The Equivalence Modulo Non-stationary Ideals and Shelah’s Main Gap Theorem

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Outline

1. Classifying First-order countable Theories

2. The Equivalence Modulo Non-stationary Ideals
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Classifying First-order countable Theories

The spectrum problem

Let $I(T, \alpha)$ denote the number of non-isomorphic models of $T$ with cardinality $\alpha$.

What is the behavior of $I(T, \alpha)$?

- **Löwenheim-Skolem Theorem:**
  \[ \exists \alpha \geq \omega \ I(T, \alpha) \neq 0 \Rightarrow \forall \beta \geq \omega \ I(T, \beta) \neq 0. \]

- **Morley’s categoricity:**
  \[ \exists \alpha > \omega \ I(T, \alpha) = 1 \Rightarrow \forall \beta > \omega \ I(T, \beta) = 1 \]

- **Shelah’s Main Gap Theorem:**
  Either, for every uncountable cardinal $\alpha$, $I(T, \alpha) = 2^\alpha$, or $\forall \alpha > 0 \ I(T, \aleph_\alpha) < \beth_1(\| \alpha \|)$. 

Approaches

• Shelah’s stability theory.
  Classify the models of $T$ by cardinal invariants and clearly differentiate between the theories that can be classified and those that cannot.

• Descriptive set theory:
  It uses Borel-reducibility and the isomorphism relation to define a partial order on the set of all first-order complete countable theories.
The topology

$\kappa$ is an uncountable cardinal that satisfies $\kappa^{<\kappa} = \kappa$.

We equip the set $\kappa^\kappa$ with the bounded topology. For every $\zeta \in \kappa^{<\kappa}$, the set

$$[\zeta] = \{ \eta \in \kappa^\kappa \mid \zeta \subset \eta \}$$

is a basic open set.
Let $E_1$ and $E_2$ be equivalence relations on $\kappa^\kappa$. We say that $E_1$ is \textit{continuous reducible} to $E_2$, if there is a continuous function $f : \kappa^\kappa \to \kappa^\kappa$ that satisfies $(x, y) \in E_1 \iff (f(x), f(y)) \in E_2$.

We write $E_1 \leq_{\kappa}^c E_2$. 
Coding structures

Fix a language $\mathcal{L} = \{ P_n | n < \omega \}$

Definition

Let $\pi$ be a bijection between $\kappa^{<\omega}$ and $\kappa$. For every $f \in \kappa^\kappa$ define the structure $A_f$ with domain $\kappa$ by: for every tuple $(a_1, a_2, \ldots, a_n)$ in $\kappa^n$

$$(a_1, a_2, \ldots, a_n) \in P_m A_f \iff f(\pi(m, a_1, a_2, \ldots, a_n)) > 0$$

Definition (The isomorphism relation)

Given $T$ a first-order complete countable theory in a countable vocabulary, we say that $f, g \in \kappa^\kappa$ are $\cong^\kappa_T$ equivalent if

- $A_f \models T, A_g \models T, A_f \cong A_g$
  
  or

- $A_f \not\models T, A_g \not\models T$
The complexity

We can define a partial order on the set of all first-order complete countable theories

\[ T \leq_{\kappa} T' \text{ iff } \cong_{T}^{\kappa} \leq_{c}^{\kappa} \cong_{T'}^{\kappa} \]
The subspace $2^\kappa$

In the subspace $2^\kappa$, we can define the following notions in the same way:

- $E_1 \leq^2 c E_2$.
- $f \sim^2_T g$.
- $T \leq^2_{\kappa} T'$.
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Shelah’s Main Gap Theorem

Theorem (Shelah)

If $T$ is classifiable and $T'$ is not, then $T$ is less complex than $T'$ and their complexity are not close.

Question:

Is there a Borel reducibility counterpart of the Main Gap Theorem in the spaces $\kappa^\kappa$ and $2^\kappa$?
For every regular cardinal \( \lambda < \kappa \), the relations \( E^\kappa_{\lambda\text{-club}} \) and \( E^2_{\lambda\text{-club}} \) are defined as follow.

