Beyond the circle method

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Hausdorff School: The Circle Method
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Diophantine equations

Goal: Study Diophantine equations, as e.g.

$$a_1x_1^3 + a_2x_2^3 + \ldots + a_sx_s^3 = 1.$$

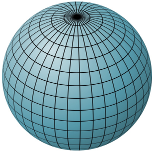
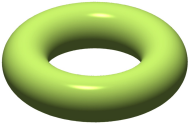
Main challenges:

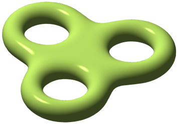
- Existence of integer/rational points?
- Obstructions to the existence of integer/rational points?
- Distribution of integer/rational points, e.g. how many solutions do we find in a bounded domain?

Geometry determines arithmetic

Let C be a curve of genus g over $\mathbb{Q}$.

- Genus 0: no points or infinitely many points
- Genus 1: elliptic curves
- Genus $g > 1$: C has only a finite number of rational points (Faltings, 1983)



What about higher dimensions?

Example (low degree hypersurface)

$$X \subset \mathbb{P}^3_{\mathbb{Q}} : x_0^2 + x_1^2 - x_2^2 - x_3^2 = 0.$$

Can count points of bounded height, e.g. using the circle method.

Example (K3 surface)

$$X \subset \mathbb{P}^3_{\mathbb{Q}} : \; x_0^4 + x_1^4 - x_2^4 - x_3^4 = 0.$$

Conjecturally logarithmic growth of points outside Zariski-closed subset.

Example (General type)

$$X \subset \mathbb{P}^3_{\mathbb{Q}} : \; x_0^5 + x_1^5 + x_2^5 + x_3^5 = 0.$$

According to Lang's conjecture, rational points in X are not Zariski-dense.

Manin's conjecture

Conjecture (Manin-Peyre)

Let V be a smooth projective Fano variety over a number field k such that $V(k)$ is dense in V. Then there exists a thin subset Z such that

$$\sharp\{x \in V(k) \setminus Z : H_{\omega_V^{-1}}(x) \leq B\} \sim cB(\log B)^{\mathrm{rk}(\mathrm{Pic}(X))-1}.$$

Examples

Known for

- flag varieties
- toric varieties
- equivariant compactifications of vector groups
- hypersurfaces $X \subset \mathbb{P}^n_{\mathbb{Q}}$ of degree d with $n > (d-1)2^d$.
- ...

Hasse principle

Example

$x^2 + y^2 = 3z^2$ has no non-trivial solution in $\mathbb{Q}$, as there are no non-trivial solutions in $\mathbb{Q}_3$.

Definition

Let $\mathcal{F}$ be a family of smooth projective varieties defined over $\mathbb{Q}$. We say that the Hasse principle holds for $\mathcal{F}$, if for any $X \in \mathcal{F}$ one has

$$X(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset \Rightarrow X(\mathbb{Q}) \neq \emptyset.$$

Conjecture (Colliot-Thélène, 2003)

Let X be a smooth, projective, geometrically irreducible and (geometrically) rationally connected variety over a number field. Then the Brauer-Manin obstruction is the only obstruction to the Hasse principle and weak approximation.

Counting integral points

Set-up: X a smooth projective variety over $\mathbb{Q}$, D a strict normal crossing divisor. Take $\mathcal{U}$ an integral model of $X \setminus D$, and H a height function on $X(\mathbb{Q})$.

Questions

- *Under what conditions is $\mathcal{U}(\mathbb{Z}) \neq \emptyset$? Obstructions?*
- *How does the counting function $\sharp\{x \in \mathcal{U}(\mathbb{Z}) : H(x) \leq B\}$ behave?*

Example (Waring's problem)

Take $X \subset \mathbb{P}^n_{\mathbb{Q}} : x_1^k + \ldots + x_n^k = mx_0^k$ with $m \in \mathbb{N}$ and $D = \{x_0 = 0\}$.Then for the naive height function we would count

$$\sharp\{x_1, \ldots, x_n \in \mathbb{Z} : \max_{1 \leq i \leq n} |x_i| \leq B,\ x_1^k + \ldots + x_n^k = (\pm 1)^k m\}.$$

More obstructions to integral points

Markoff type cubic surfaces

Example (Ghosh-Sarnak 2017)

Let $l \geq 13$ be a prime with $l \equiv \pm 4 \bmod 9$ and $l \not\equiv \pm 1 \bmod 8$.

$$V_{4+2l^2}: \quad x_1^2 + x_2^2 + x_3^2 - x_1 x_2 x_3 = 4 + 2l^2.$$

Then V_{4+2l^2} violates the integral Hasse principle and the violation is not explained by an integral Brauer-Manin obstruction.

Remark

- Is (x_1, x_2, x_3) a solution, then so is $(x_1, x_2, x_1 x_2 - x_3)$, as well as permutations.
- After a finite number of Markoff moves every solution can be moved into a fundamental domain of finite volume.

Campana points

Motivation

Interpolate between integral and rational points on varieties over number fields.

Definition

Let $m \geq 1$. We call an integer $a \in \mathbb{Z}$ m-full if

$$p \mid a \Rightarrow p^m \mid a, \quad p \text{ prime}$$

Example

Let $[x_0 : x_1]$ be homogeneous coordinates for $\mathbb{P}^1_{\mathbb{Z}}$ and $\mathcal{D} = \{x_1 = 0\}$.

