z) = \sum_{n=1}^{\infty} \tau(n)\chi(n)e(nz) \in M_{12}(M^2, \chi^2)$ for χ prim mod M.

Hecke L-function

- $\tau(mn) = \tau(m)\tau(n)$ for $(m,n) = 1$
- $\tau(p^{r+1}) = \tau(p)\tau(p^r) - p^{11}\tau(p^{r-1})$
- $|\tau(p)| < 2p^{\frac{11}{2}}$

$L(s,\Delta)$

Set $\tau_0(n) = \tau(n)/n^{\frac{11}{2}}$

$$L(s,\Delta) = \sum_{n=1}^{\infty} \frac{\tau_0(n)}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{\tau_0(p)}{p^s} + \frac{1}{p^{2s}}\right)^{-1}$$

- Absolute convergence in $\sigma > 1$.
- Local L-factor $L_p(s,\Delta)^{-1} = 1 - \tau_0(p)p^{-s} + p^{-2s}$ is of degree two in p^{-s}.

Hecke's integral representation

$$\Lambda(s,\Delta) := (1/2\pi)^s\Gamma(s+\tfrac{11}{2})L(s,\Delta) = (2\pi)^{11/2}\int_0^\infty \Delta(iy)\, y^{s+\frac{11}{2}-1}\mathrm{d}y.$$

- The automorphy relation $\Delta(-\frac{1}{z}) = z^{12}\Delta(z)$ gives analytic continuation and functional equation
$$\Lambda(s,\Delta) = \Lambda(1-s,\Delta).$$
- In general the functional equation is given by
$$\Lambda(s,f) := (1/2\pi)^s N^{s/2}\Gamma(s+\tfrac{k-1}{2})L(s,f) = i^k\Lambda(1-s,\Delta).$$
- Conductor = $N(k+t)^2$ (recall $t>2$). Convexity bound
$$L(\tfrac{1}{2}+it,f) \ll N^{1/4}(k+t)^{1/2}.$$

  There are three components - t, k (spectral), N (level).

Landmark subconvex bounds

Theorem (Burgess 1960's)

$L(\frac{1}{2}, \chi) \ll M^{\frac{3}{16}+\varepsilon}$ for χ primitive mod M.

Theorem (Good 1980's)

$L(\frac{1}{2} + it, \Delta) \ll t^{\frac{1}{3}+\varepsilon}.$

Theorem (Duke-Friedlander-Iwaniec 1990's)

$L(\frac{1}{2}, \Delta \otimes \chi) \ll M^{\frac{1}{2}-\frac{1}{22}+\varepsilon}$ for χ primitive mod M.

Methods

- To estimate $\sum_{n\sim\sqrt{M}}\chi(n)$ Burgess used Weyl differencing with factorizable shifts and Holder's inequality to complete the character sum and then appealed to Riemann Hypothesis for curves over finite fields (Weil).

- Good employed the Spectral theory of Laplacian on the upper half plane to compute the second moment

$$\int_T^{2T} |L(\tfrac{1}{2}+it,\Delta)|^2 \mathrm{d}t.$$

- D-F-I compute amplified second moment and reduce to problem to getting cancellation in a shifted convolution sum

$$\sum_n \tau_0(n)\tau_0(n+1),$$

where they applied the delta method (circle method).

Tools: Summation Formulae

- We will give an uniform treatment - separating the oscillatory terms using the circle method. Summation formulas will play a crucial role.
- Poisson summation formula: $a(n)$ periodic with period M (say $|a(n)| \leq 1$); and W a nice bump function supported on $[1, 2]$. Then

$$\sum_{n\in\mathbb{Z}} a(n)W\left(\frac{n}{N}\right) = \sum_{\beta \bmod M} a(\beta) \sum_{n\in\mathbb{Z}} W\left(\frac{\beta + nM}{N}\right)$$
$$= \sum_{\beta \bmod M} a(\beta) \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} W\left(\frac{\beta + xM}{N}\right) e(-nx)\mathrm{d}x$$
$$= \frac{N}{M} \sum_{n\in\mathbb{Z}} \left[\sum_{\beta \bmod M} a(\beta) e_M(n\beta)\right] \int_{\mathbb{R}} W(y)\, e_M(-nNy)\mathrm{d}y.$$

- Can restrict to $|n| \ll MN^{\varepsilon}/N$ at the cost of error term $O(N^{-A})$ for any $A > 0$. Saving?

