Recent Progress on Instanton Partition Functions

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Based on the following papers:

- [arXiv:1205.4741] with Amihay Hanany and Shlomo Razamat
- [arXiv:1111.5624] with Christoph Keller, Jaewon Song and Yuji Tachikawa
- [arXiv:1005.3026] with Sergio Benvenuti and Amihay Hanany

(Please see also

- [arXiv:1205.4722] by Christoph Keller and Jaewon Song

for a closely related work.)
Part I: Introduction
Consider instantons in a pure Yang-Mills theory with gauge group $G$

**The moduli space of $k$ $G$-instantons on $\mathbb{C}^2$:**

The space of solutions to the self-dual Yang-Mills equations, modulo gauge transformations, in a given winding sector $k$ and gauge group $G$

For a classical gauge group $G$, $SU(N)$, $Sp(N)$ or $SO(N)$, such instanton solutions can be constructed using linear algebra!

Such a simple method of constructions is known as the ADHM construction

(Atiyah, Drinfeld, Hitchin, Manin '78)
The ADHM construction from string theory
(Douglas, Moore, Witten '94-'96)

- Can be realised on a system of D3-branes and D7-branes (possibly with an O-plane).
- **D3’s on top of D7’s:** D3-branes ≡ instantons in the w.v. of D7-branes.
- The w.v. theory of the D3-branes has 8 SUSYs (4d $\mathcal{N} = 2$). Can be represented by the ADHM quivers:

  ![ADHM Quiver Diagram](image)

  - **D3-branes on top of D7-branes** $\leftrightarrow$ **Higgs branch** of the ADHM quiver.
  - Identified with the moduli space of $k$ $SU(N)$, $SO(N)$ or $Sp(N)$ instantons on $\mathbb{C}^2$. 

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Comments on the ADHM construction

- The $F$ and $D$ terms give rise to the moment map equations for hyperKähler quotients of the instanton moduli spaces.
- For **classical** gauge groups, the moment map equations follow from the Langrangian of the corresponding gauge theory.
- For **exceptional** gauge groups, no ADHM construction is known!
  - Although brane constructions are known, they do not admit a perturbative description and hence there is no Lagrangian.
Comments on the ADHM construction (continued)

- **For $E$-type groups:** One way is to look at the Higgs branch of theories on M5-branes wrapping Riemann spheres with punctures (Gaiotto '09) and with appropriate boundary conditions. (Benini-Benvenuti-Tachikawa '09, Gaiotto-Razamat '12)
  - For 1 instanton, this construction gives rise to theories with $E$-type global sym. proposed by [Minahan-Nemeschansky '96].
  - For $F_4$ and $G_2$, there is no known construction of this type so far!

**Even though the ADHM construction is not available, it is still possible to compute instanton partitions function exactly and explicitly!**
Symmetry of an instanton moduli space

The moduli space of $k \ G$ instantons on $\mathbb{C}^2$

- is a singular hyperKähler cone
- possesses a symmetry
  
  \[ U(2)_{\mathbb{C}^2} \times G \]

  where $U(2)_{\mathbb{C}^2}$ is a symmetry of $\mathbb{C}^2$, the overall position of the instantons

Symmetry

- The $U(1)_{\mathbb{C}^2}$ subgroup of $U(2)_{\mathbb{C}^2}$ can be identified with the Cartan of the $R$ symmetry $SU(2)_R$
- The $SU(2)_{\mathbb{C}^2}$ subgroup of $U(2)_{\mathbb{C}^2}$ rotates the two chiral multiplets in the $\{\text{adjoint, A, S}\}$ hypermultiplet of the ADHM quiver for $\{SU(N), SO(N), Sp(N)\}$ instantons
Part II: Hilbert series for instanton moduli spaces
Hilbert series for instanton moduli spaces

- In order to study the instanton moduli space, we compute a partition function that counts holomorphic functions on the space wrt. the global $U(1)_{\mathbb{C}^2}$ charge.