- Integral points on $\mathbb{P}^1_{\mathbb{Z}} \setminus \mathcal{D}$ correspond to $x_0 \in \mathbb{Z}$ and $x_1 \in \{\pm 1\}$
- Campana points $(\mathbb{P}^1_{\mathbb{Z}}, \mathcal{D}, m)$ correspond to $x_0 \in \mathbb{Z}$ and $x_1 \in \mathbb{Z}$ m-full, $\gcd(x_0, x_1) = 1$.

Campana points

Definition

Let X be a smooth proper variety over a number field k and $D = \cup_{i=1}^n D_i$ a strict normal crossing divisor. Let $S \subset \Omega_k$ be a finite set of places of k such that there exists a smooth proper $\mathcal{O}_{k,S}$-model $(\mathcal{X}, \mathcal{D})$ of (X, D) and such that $\mathcal{D} = \cup_{i=1}^n \mathcal{D}_i$ is a strict normal crossing divisor modulo primes p not contained in S. Let $\mathbf{m} = (m_1, \dots, m_n) \in (\mathbb{Z}_{\geq 1})^n$. Define

$$(\mathcal{X}, \mathcal{D}, \mathbf{m})(\mathcal{O}_{k,S}) = \{x \in \mathcal{X}(\mathcal{O}_{k,S}), x \notin D,$$
$$\forall_{p \notin S} \nu_p(x^*\mathcal{D}_i) > 0 \Rightarrow \nu_p(x^*\mathcal{D}_i) \geq m_i,\ 1 \leq i \leq n\}.$$

Campana points

Definition

Let X be a smooth proper variety over a number field k and $D = \cup_{i=1}^n D_i$ a simple normal crossing divisor. Let $S \subset \Omega_k$ be a finite set of places of k such that there exists a smooth proper $\mathcal{O}_{k,S}$-model $(\mathcal{X}, \mathcal{D})$ of (X, D) and such that $\mathcal{D} = \cup_{i=1}^n \mathcal{D}_i$ is a simple normal crossing divisor modulo primes p not contained in S. Let $\mathbf{m} = (m_1, \ldots, m_n) \in (\mathbb{Z}/N\mathbb{Z})^n$. Define

$$(\mathcal{X}, \mathcal{D}, \mathbf{m})(\mathcal{O}_{k,S}) = \{x \in \mathcal{X}(\mathcal{O}_{k,S}), x \notin D, \\ \forall_{p \notin S} \nu_p(x^* \mathcal{D}_i) > 0 \Rightarrow \nu_p(x^* \mathcal{D}_i) \geq m_i,\ 1 \leq i \leq n\}.$$

Remark

If f_i is a local equation of $\mathcal{D}_i$ around x then

$$\nu_p(x^* \mathcal{D}_i) = \nu_p(f_i(x)).$$

Campana points, an example

Example

Let $m \geq 1$. Let $[x_0 : x_1]$ be homogeneous coordinates for $\mathbb{P}^1_{\mathbb{Z}}$ and $\mathcal{D} = \{x_1 = 0\}$. Then

$$(\mathbb{P}^1_{\mathbb{Z}}, \{x_1 = 0\}, m)(\mathbb{Z}) = \{(x_0 : x_1), x_0, x_1 \in \mathbb{Z} \text{ coprime }, x_1 \text{ is } m\text{−full}\}.$$

One has the inclusions

$$(\mathbb{P}^1_{\mathbb{Z}} \setminus \mathcal{D})(\mathbb{Z}) \subset (\mathbb{P}^1_{\mathbb{Z}}, \mathcal{D}, m)(\mathbb{Z}) \subset (\mathbb{P}^1_{\mathbb{Q}} \setminus D)(\mathbb{Q}).$$

Counting Campana points

Question

Conjectures for the growth of the number of Campana points of bounded height?

Theorem (Erdos-Szekeres 1935)

Let $m \geq 1$. Then

$$\sharp\{1 \leq y \leq B : y \text{ is } m - \text{full}\} \sim c_m B^{\frac{1}{m}}$$

Idea: parametrize m-full numbers by products $\prod_{r=0}^{m-1} y_r^{m+r}$ with $y_1, \ldots, y_{m-1}$ square-free and pairwise coprime.

Counting Campana points

Let $X = \mathbb{P}^{n-1}$, $\Delta = \cup_{i=0}^{n} D_i$ with $D_i = \{x_i = 0\}$, $0 \leq i \leq n-1$, and $D_n = \{c_0 x_0 + \ldots + c_{n-1} x_{n-1} = 0\}$, for $c_0, \ldots, c_{n-1} \in \mathbb{Z} \setminus \{0\}$.

Theorem (Browning-Yamagishi 2019)

Assume that $m_0, \ldots, m_n \geq 2$ *such that there exists* $j \in \{0, \ldots, n\}$ *with* $\sum_{\substack{0 \leq i \leq n \\ i \neq j}} \frac{1}{m_i(m_i+1)} \geq 1$. *Then*

$$\sharp\{x \in (\mathcal{X}, \Delta, \mathbf{m})(\mathbb{Z}) : H_{naiv}(x) \leq B\} \sim cB^{\sum_{i=0}^{n} \frac{1}{m_i} - 1}.$$

Remark

- Reduce to counting m_i-full solutions to $c_0 x_0 + \ldots + c_{n-1} x_{n-1} = x_n$.
- Note $K_{\mathbb{P}^{n-1}} + \sum_{i=0}^{n} \left(1 - \frac{1}{m_i}\right) D_i \sim \left(1 - \sum_{i=0}^{n} \frac{1}{m_i}\right) H$.

