Generalized descriptive set theory
and the classification of uncountable structures
and non-separable spaces

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Workshop on Generalized Baire spaces
Two premises... and a question

**Premises**

1. Most of the people in this room already know what generalized DST is, its basics, and the most important results in this area.
2. Such results are not due to me, and I’m definitely not an expert in the field.

**The question**

What am I doing here? What am I supposed to talk about in this “tutorial”?

(I’m not joking! I seriously asked this question to myself a hundred of times in the past weeks...)
Part 1

Generalized DST and classification problems
Descriptive set theory is the study of “definable sets” in Polish (i.e., separable completely metrizable) spaces. In this theory, sets are classified in hierarchies, according to the complexity of their definitions, and the structure of the sets in each level of these hierarchies is systematically analyzed.

Kechris, Classical descriptive set theory

Polish spaces were isolated as those spaces retaining the fundamental properties of the Euclidean spaces $\mathbb{R}^n$ that are needed to carry out a great part of the analysis of their definable sets/functions.

Among such spaces, a prominent role is occupied by the Baire space $\omega^\omega$ and the Cantor space $2^\omega$, both endowed with the product of the discrete topology.
Classical DST has gained popularity also outside the logic community because of its effectiveness in tackling (and often solving) problems from other areas of mathematics. Part of this success is due to the combination of the following facts:

- Polish spaces are ubiquitous, most spaces considered in ordinary mathematics belong (up to coding) to such class;
- Polish spaces are quite manageable, and they avoid annoying (topological) pathologies;
- at least when the problem at hand is invariant under Borel isomorphism, it is not restrictive to work with well-understood spaces like $\omega^\omega$ or $2^\omega$. 

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Classification problems

Let $\mathcal{X}$ be a set of mathematical objects and $E$ be an equivalence relation on $\mathcal{X}$. A solution to the classification problem for $\mathcal{X}$ up to $E$ is just an assignment of complete invariants to elements of $\mathcal{X}$ up to $E$.

This means that we want to find:

- a set $I$ of invariants, and
- an assignment $\varphi: \mathcal{X} \to I$

such that for all $x_0, x_1 \in \mathcal{X}$

$$x_0 \ E \ x_1 \iff \varphi(x_0) = \varphi(x_1).$$

In order to have a meaningful solution, both $I$ and $\varphi$ must be as “natural” and “concrete” as possible.
Some examples

- Classification up to similarity of all the $n$-by-$n$ matrices over the complex numbers.
- Classification of linear orders up to isomorphism or equimorphism (= bi-embeddability).
- Classification of algebraic structures (groups, fields, ...) up to isomorphism or bi-embeddability.
- Classification of topological spaces/groups up to topological homeomorphism or bi-embeddability.
- Classification of metric spaces up to isometry or isometric bi-embeddability.
- Classification of more complicated structures (Banach spaces, von Neumann algebras, ...) up to the relevant notions of isomorphism/bi-embeddability.
The DST point of view on classification: Borel reducibility

Suppose that the space $\mathcal{X}$ carries a natural Polish topological structure: then a **concrete classification** of the objects of $\mathcal{X}$ up to $E$ consists of a Polish space $I$ and a Borel map $f: \mathcal{X} \rightarrow I$ such that for all $x_0, x_1 \in \mathcal{X}$

$$x_0 \ E \ x_1 \iff f(x_0) = f(x_1).$$

**Example**

Classification of $n$-by-$n$ complex matrices up to similarity by means of the associated canonical Jordan form (up to permutation of the blocks).

When a *concrete* classification is not possible, one may still be satisfied with a classification map assigning

- invariants up to countably many mistakes,
- invariants up to some natural equivalence (e.g. linear orders up to isomorphism),
- and so on.
Let $\mathcal{X}, \mathcal{Y}$ be Polish spaces, and $E, F$ be equivalence relations on $\mathcal{X}$ and $\mathcal{Y}$.

**Definition (Borel reducibility)**

$E$ is **Borel reducible** to $F$ ($E \leq_B F$) if there is a Borel map $f : \mathcal{X} \to \mathcal{Y}$ such that for all $x_0, x_1 \in \mathcal{X}$

$$x_0 E x_1 \iff f(x_0) F f(x_1).$$

$E \leq_B F \iff$ The objects in $\mathcal{X}$ may be classified up to $E$-equivalence using the $F$-classes as invariants.

For example:

- there is a **concrete classification** for the elements of $\mathcal{X}$ up to $E$ equivalence if and only if $E \leq_B \text{id}(\mathbb{R})$;
- $(\mathcal{X}, E)$ is **classifiable by countable structures** if $E \leq_B \cong \upharpoonright \text{Mod}_L^\omega$ for $L$ some countable first-order relational language (we can equivalently use as invariants graphs, linear orders, ... up to isomorphism).
Borel reducibility can also be used as a tool for comparing the complexity of various classification problems.