- Such a partition function is known as the Hilbert series (HS) of instanton moduli space. It takes the form

$$g(t; x; y_1, \ldots, y_r) = \sum_{k=0}^{\infty} R^{(k)}_{SU(2)_{\mathbb{C}^2}}(x) \ r^{(k)}_G (y_1, \ldots, y_r) \ t^k$$

- The variable (fugacity) $t$ keeps track of the charge $k$ under $U(1)_{\mathbb{C}^2}$
- $R^{(k)}_{SU(2)_{\mathbb{C}^2}}(x)$ is the character of the rep $R^{(k)}$ of $SU(2)_{\mathbb{C}^2}$
- $r^{(k)}_G (y_1, \ldots, y_r)$, with $r = rk \ G$, is the character of the rep $r^{(k)}$ of $G$
Hilbert series for instanton moduli spaces (continued)

\[ g(t; x; y_1, \ldots, y_r) = \sum_{k=0}^{\infty} R_{SU(2)_C^2}^{(k)}(x) \cdot r_{G}^{(k)}(y_1, \ldots, y_r) \cdot t^k \]

- **Interpretation:** Holomorphic functions carrying $U(1)_C^2$ charge $k$ transform under the rep $[R_{SU(2)_C^2}^{(k)}; r_{G}^{(k)}]$ of $SU(2)_C^2 \times G$.

- The number of such functions are $\dim R_{SU(2)_C^2}^{(k)} \times \dim r_{G}^{(k)}$.

- **Dimension of the moduli space.** Setting $x = y_1 = \ldots = y_r = 1$, we have

\[ g(t; x = 1; \{y_i = 1\}) = \sum_{k=0}^{\infty} \dim R_{SU(2)_C^2}^{(k)} \times \dim r_{G}^{(k)} \cdot t^k \]

\[ \sim \frac{1}{(1-t)^{2kh_G^\vee}}, \quad t \to 1. \]

The cplx dim of the moduli space of $k \ G$ instantons is $2kh_G^\vee$. 
Example 1: Hilbert series of \( \mathbb{C}^2 \)

- Holomorphic coordinates are \( z_1, z_2 \)
- \( U(2) = U(1) \times SU(2) \). Both \( z_1 \) and \( z_2 \) carry charge +1 under \( U(1) \) and transform under the fund. rep. \([1]\) of \( SU(2) \). Note the character \([1]_{SU(2)}(x) = x + x^{-1}\)
- Assign the fugacities \( t \) of \( U(1) \) and \( x \) of \( SU(2) \) to \( z_{1,2} \):
  \[
  z_1 \to tx, \quad z_2 \to tx^{-1}
  \]
- Any holomorphic function on \( \mathbb{C}^2 \) takes the form \( z_1^{k_1} z_2^{k_2} \), with \( k_1, k_2 \geq 0 \).
- The Hilbert series is
  \[
  g(t, x) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} t^{k_1+k_2} x^{k_1-k_2} = \sum_{n=0}^{\infty} [n]_{SU(2)}(x) t^n
  \]
- This can also be written as \( g(t, x) = \text{PE} \left[ t \ [1]_{SU(2)}(x) \right] \), where
  \[
  \text{PE}[f(t_1, \ldots, t_n)] = \exp \left( \sum_{k=1}^{\infty} \frac{1}{k} f(t_1^k, \ldots, t_n^k) \right).
  \]
Example 2: Hilbert series of $\mathbb{C}^2/\mathbb{Z}_2$

- Holomorphic coordinates are $z_1, z_2$, with $\mathbb{Z}_2$ action $(z_1, z_2) \mapsto (-z_1, -z_2)$.

- Focus on the $\mathbb{Z}_2$ invariant quantities:
  
  - 3 generators: $G_1 = z_1^2$, $G_2 = z_1 z_2$, $G_3 = z_2^2$
  
  - 1 relation: $G_1 G_3 - G_2^2 = 0$ (defining equation for $\mathbb{C}^2/\mathbb{Z}_2$)

- The symmetry of $\mathbb{C}^2/\mathbb{Z}_2$ is $U(2) = U(1) \times SU(2)$.