Manin-type conjecture for Campana points

Assume that $(\mathcal{X}, \mathcal{D}, \mathbf{m})(\mathcal{O}_{k,S})$ is Zariski-dense in X (and not thin), and let L be an ample line bundle on X.

Conjecture (Pieropan-Smeets-Tanimoto-Varilly-Alvarado 2019)

There exists a thin set Z such that

$$\sharp\{x \in (\mathcal{X}, \mathcal{D}, \mathbf{m})(\mathcal{O}_{k,S}) \setminus Z : H_L(x) \leq B\} \sim cB^a (\log B)^{b-1},$$

where c is a product of local densities,

$$a = \inf\{t \in \mathbb{R} : tL + K_X + \sum_{i=1}^{n} \left(1 - \frac{1}{m_i}\right) D_i \textit{ is effective } \},$$

and b is the codimension of the minimal face of the effective cone that contains $aL + K_X + \sum_{i=1}^{n} \left(1 - \frac{1}{m_i}\right) D_i$.

Theorem (Pieropan-S. 2020)

Let X be a split smooth proper toric variety over $\mathbb{Q}$ with boundary divisor $D = \cup_{i=1}^s D_i$. Let $m_i \geq 2$ for $1 \leq i \leq s$ and assume that $L = -\left(K_X + \sum_{i=1}^s \left(1 - \frac{1}{m_i}\right) D_i\right)$ is ample + a technical condition on L. Let $r = \mathrm{rank}\,\mathrm{Pic}(X)$. Then

$$\sharp\{x \in (\mathcal{X}, \mathcal{D}, \mathbf{m})(\mathbb{Z}) : H_L(x) \leq B\} \sim cB(\log B)^{r-1}.$$

where c is compatible with the conjectured constant.

Remark

The technical condition holds for e.g. projective space, products of projective spaces, blow-up of $\mathbb{P}^2$ in one point, and all smooth projective toric varieties with $\mathrm{rank}\,\mathrm{Pic}(X) \geq \dim X + 2$.

Campana points on toric varieties

Proof strategy

- *Use Cox rings/universal torsor method*
- *Generalized version of the Blomer-Brüdern hyperbola method*

Let $Y \to X$ be the universal torsor of X. Then

$$Y \subset \mathbb{A}^s_{\mathbb{Q}} = \mathrm{Spec}(\mathbb{Q}[y_{\rho_1}, \ldots, y_{\rho_s}])$$

is the open subvariety given by the complement of

$$\langle \prod_{\rho \notin \sigma} y_\rho = 0, \ \sigma \in \Sigma_{\max} \rangle.$$

Campana points on toric varieties

Let $\pi : \mathcal{Y} \to \mathcal{X}$ be an integral model of $Y \to X$. By work of Salberger one can reduce to the following problem

Goal

Asymptotically evaluate the counting function

$$N(B) = \frac{1}{2^r} \sharp \{ \mathbf{y} \in \mathcal{Y}(\mathbb{Z}) : y_i \neq 0, \ y_i \text{ is } m_i - \text{full}, 1 \leq i \leq s,$$

$$\max_{\sigma \in \Sigma_{\max}} \prod_{i=1}^{s} |y_i|^{\alpha_{\sigma,i}} \leq B \}.$$

An example

Let $X = \mathbb{P}^1 \times \mathbb{P}^1$.
The universal torsor $Y \subset \mathbb{A}^4 = \mathrm{Spec}(\mathbb{Q}[x_0, y_0, x_1, y_1])$ is given by the complement of the subvariety given by $\langle x_0 y_0, y_0 x_1, x_1 y_1, y_1 x_0 \rangle$, i.e.

$$Y = \mathbb{A}^4 \setminus (\{x_0 = x_1 = 0\} \cup \{y_0 = y_1 = 0\}) .$$

Take $m_1 = \ldots = m_4 = 2$ and $K_x = -\sum_{i=1}^4 D_i$, i.e. $L = \frac{1}{2} \sum_{i=1}^4 D_i$. Then the height function for integral points $(x_0, y_0, x_1, y_1) \in \mathcal{Y}(\mathbb{Z})$ is given by

$$\begin{aligned} H_L(\pi(x_0, y_0, x_1, y_1)) &= \max (|y_0 x_0|, |y_0 x_1|, |y_1 x_0|, |y_1 x_1|) \\ &= \max (|x_0|, |x_1|) \max(|y_0|, |y_1|). \end{aligned}$$

Expectation for the growth of $N(B)$

$$N(B) = \frac{1}{2^r} \sharp\{\mathbf{y} \in \mathcal{Y}(\mathbb{Z}) : y_i \neq 0,\ y_i \text{ is } m_i - \text{full}, 1 \leq i \leq s,$$

$$\max_{\sigma \in \Sigma_{\max}} \prod_{i=1}^{s} |y_i|^{\alpha_{\sigma,i}} \leq B\}.$$

Idea: consider the contribution of a dyadic box

$$B_i \leq y_i < 2B_i, \quad 1 \leq i \leq s.$$

Let $B_i = B^{t_i}$ for $t_i \geq 0$. Then

$$\sharp\{(y_1, \ldots, y_s) \in \mathbb{Z}^2 : y_i \sim B_i, m_i - \text{full},\ 1 \leq i \leq s\}$$

$$\sim C \prod_{i=1}^{s} B_i^{\frac{1}{m_i}} \sim CB^{\sum_{i=1}^{s} \frac{1}{m_i} t_i}.$$