- Voronoi summation formula for $d(n)$:

$$\sum_{n=1}^{\infty} d(n)e_c(an)g(n) = \frac{1}{c}\int_0^{\infty}(\log x + 2\gamma - 2\log c)g(x)\mathrm{d}x + \frac{1}{c}\sum_{\pm}(\pm 1)\sum_{n=1}^{\infty} d(n)e_c(\pm\bar{a}n)\tilde{G}_{\pm}(n)$$

where

$$\tilde{G}_{+}(n) = 4\int K_0\left(\frac{4\pi\sqrt{nx}}{c}\right)g(x)\mathrm{d}x, \qquad \tilde{G}_{-}(n) = 2\pi\int Y_0\left(\frac{4\pi\sqrt{nx}}{c}\right)g(x)\mathrm{d}x.$$

- Voronoi summation formula for cuspform Δ:

$$\sum_{n=1}^{\infty} \tau_0(n)e_c(an)g(n) = \frac{2\pi}{c}\sum_{n=1}^{\infty} \tau_0(n)e_c(-\bar{a}n)\int_0^{\infty} J_{11}\left(\frac{4\pi\sqrt{nx}}{c}\right)g(x)\mathrm{d}x.$$

[From functional equation or from modularity $z = -\frac{d}{c} + \frac{i}{cy}$.]

Kloosterman circle method

Theorem

For any $Q \geq 1$, we have

$$\delta(n) = 2\text{Re} \int_0^1 \sum_{q \leq Q < d \leq q+Q} \sideset{}{^\star}\sum \frac{1}{dq} e_q(\bar{d}n)\, e_{dq}(-nx)\mathrm{d}x.$$

Splitting into Farey segments of order Q, we get

$$\delta(n) = \int_0^1 e(nx)\mathrm{d}x = \sum_{0 \leq a < q \leq Q} \sideset{}{^\star}\sum \int_{\mathfrak{m}(a/q)} e(nx)\mathrm{d}x.$$

Now

$$\mathfrak{m}(a/q) = \left[\frac{a}{q} - \frac{1}{q(q+q')}, \frac{a}{q} + \frac{1}{q(q+q'')}\right]$$

where $Q - q < q', q'' \leq Q$ are given by $aq' \equiv 1 \equiv -aq'' \bmod q$

$L(\frac{1}{2}+it,\Delta) \ll t^{\frac{1}{3}+\varepsilon}$

- We need to estimate

$$S(N) = \sum_{n\sim N} \lambda(n) n^{it}, \qquad (\lambda(n) = \tau_0(n)),$$

for $N \ll t^{1+\varepsilon}$. By approx functional eqn we have

$$L(\tfrac{1}{2}+it,\Delta) \ll t^{\varepsilon} \sup_{N \ll t^{1+\varepsilon}} \frac{|S(N)|}{\sqrt{N}}.$$

- Separate oscillation using Kloosterman's CM

$$\int_0^1 \sum_{1\leq q\leq Q<a\leq Q+q} \sideset{}{^\star}\sum \frac{1}{aq} \sum\sum_{n,m\sim N} \lambda(n) m^{it} e\left(\frac{\bar{a}(n-m)}{q} - \frac{(n-m)x}{aq}\right)$$
$$= \int_0^1 \sum_{1\leq q\leq Q<a\leq Q+q} \sideset{}{^\star}\sum \frac{1}{aq} \sum_{n\sim N} \lambda(n) e\left(\frac{\bar{a}n}{q} - \frac{nx}{aq}\right) \sum_{m\sim N} m^{it} e\left(-\frac{\bar{a}m}{q} + \frac{mx}{aq}\right)$$