- $G_1, G_2, G_3$ transform as a triplet $[2]$ under the isometry $SU(2)$ and each carries charge $+2$ under $U(1)$.

- **Hilbert series:** Let $t$ be a fugacity for $U(1)$, and $x$ be an $SU(2)$ fugacity

\[
g(t, x) = (1 - t^4) \mathrm{PE} \left([2]_{SU(2)}(x)t^2\right)
\]

\[
= \sum_{n=0}^{\infty} [2n]_{SU(2)}(x) t^{2n}
\]

- Observe that the symmetry $U(2)$ is manifest in this expression.
Example 3: One $SU(N)$ instanton on $\mathbb{C}^2$

- Translate the ADHM quiver from $\mathcal{N} = 2$ language to $\mathcal{N} = 1$ language
- In $\mathcal{N} = 1$ notation, the quiver looks like

\begin{align*}
\text{Superpotential } W &= \tilde{Q}^i \varphi Q_i \quad \longrightarrow \quad F \text{ terms: } \tilde{Q}^i Q_i = 0 \\
\text{Global symmetry: } SU(2)_{\mathbb{C}^2} \times U(1)_{\mathbb{C}^2} \times SU(N) \\
\phi_1, \phi_2 \text{ transform as a doublet under } SU(2)_{\mathbb{C}^2}
\end{align*}

\[
\begin{multline*}
\left[ [1]_{SU(2)_{\mathbb{C}^2}} (x) t + [1, 0, \ldots, 0]_{SU(N)} (y) tz^{-1} + [0, \ldots, 0, 1]_{SU(N)} (y) tz \right] \\
\left[ \tilde{Q} \right] \\
\left[ Q \right]
\end{multline*}

\[
\begin{align*}
g_1, SU(N) (t; x, y_1, \ldots, y_{n-1}) &= \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{2\pi iz} \\
\text{D-terms and modding out by } U(1) \\
\text{F-terms} \\
\end{align*}
\]
Example 3: One $SU(N)$ instanton on $\mathbb{C}^2$ (continued)

- The result of the integration gives the HS:

\[
g_{1, SU(N)}(t; x, y) = \frac{1}{(1 - tx)(1 - tx^{-1})} \tilde{g}_{1, SU(N)}(t, y)
\]

Note: $\tilde{g}_{1, SU(N)}(t, y) = \sum_{n=0}^{\infty} \left[ n, 0, \ldots, 0, n \right]_{SU(N)}(y) t^{2n}$

- $\tilde{g}_{1, SU(N)}(t, y)$ is said to be the HS of the reduced instanton moduli space

  (i.e. excluding the $\mathbb{C}^2$ component corresponding to the position of the instanton)

- An example of holomorphic function at $t^2$: This can be written as $M_{ij}^i = \tilde{Q}_a^i Q_j^a$,

  with a constraint $M_{ii}^i = 0$ due to the $F$ term. Hence $M_{ij}^i$ transform under the

  adjoint rep $[1, 0, \ldots, 0, 1]$ of $SU(N)$. 
Example 4: One $G$ instanton on $\mathbb{C}^2$ (with any simple group $G$)

- Repeat this computation for $G = SO(N), Sp(N)$ using the ADHM quivers. The HS of the reduced instanton moduli space take the same form as before:

\[
\tilde{g}_{1,G}(t; y) = \sum_{n=0}^{\infty} n(\text{highest weight of Adj}) t^{2n}
\]

- **Claim:** This holds for any simple group $G$, i.e. the $ABCDEFG$ type groups!
  (Benvenuti, Hanany, NM '08)

- The symmetry $G$ is manifest in this expression.