**Definition**

- **On the space** \( \kappa^\kappa \), we say that \( f, g \in \kappa^\kappa \) are \( E^\kappa_{\lambda\text{-club}} \) equivalent if the set \( \{ \alpha < \kappa | f(\alpha) = g(\alpha) \} \) contains an unbounded set that is closed under \( \lambda \)-limits.

- **On the space** \( 2^\kappa \), we say that \( f, g \in 2^\kappa \) are \( E^2_{\lambda\text{-club}} \) equivalent if the set \( \{ \alpha < \kappa | f(\alpha) = g(\alpha) \} \) contains an unbounded set that is closed under \( \lambda \)-limits.
Looking above the Gap

Theorem (Friedman, Hyttinen, Kulikov)

Suppose \( \kappa = \lambda^+ = 2^\lambda \) and \( \lambda^{<\lambda} = \lambda \).

- If \( T \) is an unstable or superstable with OTOP, then \( E^{2}_{\lambda\text{-club}} \leq^2 c \equiv^2 T \).
- If \( \lambda \geq 2^\omega \) and \( T \) is a superstable with DOP, then \( E^{2}_{\lambda\text{-club}} \leq^2 c \equiv^2 T \).

Theorem (Friedman, Hyttinen, Kulikov)

Suppose that for all \( \gamma < \kappa, \gamma^\omega < \kappa \) and \( T \) is a stable unsuperstable. Then \( E^{2}_{\omega\text{-club}} \leq^2 c \equiv^2 T \).
Looking below the Gap

Theorem (Friedman, Hyttinen, Kulikov)

*If T is a classifiable theory, then for all regular cardinal \( \lambda < \kappa \),
\[
E^2_{\lambda\text{-club}} \not\leq^2_c \sim^2_T
\]*

Theorem (Hyttinen, Moreno)

*Suppose T is a classifiable theory and \( \lambda < \kappa \) is a regular cardinal.
Then \( \sim^\kappa_T \leq^\kappa_c E^\kappa_{\lambda\text{-club}} \).*

Theorem (Hyttinen, Kulikov, Moreno)

Denote by \( S^{\kappa}_{\lambda} \) the set \( \{ \alpha < \kappa | cf(\alpha) = \lambda \} \).
*Suppose T is a classifiable theory and \( \lambda < \kappa \) is a regular cardinal. If
\( \Diamond (S^{\kappa}_{\lambda}) \) holds, then \( \sim^2_T \leq^2_c E^2_{\lambda\text{-club}} \).*
The Gap in ZFC

Theorem (Hyttinen, Moreno)

Suppose $T$ is a classifiable theory, $T'$ is an stable theory with the OCP, and $\kappa$ an inaccessible cardinal. Then $\sim_T^\kappa \leq^c E_{\omega-club}^\kappa \leq^c \sim_T^\kappa$.

Theorem (Moreno)

Suppose $T$ is a classifiable theory, $T'$ is a superstable theory with the S-DOP, $\lambda \geq 2^\omega$, and $\kappa$ an inaccessible cardinal. Then

$\sim_T^\kappa \leq^c E_{\lambda-club}^\kappa \leq^c \sim_T^\kappa$.

Theorem (Hyttinen, Kulikov, Moreno)

Suppose $\kappa = \lambda^+$ and $\lambda^\omega = \lambda$. If $T$ is a classifiable theory and $T'$ is a stable unsuperstable theory, then $\sim_T^2 \leq^c E_{\omega-club}^2 \leq^c \sim_T^2$.
Consistency

Let $H(\kappa)$ be the following property: If $T$ is classifiable and $T'$ is not, then $T \leq_{2\kappa} T'$ and $T' \not\leq_{2\kappa} T$.

Theorem

Suppose $\kappa = \lambda^+$, $2^\lambda > 2^\omega$ and $\lambda^{<\lambda} = \lambda$.

1. If $V = L$, then $H(\kappa)$ holds.
2. It is consistent that $H(\kappa)$ holds and there are $2^\kappa$ equivalence relations strictly between $\cong^{2T_1}$ and $\cong^{2T_2}$. 
References