Expectation for the growth of $N(B)$

Idea: consider the contribution of a dyadic box

$$B_i \leq y_i < 2B_i, \quad 1 \leq i \leq s.$$

For the height condition to hold

$$\max_{\sigma \in \Sigma_{\max}} \prod_{i=1}^{s} |y_i|^{\alpha_{\sigma,i}} \leq B$$

we consider boxes for which

$$\prod_{i=1}^{s} B_i^{\alpha_{\sigma,i}} \leq B, \quad \forall \sigma \in \Sigma_{\max}.$$

I.e. we consider $B_i = B^{t_i}$, with

$$\sum_{i=1}^{s} \alpha_{\sigma,i} t_i \leq 1, \quad \sigma \in \Sigma_{\max},$$

$$t_i \geq 0, \quad 1 \leq i \leq s.$$

Maximizing a linear function on a polytope

Let $\mathcal{P} \subset \mathbb{R}^s$ be the polytope given by

$$\sum_{i=1}^{s} \alpha_{\sigma,i} t_i \leq 1, \quad \sigma \in \Sigma_{\max},$$
$$t_i \geq 0, \quad 1 \leq i \leq s.$$

Goal

Maximize the function $\sum_{i=1}^{s} \frac{1}{m_i} t_i$ on the polytope $\mathcal{P}$.

- linear programming problem

Remark

Expected log exponent = dimension of the face of the polytope $\mathcal{P}$ where the max is attained.

The conjectured exponent

The conjectured exponent

$$a = \inf\{t \in \mathbb{R} : tL + K_X + \sum_{i=1}^{s} \left(1 - \frac{1}{m_i}\right) \text{ is effective}\},$$

leads to the following linear programming problem.

Minimize the linear function $\sum_{\sigma \in \Sigma_{\max}} \lambda_\sigma$ subject to the conditions

$$\lambda_\sigma \geq 0, \quad \sigma \in \Sigma_{\max}$$
$$\sum_{\sigma \in \Sigma_{\max}} \lambda_\sigma \alpha_{i,\sigma} \geq \frac{1}{m_i}, \quad 1 \leq i \leq s.$$

Duality in linear programming

Theorem (Strong duality in linear programming)

Let $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$, $\mathbf{b} \in \mathbb{R}^m$ *and* $\mathbf{c} \in \mathbb{R}^n$.

$\mathcal{P}$: *Maximize* $\mathbf{c}^t\mathbf{x}$ *subject to*

$$A\mathbf{x} \leq \mathbf{b}, \quad \mathbf{x} \geq 0.$$

$\mathcal{D}$: *Minimize* $\mathbf{b}^t\mathbf{y}$ *subject to*

$$A^t\mathbf{y} \geq \mathbf{c}, \quad \mathbf{y} \geq 0.$$

If $\mathcal{P}$ *has a finite optimal solution then so does* $\mathcal{D}$ *and these two are equal.*

A pair of dual linear programming problems

The exponent that we compute

Maximize the function $\sum_{i=1}^{s} \frac{1}{m_i} t_i$ *subject to*

$$\sum_{i=1}^{s} \alpha_{\sigma,i} t_i \leq 1, \quad \sigma \in \Sigma_{\max},$$

$$t_i \geq 0, \quad 1 \leq i \leq s.$$

The conjectured exponent

Minimize the linear function $\sum_{\sigma \in \Sigma_{\max}} \lambda_\sigma$ *subject to the conditions*

$$\lambda_\sigma \geq 0, \quad \sigma \in \Sigma_{\max}$$

$$\sum_{\sigma \in \Sigma_{\max}} \lambda_\sigma \alpha_{i,\sigma} \geq \frac{1}{m_i}, \quad 1 \leq i \leq s.$$

From box counting to hyperbola shapes

Let $f : \mathbb{N}^s \to \mathbb{R}_{\geq 0}$ be an arithmetic function. Assume that we understand sums of f over boxes. Let B be a large real parameter, $\mathcal{K}$ a finite index set and $\alpha_{i,k} \geq 0$ for $1 \leq i \leq s$ and $k \in \mathcal{K}$.

Goal

Find an asymptotic for

$$S^f := \sum_{\substack{\prod_{i=1}^s y_i^{\alpha_{i,k}} \leq B,\ \forall k \in \mathcal{K} \\ y_i \in \mathbb{N}, 1 \leq i \leq s}} f(\mathbf{y}).$$

Remark

We don't assume any multiplicative structure for f.

From box counting to hyperbola shapes

Property I

Assume that there are non-negative real constants $C_{f,M} \leq C_{f,E}$ and $\delta > 0$ and $\varpi_i > 0$, $1 \leq i \leq s$ such that for all $B_1, \ldots, B_s \in \mathbb{R}_{\geq 1}$ we have

$$\sum_{\substack{1 \leq y_i \leq B_i \\ 1 \leq i \leq s}} f(\mathbf{y}) = C_{f,M} \prod_{i=1}^{s} B_i^{\varpi_i} + O\left(C_{f,E} \prod_{i=1}^{s} B_i^{\varpi_i} \left(\min_{1 \leq i \leq s} B_i \right)^{-\delta} \right)$$

where the implied constant is independent of f.