- **This claim can be mathematically proven.** The proof relies on a special property of the moduli space of one instanton:
  - It is the orbit of the highest weight vector in the Lie algebra of $G_{\mathbb{C}}$ (Kronheimer '90).
  - The space of holomorphic functions on such a space is known (e.g. Vinberg-Popov '72 and Garfinkle '73); from which the HS can be deduced.
  (See also Gaiotto, Neitzke, Tachikawa '08)
Example 5: Two $Sp(N)$ instantons on $\mathbb{C}^2$

The Hilbert series can be computed from the ADHM quiver and can be written in terms of $U(2) \times Sp(N)$ character expansion as

$$\tilde{g}_{2,Sp(N)}(t, x, y_1, \ldots, y_N) = f(0; 0, \ldots, 0) + f(0; 0, 1, 0, \ldots, 0)t^4$$
$$+ [f(1; 2, 0, 0, \ldots, 0) + f(1; 2, 1, 0, \ldots, 0)] t^5,$$

where the function $f$ is defined as

$$f(a; b_1, b_2, \ldots, b_N) = \frac{1}{1 - t^4} \sum_{m_2=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} t^{2m_2+2n_2+3n_3+4n_4} \times$$
$$[2m_2 + n_3 + a; 2n_2 + 2n_3 + b_1, 2n_4 + b_2, b_3, \ldots, b_N].$$

Observe the lattice spanned by certain highest weight vectors associated with $SU(2) \times SU(N)$ reps.
Example 6: Two $SO(8)$ instantons on $\mathbb{C}^2$

The Hilbert series can be computed from the ADHM quiver and can be written in terms of $U(2) \times SO(8)$ character expansion as

$$
\bar{g}_{2,SO(8)}(t, x, y_1, \ldots, y_4) = \sum_{k_8=0}^{\infty} \left\{ f(0; 0, 2k_8, 0, 0)t^{8k_8} + f(1; 0, 2k_8 + 1, 0, 0)t^{8k_8+5} + f(1; 1, 2k_8, 1, 1)t^{8k_8+7}
+ f(0; 1, 2k_8 + 1, 1, 1)t^{8k_8+10} \right\} + \sum_{k_5=0}^{\infty} \left\{ f(k_5 + 1; k_5 + 1, 0, k_5 + 1, k_5 + 1)t^{5k_5+5}
+ f(k_5 + 2; k_5 + 2, 0, k_5 + 2, k_5 + 2)t^{5k_5+12} \right\}.
$$

where the function $f$ is defined as

$$
f(a; b_1, \ldots, b_r) = \frac{1}{1 - t^4} \sum_{m_2=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} \sum_{m_4=0}^{\infty} \sum_{l_4=0}^{\infty} \sum_{n_6=0}^{\infty} t^{2m_2 + 2n_2 + 3n_3 + 4n_4 + 4m_4 + 4l_4 + 6n_6} \times [2m_2 + n_3 + a; 2n_4 + n_6 + b_1, n_2 + n_3 + b_2, n_6 + 2l_4 + b_3, 2m_4 + n_6 + b_4],
$$

\[\text{Noppadol Mekareeya (MPI)}\]
Detailed study of the lattice structures allows for general and explicit expressions for the Hilbert series for 1 and 2 instantons in any simple group.

- **Universal lattice:** denoted by $n$'s, $m$'s and $l$'s.
- **Non-universal lattices:** denoted by $k$'s.
- **Shifts:** denoted by $a$ and $b$'s.

The HS for two instantons in any simple group on $\mathbb{C}^2$ can be found at [arXiv:1205.4741].
Part III: Hilbert series as instanton partition functions
Hilbert series and Nekrasov’s partition functions

- **HS:** \( g(t; x; y_1, \ldots, y_r) = \sum_{m=0}^{\infty} R_{SU(2)_{C^2}}^{(m)}(x) r_G^{(m)}(y_1, \ldots, y_r) \ t^m. \)

- **Nekrasov’s partition function:** Nekrasov’s partition function for \( k \ G \) instantons can be obtained from the HS (Bruzzo-Fucito-Morales-Tanzini ’02, Nakajima-Yoshioka ’03):

  \[
  Z_k(\epsilon_1, \epsilon_2, a) = \lim_{{\beta \to 0}} \beta^{2k h^\vee} G g(e^{-\frac{1}{2} \beta (\epsilon_1+\epsilon_2)}; e^{-\frac{1}{2} \beta (\epsilon_1-\epsilon_2)}; e^{-\beta a_1}, \ldots, e^{-\beta a_r}).
  \]

- **One \( G \) instanton:** Nekrasov’s partition function is

  \[
  Z_{k=1}(\epsilon_1, \epsilon_2, a) = -\frac{1}{\epsilon_1 \epsilon_2} \sum_{\gamma \in \Delta_l} \frac{1}{(\epsilon_1 + \epsilon_2 + \gamma \cdot a)(\gamma \cdot a)} \prod_{\alpha \in \Delta} \gamma^\vee = 2\gamma \gamma^\vee.
  \]

  where \( \Delta \) and \( \Delta_l \) are the sets of the roots and the long roots, and \( \gamma^\vee = \frac{2\gamma}{\gamma \cdot \gamma}. \)

  (Keller, NM, Song, Tachikawa ’11; thanks to A. Bondal and S. Carnahan)

- **AGT relation:** This is equal to the norm of a certain coherent state of the W-algebra. For non-simply laced \( G \), the coherent state is in the twisted sector of a simply-laced W-algebra.

  (Keller, NM, Song, Tachikawa ’11)
Hilbert series and superconformal indices

- The superconformal index (SCI) is a partition function of a SCFT on $S^3 \times S^1$ with periodic BCs for fermions around $S^1$.
  (e.g. Römelsberger '05,'07, Kinney-Maldacena-Minwalla-Raju '05, Dolan-Osborn '09, Spiridonov-Vatanov '08-'09, Gadde, Pomoni, Rastelli, Razamat, Yan '10-onwards)

- For a 4d $\mathcal{N} = 2$ SCFT, the SCI can be thought as a trace over the states of the theory on $S^3$. It gets contribution from all states annihilated by one of the supercharges. (Any choice of supercharges yields the same result.)

- One can assign to such states certain combinations of global charges that commute with this supercharge. There are fugacities associated with those global charges.

- Some of such fugacities can be set to zero and the SCI simplifies tremendously. A special case of our interests is known as the Hall-Littlewood (HL) index.
  (Gadde, Rastelli, Razamat, Yan '11)
Hilbert series and Hall-Littlewood indices

- For a theory with Lagrangian, the HL index gets contributions only from one of the complex scalars in the h-plet and one of the fermions in the $\mathcal{N} = 2$ v-plet.

- For a 4d $\mathcal{N} = 2$ gauge theory arise from M5-branes wrapping a Riemann sphere (i.e. genus 0) with punctures, it is conjectured that the HL index is equal to the HS. (Gadde, Rastelli, Razamat, Yan '11)

- $E_{6,7,8}$ instantons can be realised in this way! For $F_4$ and $G_2$, there is no known construction of this type.
Instantons in $E$-type groups from M5 branes on Riemann spheres

- One, two and three $E_6$ instantons

- One and two $E_7$ instantons

- One and two $E_8$ instantons

The HL indices for these theories can be computed **exactly** in terms of HL polynomials (Gaiotto-Razamat '12). But the $E_{6,7,8}$ symmetry are not manifest.
Example 7: Two $E_6$ instantons

- The HS is equal to the HL index and can be rewritten in terms of $U(2) \times E_6$ character expansion as

$$\tilde{g}_{2,E_6}(t, x, y) = \sum_{k_8=0}^{\infty} \left\{ f(0; 0, 2k_8, 0, 0, 0) t^{8k_8} + f(1; 0, 2k_8 + 1, 0, 0, 0) t^{8k_8+5} + f(1; 0, 2k_8, 1, 0, 0) t^{8k_8+7} + f(0; 0, 2k_8 + 1, 0, 1, 0) t^{8k_8+10} \right\} + \sum_{k_5=0}^{\infty} \left\{ f(k_5 + 1; 0, 0, 0, k_5 + 1, 0, 0) t^{5k_5+5} + f(k_5 + 2; 0, 0, 0, k_5 + 2, 0, 0) t^{5k_5+12} \right\},$$