From box counting to hyperbola shapes

Property II

Assume that there are positive real numbers D and ν such that the following holds. Let $\mathcal{I} \subsetneq \{1, \dots, s\}$ be a non-empty subset of indices and fix some $(y_i)_{i\in\mathcal{I}} \in \mathbb{N}^{|\mathcal{I}|}$. Write $\mathbf{y}_\mathcal{I}$ for the vector $(y_i)_{i\in\mathcal{I}}$ and $|\mathbf{y}_\mathcal{I}|$ for its maximums norm. Then there is a non-negative constant $C_{f,M,\mathcal{I}}(\mathbf{y}_\mathcal{I})$ such that for all $B_i \in \mathbb{R}_{\geq 1}$, $i \in \{1, \dots, s\} \setminus \mathcal{I}$ one has

$$\sum_{1\leq y_i \leq B_i,\ i\notin\mathcal{I}} f(\mathbf{y}) = C_{f,M,\mathcal{I}}(\mathbf{y}_\mathcal{I}) \prod_{i\notin\mathcal{I}} B_i^{\varpi_i} + O(C_{f,E}|\mathbf{y}_\mathcal{I}|^D \prod_{i\notin\mathcal{I}} B_i^{\varpi_i} (\min_{i\notin\mathcal{I}} B_i)^{-\delta}),$$

uniformly in $|\mathbf{y}_\mathcal{I}| \leq (\prod_{i\notin\mathcal{I}} B_i)^\nu$.

From box counting to hyperbola shapes

Recall

$$S^f := \sum_{\substack{\prod_{i=1}^s y_i^{\alpha_{i,k}} \leq B,\ \forall k \in \mathcal{K} \\ y_i \in \mathbb{N}, 1 \leq i \leq s}} f(\mathbf{y}).$$

Define the polyhedron $\mathcal{P} \subset \mathbb{R}^s$ by

$$\sum_{i=1}^{s} \alpha_{i,k} \varpi_i^{-1} t_i \leq 1, \quad k \in \mathcal{K} \tag{0.1}$$

and

$$t_i \geq 0, \quad 1 \leq i \leq s. \tag{0.2}$$

The linear function $\sum_{i=1}^s t_i$ takes its maximal value on a face of $\mathcal{P}$ which we call F. Write a for its maximal value.

From box counting to hyperbola shapes

Theorem (Pieropan-S. 2020)

Let $f : \mathbb{N}^s \to \mathbb{R}_{\geq 0}$ satisfy Property I and Property II. Assume that $\mathcal{P}$ is bounded and non-degenerate, that F is not contained in a coordinate hyperplane of $\mathbb{R}^s$ + a technical condition on $\mathcal{P}$. Let $k = \dim F$. Then we have*

$$S^f = (s-1-k)! C_{f,M} c_P (\log B)^k B^a + O\left(C_{f,E} (\log \log B)^s (\log B)^{k-1} B^a \right).$$

Remark

The case $|\mathcal{K}| = 1$, $\alpha_{i,k} = \alpha > 0$ for all $1 \leq i \leq s$ and $k = s-1$ is contained in the original work of Blomer and Brüdern on the hyperbola method.

Dimension growth conjecture

Let $n \geq 2$ and $V \subset \mathbb{P}^n_{\mathbb{Q}}$ an irreducible variety of degree d.
For a point $x = (x_0 : \ldots : x_n) \in \mathbb{P}^n(\mathbb{Q})$ with $x_0, \ldots x_n \in \mathbb{Z}$ and $\gcd(x_0, \ldots, x_n) = 1$ set

$$H(x) = \max_{0 \leq i \leq n} |x_i|.$$

Question

Define for $B \in \mathbb{R}_{\geq 0}$ the counting function

$$N_V(B) := \sharp\{x \in V(\mathbb{Q}) : H(x) \leq B\}.$$

What can we say about upper bounds for $N_V(B)$?

Trivial upper bound: $N_V(B) \ll_V B^{\dim V + 1}$.

Dimension growth conjecture

Example

Rational linear subspaces, e.g. $V \subset \mathbb{P}^n_{\mathbb{Q}}$ given by

$$a_0x_0 + \ldots + a_nx_n = 0,$$

with $a_i \in \mathbb{Z}$, $1 \leq i \leq n$ and $a_0 \neq 0$. Then

$$N_V(B) \sim B^n.$$

Example

Let $a_0, \ldots, a_n \in \mathbb{Z} \setminus \{0\}$ and $V \subset \mathbb{P}^n_{\mathbb{Q}}$ given by

$$a_0x_0^2 + \ldots + a_nx_n^2 = 0.$$

Then, for $n \geq 4$ we have

$$N_V(B) \sim cB^{n-1}.$$

Dimension growth conjecture

Example

Let $F_0(\mathbf{x}), F_1(\mathbf{x}) \in \mathbb{Z}[x_0, \ldots, x_n]$ and $V \subset \mathbb{P}^n_{\mathbb{Q}}$ given by

$$x_0 F_0(\mathbf{x}) - x_1 F_1(\mathbf{x}) = 0.$$

Then

$$\begin{aligned} N_V(B) &\geq \frac{1}{2}\sharp\{(x_2, \ldots, x_n) \in \mathbb{Z}^{n-1} : \max_{1 \leq i \leq n} |x_i| \leq B \\ &\quad \gcd(x_2, \ldots, x_n) = 1\} \\ &\gg B^{n-1} \end{aligned}$$

Dimension growth conjecture

Conjecture (Weak Dimension growth conjecture)

Let $V \subset \mathbb{P}^n_{\mathbb{Q}}$ *be an irreducible projective variety of degree* $\deg(V) \geq 2$. *Then*

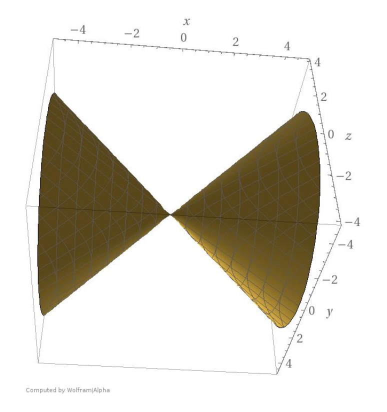
$$N_V(B) \ll_V B^{\dim V + \varepsilon},$$

for any $\varepsilon > 0$.

Remark

- Sharper version with uniformity in V and implicit constants only depending on n, d, ε.
- Solved in a series of articles by Browning, Heath-Brown and Salberger, using the determinant method.

Example



The surface $\mathcal{S}$ given by the quadratic form $Q(x,y,z) = x^2 - y^2 - \sqrt{2}z^2$.

Rational points close to manifolds

Let $\mathcal{M} \subset \mathbb{R}^n$ be a bounded submanifold with $\dim \mathcal{M} = m$.

A counting function

Let $Q > 1$ and $0 \leq \delta < \frac{1}{2}$. We define the counting function

$$N_{\mathcal{M}}(Q, \delta) := \sharp\{\frac{\mathbf{p}}{q} \in \mathbb{Q}^n : 1 \leq q \leq Q : \mathrm{dist}(\frac{\mathbf{p}}{q}, \mathcal{M}) \leq \frac{\delta}{q}\},$$

where $\mathbf{p} \in \mathbb{Z}^n$.

Trivial upper bound

$$N_{\mathcal{M}}(Q, \delta) \ll Q^{m+1}.$$

Example

For $\mathcal{M}$ a piece of a rational linear subspace we have $N_{\mathcal{M}}(Q,\delta) \gg Q^{m+1}$.

Question

Typical expectation for $N_{\mathcal{M}}(Q,\delta)$?

Set $k = n - m$. A heuristic argument via volume computation:

$$N_{\mathcal{M}}(Q,\delta) \sim \left(\frac{\delta}{Q}\right)^k QQ^n \sim \delta^k Q^{m+1}.$$

Rational points close to manifolds

Heuristic

$$N_{\mathcal{M}}(Q,\delta) \sim \delta^k Q^{m+1}.$$

For what size of δ is this realistic?

Example

a) Let $\mathcal{P} \subset \mathbb{R}^2$ be given by $y = x^2$ for $0 \leq x \leq 1$. Then

$$N_{\mathcal{P}}(Q,\delta) \geq N_{\mathcal{P}}(Q,0) \geq \sum_{q \leq \sqrt{Q}} q \gg Q.$$

This suggests $\delta \gg Q^{-1}$ for the 'volume term' to dominate.
b) Let $\mathcal{S} = S^{n-1} \subset \mathbb{R}^n$, $n \geq 3$ be given by $x_1^2 + \ldots + x_n^2 = 1$. Then

$$N_{\mathcal{S}}(Q,0) \sim c_n Q^{n-1}.$$

Rational points close to manifolds

Conjecture

If $\mathcal{M}$ is a bounded submanifold of $\mathbb{R}^m$ with boundary and proper curvature conditions, then for $\varepsilon > 0$

$$N_{\mathcal{M}}(Q, \delta) \ll \delta^k Q^{m+1} + Q^{m+\varepsilon},$$

for all $0 \leq \delta < 1/2$ and $Q \geq 1$. Moreover, if $\delta \geq Q^{-\frac{1}{k}+\varepsilon}$ for some $\varepsilon > 0$, then

$$N_{\mathcal{M}}(Q, \delta) \sim c\delta^k Q^{m+1},$$

for some constant c depending on $\mathcal{M}$, where $Q \to \infty$.

Example (The curvature condition is necessary)

For the Fermat curve $\mathcal{F}_d: \; x^d + y^d = 1$ one has

$$N_{\mathcal{F}_d}(Q, \delta) \gg \delta^{\frac{1}{d}} Q^{2-\frac{1}{d}}.$$

Rational points close to manifolds

Theorem (J.-J. Huang 2019)

The main conjecture holds for smooth compact hypersurfaces $\mathcal{S} \subset \mathbb{R}^n$ with Gaussian curvature bounded away from 0.

Previous work: Beresnevich, Dickinson, Velani, Huxley, Vaughan, Velani, Zorn...

Corollary

Analogue of dimension growth conjecture for smooth hypersurfaces with non-vanishing Gaussian curvature.

Question

What happens in higher codimension?

Rational points close to manifolds

Let $\mathcal{D} \subset \mathbb{R}^m$, $m \geq 2$ be a connected open set and $f_j \in C^{m+3}(\mathcal{D})$, $1 \leq j \leq k$. Let $w \in C_0^\infty(\mathbb{R}^m)$ be a non-negative weight function with $\mathrm{supp}(w) \subseteq \mathcal{D}$. Fix $\mathbf{x}_0 \in \mathcal{D}$. For $Q \in \mathbb{N}$ and $\delta \geq 0$, we define

$$\mathcal{N}_w(Q,\delta) = \sum_{\substack{\mathbf{a}\in\mathbb{Z}^m \\ q\leq Q \\ \|qf_1(\mathbf{a}/q)\|\leq\delta \\ \vdots \\ \|qf_k(\mathbf{a}/q)\|\leq\delta}} w\left(\frac{\mathbf{a}}{q}\right),$$

Assumption *

Given any $(t_1,\dots,t_k) \in \mathbb{R}^k \setminus \{\mathbf{0}\}$,

$$\det H_{t_1 f_1+\dots+t_k f_k}(\mathbf{x}_0) \neq 0$$

Rational points close to manifolds

Theorem (S.-Yamagishi 2020)

Assume that Assumption holds and that the support of w is sufficiently small. Then for any $\varepsilon > 0$*

$$\left|\mathcal{N}_w(Q,\delta) - \delta^R \sigma Q^{m+1}\right| \ll Q^{m+1-\frac{km}{m+2(k-1)}+\varepsilon},$$

where $\sigma > 0$.