where the function $f$ is defined as

$$f(a; b_1, \ldots, b_6) = \frac{1}{1 - t^4} \sum_{m_2=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} \sum_{n_6=0}^{\infty} \sum_{n_8=0}^{\infty} \sum_{n_{12}=0}^{\infty} t^{2m_2+2n_2+3n_3+4n_4+6n_6+8n_8+12n_{12}}$$

$$[2m_2 + n_3 + a; n_4 + b_1, n_2 + n_3 + b_2, n_8 + b_3, n_6 + 2n_{12} + b_4, n_8 + b_5, n_4 + b_6].$$

This form of HS provides a way to generalise this to $E_7$ and $E_8$. 
Example 8: Two $E_7$ instantons

The Hilbert series can be rewritten in terms of $U(2) \times E_7$ character expansion as

$$\tilde{g}_{2,E_7}(t, x, y_1, \ldots, y_7) = \sum_{k_8=0}^{\infty} \left\{ f(0; 2k_8, 0, 0, 0, 0, 0) t^{8k_8} + f(1; 2k_8 + 1, 0, 0, 0, 0, 0) t^{8k_8+5} + f(1; 2k_8, 0, 1, 0, 0, 0, 0) t^{8k_8+7} + f(0; 2k_8 + 1, 0, 1, 0, 0, 0) t^{8k_8+10} \right\} +$$

$$\sum_{k_5=0}^{\infty} \left\{ f(k_5 + 1; 0, 0, k_5 + 1, 0, 0, 0, 0) t^{5k_5+5} + f(k_5 + 2; 0, 0, k_5 + 2, 0, 0, 0, 0) t^{5k_5+12} \right\},$$

where the function $f$ is defined as

$$f(a; b_1, \ldots, b_7) = \frac{1}{1 - t^4} \sum_{m_2=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} \sum_{n_6=0}^{\infty} \sum_{n_8=0}^{\infty} \sum_{n_{12}=0}^{\infty} t^{2m_2+2n_2+3n_3+4n_4+6n_6+8n_8+12n_{12}} \times$$

$$[2n_2 + n_3 + a; n_2 + n_3 + b_1, b_2, n_6 + 2n_{12} + b_3, n_8 + b_4, b_5, n_4 + b_6, b_7].$$
Example 9: Two $E_8$ instantons

The Hilbert series can be rewritten in terms of $U(2) \times E_8$ character expansion as

$$\tilde{g}_{2, E_8}(t, x, y_1, \ldots, y_8)$$

$$= \sum_{k_8=0}^{\infty} \left\{ f(0; 0, 0, 0, 0, 0, 0, 2k_8) t^{8k_8} + f(1; 0, 0, 0, 0, 0, 0, 2k_8 + 1) t^{8k_8+5} + f(1; 0, 0, 0, 0, 0, 1, 2k_8 + 1) t^{8k_8+7} + f(0; 0, 0, 0, 0, 0, 1, 2k_8 + 1) t^{8k_8+10} \right\} +$$

$$\sum_{k_5=0}^{\infty} \left\{ f(k_5 + 1; 0, 0, 0, 0, 0, 0, k_5 + 1, 0) t^{5k_5+5} + f(k_5 + 2; 0, 0, 0, 0, 0, k_5 + 2, 0) t^{5k_5+12} \right\},$$

where the function $f$ is defined as

$$f(a; b_1, \ldots, b_8) = \frac{1}{1 - t^4} \sum_{m_2=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} \sum_{n_6=0}^{\infty} \sum_{n_8=0}^{\infty} \sum_{n_{12}=0}^{\infty} t^{2m_2 + 2n_2 + 3n_3 + 4n_4 + 6n_6 + 8n_8 + 12n_{12}} \times [2n_2 + n_3 + a; n_4 + b_1, b_2, b_3, b_4, b_5, n_8 + b_6, n_6 + 2n_{12} + b_7, n_2 + n_3 + b_8].$$
Example 10: Two $G_2$ instantons

- The Dynkin diagram $G_2$ can be obtained by folding the Dynkin diagram of $SO(8)$ via a $\mathbb{Z}_3$ outer-automorphism.