Question

What to expect in the dimension growth conjecture in higher codimension?

Example

Examples for Assumption* are given by a construction of Suslin and results from determinantal representations.

Approximating the indicator function

For $0 < \delta \leq 1/2$ write

$$\chi_\delta(\theta) = \begin{cases} 1 & \text{if } \|\theta\| \leq \delta \\ 0 & \text{otherwise.} \end{cases} \tag{0.3}$$

Then

$$\mathcal{N}_w(Q,\delta) = \sum_{\substack{\mathbf{a}\in\mathbb{Z}^n \\ q\leq Q}} w\left(\frac{\mathbf{a}}{q}\right) \prod_{r=1}^{R} \chi_\delta\left(qf_r\left(\frac{\mathbf{a}}{q}\right)\right).$$

Question

Find a good approximation to $\chi_\delta(\theta)$ in the form of trigonometric polynomials of low degree.

Beurling polynomials

For $x \in \mathbb{R}$, consider the saw-tooth function

$$s(x) = \begin{cases} \{x\} - \frac{1}{2} & \text{if } x \notin \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases} \tag{0.4}$$

Let $J \in \mathbb{N}$ and write $F_J(x)$ for the Fejer-kernel

$$F_J(x) = \sum_{j=-J}^{J} \left(1 - \frac{|j|}{J}\right) e(jx) = \frac{1}{J} \left(\frac{\sin \pi Jx}{\sin \pi x}\right)^2$$

The Jth Beurling polynomial is given by

$$B_J(x) := \frac{1}{2(J+1)} F_{J+1}(x) + \frac{1}{J+1} \sum_{j=1}^{J} \left(\frac{j}{J+1} - \frac{1}{2}\right) F_{J+1}\left(x - \frac{j}{J+1}\right) + \frac{1}{2\pi(J+1)} \sin 2\pi(J+1)x - \frac{1}{2\pi} F_{J+1}(x) \sin 2\pi x$$

Beurling polynomials

$s(x)$: saw-tooth function
$B_J(x)$: Jth Beurling polynomial

Lemma (Vaaler's Lemma)

For $x \in \mathbb{R}$ and $J \in \mathbb{N}$ one has

$$s(x) \leq B_J(x).$$

Let $T(x)$ be a trigonometric polynomial of degree at most J with $s(x) \leq T(x)$ for all $x \in \mathbb{R}$. Then

$$\int_0^1 T(x)\,\mathrm{d}x \geq \frac{1}{2(J+1)}.$$

One obtains equality if and only if $T(x) = B_J(x)$.

Selberg polynomials

Recall

$$\chi_\delta(x) = \begin{cases} 1 & \text{if } \|x\| \leq \delta \\ 0 & \text{otherwise.} \end{cases}$$

Write

$$\chi_\delta(x) = 2\delta + s(x-\delta) + s(-\delta - x).$$

For $J \in \mathbb{N}$ set

$$S_J^+(x) = 2\delta + B_J(\delta - x) + B_J(x+\delta),$$

and

$$S_J^-(x) = 2\delta - B_J(\delta - x) - B_J(x+\delta).$$

Then

$$S_J^-(x) \leq \chi_\delta(x) \leq S_J^+(x), \quad x \in \mathbb{R}.$$

Further properties of Selberg polynomials

$S_J^{\pm}(x)$ is a trigonometric polynomial of degree $\leq J$. Write

$$S_J^{\pm}(x) = \sum_{|j|\leq J} \hat{S}_J^{\pm}(j) e(jx),$$

Observe

$$\hat{S}_J^{\pm}(0) = \int_0^1 S_J^{\pm}(x)\,\mathrm{d}x = 2\delta \pm \frac{1}{J+1}.$$

For $0 \leq |j| \leq J$ we have

$$|\hat{S}_J^{\pm}(j)| \leq b_j := \frac{1}{J+1} + \min\left(2\delta, \frac{1}{\pi|j|}\right)$$

Back to our counting problem

Recall

$$\mathcal{N}_w(Q,\delta) = \sum_{\substack{\mathbf{a}\in\mathbb{Z}^n \\ q\leq Q}} w\left(\frac{\mathbf{a}}{q}\right) \prod_{r=1}^{R} \chi_\delta\left(qf_r\left(\frac{\mathbf{a}}{q}\right)\right).$$

Using the Selberg polynomials obtain

$$\begin{aligned}\mathcal{N}_w(Q,\delta) &\leq \sum_{\substack{\mathbf{a}\in\mathbb{Z}^n \\ q\leq Q}} w\left(\frac{\mathbf{a}}{q}\right) \prod_{r=1}^{R} \left(\sum_{j_r=-J}^{J} \hat{S}_J^+(j_r) e\left(j_r q f_r\left(\frac{\mathbf{a}}{q}\right)\right)\right) \\ &= \sum_{\substack{0\leq|j_1|\leq J \\ \vdots \\ 0\leq|j_R|\leq J}} \left(\prod_{r=1}^{R} \hat{S}_J^+(j_r)\right) \sum_{\substack{\mathbf{a}\in\mathbb{Z}^n \\ q\leq Q}} w\left(\frac{\mathbf{a}}{q}\right) e\left(\sum_{r=1}^{R} j_r q f_r\left(\frac{\mathbf{a}}{q}\right)\right).\end{aligned}$$