- The Hilbert series can be rewritten in terms of $U(2) \times G_2$ character expansion as

$$
\tilde{g}_{2,G_2}(t, x, y_1, y_2) = \sum_{k_8=0}^{\infty} \left\{ f(0; 0, 2k_8)t^{8k_8} + f(1; 0, 2k_8 + 1)t^{8k_8+5} + f(1; 3, 2k_8)t^{8k_8+7} + f(0; 3, 2k_8 + 1)t^{8k_8+10} \right\} + 
\sum_{k_5=0}^{\infty} \left\{ f(k_5 + 1; 3k_5 + 3, 0)t^{5k_5+5} + f(k_5 + 2; 3k_5 + 6, 0)t^{5k_5+12} \right\},
$$

where

$$f(a; b_1, b_2) = \frac{1}{1 - t^4} \sum_{m_2=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} \sum_{n_6=0}^{\infty} t^{2m_2+2n_2+3n_3+4n_4+6n_6} \times [2m_2 + n_3 + a; 2n_4 + 3n_6 + b_1, n_2 + n_3 + b_2]$$
Example 11: Two $F_4$ instantons

- The Dynkin diagram $F_4$ can be obtained by folding the Dynkin diagram of $E_6$ via a $\mathbb{Z}_2$ outer-automorphism.

\[
\begin{array}{c}
5 \\
\downarrow \\
4 \\
\uparrow \\
3 \\
\downarrow \\
2 \\
\end{array} \rightarrow 
\begin{array}{c}
1 \\
\end{array}
\]

- The Hilbert series can be rewritten in terms of $U(2) \times F_4$ character expansion as

\[
\tilde{g}_{2,F_4}(t, x, y) = \sum_{k_8=0}^{\infty} \left\{ f(0; 2k_8, 0, 0, 0)t^{8k_8} + f(1; 2k_8 + 1, 0, 0, 0)t^{8k_8+5} + f(1; 2k_8, 1, 0, 0)t^{8k_8+7} \\
+ f(0; 2k_8 + 1, 1, 0, 0)t^{8k_8+10} \right\} + \sum_{k_5=0}^{\infty} \left\{ f(k_5 + 1; 0, k_5 + 1, 0, 0)t^{5k_5+5} \\
+ f(k_5 + 2; 0, k_5 + 2, 0, 0)t^{5k_5+12} \right\}.
\]

where

\[
f(a; b_1, \ldots, b_4) = \frac{1}{1 - t^4} \sum_{m_2=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} \sum_{n_6=0}^{\infty} \sum_{n_8=0}^{\infty} \sum_{n_{12}=0}^{\infty} t^{2m_2+2n_2+3n_3+4n_4+6n_6+8n_8+12n_{12}} \\
\left[ 2m_2 + n_3 + a; n_2 + n_3 + b_1, n_6 + 2n_{12} + b_2, 2n_8 + b_3, 2n_4 + b_4 \right] \]

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Instanton Partition Functions

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Conclusions

- Hilbert series are computed explicitly for one and two instantons in any simple group, regardless of the existence of ADHM constructions.

- Great advantages of writing a Hilbert series in terms of a character expansion:
  - The symmetry $U(2) \times G$ of the instanton moduli space is manifest.
  - The generalisation for higher rank groups or other groups can be done quite straightforwardly.

- For the groups of $E$-type, recent superconformal index results & character expansions allow for an explicit computation of the HS.

- For $G_2$ or $F_4$, discrete symmetries are enough to evaluate the HS exactly, even though neither ADHM construction nor SCI is known for these cases.

- In general, the HS for multi-instantons can also be computed from the blow-up formula due to [Nakajima-Yoshioka '03]. (see, e.g. Keller-Song '12.) It’d be interesting to obtain closed forms, in which the symmetry is manifest, from such a formula.