The expected main term

$$\mathcal{N}_w(Q,\delta) \leq \sum_{\substack{0\leq |j_1|\leq J \\ \vdots \\ 0\leq |j_R|\leq J}} \left(\prod_{r=1}^{R} \hat{S}_J^+(j_r)\right) \sum_{\substack{\mathbf{a}\in\mathbb{Z}^n \\ q\leq Q}} w\left(\frac{\mathbf{a}}{q}\right) e\left(\sum_{r=1}^{R} j_r q f_r\left(\frac{\mathbf{a}}{q}\right)\right).$$

Contribution from $j_1 = \ldots = j_R = 0$ leads to

$$\left(2\delta + \frac{1}{J+1}\right)^R N_0 = (2\delta)^R N_0 + O\left(\delta^{R-1}\frac{1}{J}Q^{n+1} + \frac{1}{J^R}Q^{n+1}\right),$$

with

$$N_0 := \sum_{\substack{\mathbf{a}\in\mathbb{Z}^n \\ q\leq Q}} w\left(\frac{\mathbf{a}}{q}\right).$$

Separating the error term

Using the upper and lower bounds via Selberg's magic functions we obtain

$$\left|\mathcal{N}_w(Q,\delta)-(2\delta)^R N_0\right| \ll \delta^{R-1}\frac{1}{J}Q^{n+1}+\frac{1}{J^R}Q^{n+1}$$

$$+\sum_{\substack{0\le|j_1|\le J\\ \vdots\\ 0\le|j_R|\le J\\ \mathbf{j}\ne\mathbf{0}}}\left(\prod_{r=1}^R b_{j_r}\right)\left|\sum_{\substack{\mathbf{a}\in\mathbb{Z}^n\\ q\le Q}} w\left(\frac{\mathbf{a}}{q}\right)e\left(\sum_{r=1}^R j_r q f_r\left(\frac{\mathbf{a}}{q}\right)\right)\right|.$$

Next: we need upper bounds for the sums

$$\sum_{\substack{\mathbf{a}\in\mathbb{Z}^n\\ q\le Q}} w\left(\frac{\mathbf{a}}{q}\right)e\left(\sum_{r=1}^R j_r q f_r\left(\frac{\mathbf{a}}{q}\right)\right).$$

Poisson summation

Define

$$I(q;\mathbf{j};\mathbf{k}) := \int_{\mathbb{R}^n} w(\mathbf{x})\, e\left(\sum_{r=1}^{R} qj_r f_r(\mathbf{x}) - q\mathbf{k}\cdot\mathbf{x}\right) d\mathbf{x}.$$

By Poisson summation

$$\begin{aligned}
&\sum_{\mathbf{a}\in\mathbb{Z}^n} w\left(\frac{\mathbf{a}}{q}\right) e\left(\sum_{r=1}^{R} j_r q f_r\left(\frac{\mathbf{a}}{q}\right)\right)\\
= \quad &\sum_{\mathbf{k}\in\mathbb{Z}^n} \int_{\mathbb{R}^n} w\left(\frac{\mathbf{z}}{q}\right) e\left(\sum_{r=1}^{R} j_r q f_r\left(\frac{\mathbf{z}}{q}\right) - \mathbf{k}\cdot\mathbf{z}\right) d\mathbf{z}\\
= \quad & q^n \sum_{\mathbf{k}\in\mathbb{Z}^n} I(q;\mathbf{j};\mathbf{k}).
\end{aligned}$$

Bounding oscillatory integrals

We consider the case $j_1 > 0$ and

$$0 \leq j_r \leq j_1, \quad 2 \leq r \leq R.$$

The other cases follow by symmetry. Define

$$F_{\mathbf{j}} = f_1 + \frac{j_2}{j_1} f_2 + \cdots + \frac{j_R}{j_1} f_R.$$

and write

$$\begin{aligned} I(q;\mathbf{j};\mathbf{k}) &= \int_{\mathbb{R}^n} w(\mathbf{x})\, e\left(\sum_{r=1}^{R} q j_r f_r(\mathbf{x}) - q\mathbf{k}\cdot\mathbf{x} \right) d\mathbf{x}. \\ &= \int_{\mathbb{R}^n} w(\mathbf{x})\, e\left(q j_1 \left(F_{\mathbf{j}}(\mathbf{x}) - \frac{\mathbf{k}\cdot\mathbf{x}}{j_1} \right) \right) d\mathbf{x}. \end{aligned}$$

Bounding oscillatory integrals

We need bounds for integrals of the type

$$I(q;\mathbf{j};\mathbf{k}) = \int_{\mathbb{R}^n} w(\mathbf{x})\, e\left(qj_1\left(F_{\mathbf{j}}(\mathbf{x}) - \frac{\mathbf{k}\cdot\mathbf{x}}{j_1}\right)\right) d\mathbf{x}.$$

Use

- Rapid decay bounds for **k** 'large'
- Stationary phase for a non-degenerate critical point
- non-trivial summation over the main terms obtained from stationary phase

Thanks!

Thank you for listening!