**Definition (Borel reducibility)**

- \( E \leq_B F \) if there is a Borel map \( f : \mathcal{X} \to \mathcal{Y} \) such that \( x_0 E x_1 \iff f(x_0) F f(x_1) \) for all \( x_0, x_1 \in \mathcal{X} \).
- \( E \) and \( F \) are **Borel bi-reducible** \((E \sim_B F)\) if \( E \leq_B F \leq_B E \).

Intuitively, \( E \leq_B F \) means: \( E \) is not more complicated than \( F \). 
\( E \sim_B F \) means: \( E \) and \( F \) have the same complexity.

Usually, one concentrates on “definable” equivalence relations \( E \) on \( \mathcal{X} \), most notably:

- \( E \) is **Borel** if it is a Borel subset of the square \( \mathcal{X} \times \mathcal{X} \);
- \( E \) is **analytic** if, as a subset of \( \mathcal{X} \times \mathcal{X} \), is a Borel image of \( \omega^\omega \).
First-order structures

Consider (for simplicity) the countable relational language $\mathcal{L} = (R_j)_{j \in J}$, where each $R_j$ has arity $n_j$. Up to isomorphism, all countable $\mathcal{L}$-structures can be assumed to have domain $\omega$, and thus they can be identified via characteristic functions of their predicates with the elements of the space

$$\text{Mod}_\omega^\mathcal{L} = \prod_{j \in J} 2^{(\omega^{n_j})}.$$

Endow $\text{Mod}_\omega^\mathcal{L}$ with the topology whose basic open sets are the collection of structures containing (as a substructure) a given finite $\mathcal{L}$-structure on $\omega$: then $\text{Mod}_\omega^\mathcal{L}$ is homeomorphic to $2^\omega$ (thus it is Polish), and the isomorphism and bi-embeddability relations are analytic equivalence relations on it.

Fixing a first-order theory (or, more generally, an $\mathcal{L}_{\omega_1 \omega}$-sentence) $T$ and considering only the collection

$$\text{Mod}_T^\mathcal{L} = \{ x \in \text{Mod}_\omega^\mathcal{L} \mid x \models T \}$$

of models of it gives us a Borel subset of $\text{Mod}_\omega^\mathcal{L}$, and thus a standard Borel space.
First-order structures

Using this coding one may obtain many (anti-)classification results:

- the classification problem for countable graphs (trees, linear orders, non-abelian groups, ...) is as complex as possible, i.e. it is $\leq B$ above any equivalence relation which is classifiable by countable structures \((H.\ Friedman-Stanley,\ Mekler);\)
- finitely generated groups are not concretely classifiable (but they can be classified up to countably many mistakes) \((Thomas?);\)
- the classification problem for torsion-free abelian countable groups of finite rank strictly increases in complexity with the rank \((Hjorth-Thomas);\)
- countable graphs cannot be classified (in any reasonable sense) up to bi-embeddability \((Louveau-Rosendal);\)
- the same is true for lattices, countable groups, ... but the classification of e.g. countable linear orders up to bi-embeddability is simpler \((Louveau-Rosendal,\ Laver).\)
Separable metric spaces

Every **Polish** (= separable complete) metric space is isometric to a closed subspace of the separable Urysohn space $\mathbb{U}$. Thus the space

$$F(\mathbb{U}) = \{ C \subseteq \mathbb{U} \mid C \text{ closed in } \mathbb{U} \}$$

endowed with the Effros-Borel structure may be regarded as the standard Borel space of all (up to isometry) Polish metric spaces.

- The classification problem for compact Polish metric spaces is concretely classifiable (*Gromov*).
- The classification problem for arbitrary Polish metric spaces is Borel bi-reducible with the most complex orbit equivalence relation (*Gao-Kechris*).
- The classification problem for Polish *ultrametric* spaces is Borel bi-reducible with graph isomorphism (*Gao-Kechris*).
- The complexity does not drop if we consider only *discrete* Polish ultrametric spaces (*Camerlo-Marcone-M.*).
- It is not possible to classify (discrete) Polish (ultra)metric spaces up to isometric bi-embeddability (*Louveau-Rosendal, Camerlo-Marcone-M.*).
Other separable spaces

In the same fashion one can consider more complicated separable spaces, as long as there is a “universal” space in the class of objects $\mathcal{X}$:

- compact metrizable spaces $\sim$ Hilbert cube;
- separable Banach spaces $\sim C[0; 1]$;
- ...

This allows us to construe $\mathcal{X}$ as a Polish (or standard Borel) space and use $\leq_B$ to classify its objects.

- It is not possible to classify separable Banach spaces up to isomorphism or bi-embeddability (Ferenczi-Louveau-Rosendal).
- There is no reasonable classification of Polish groups up to (topological) isomorphism (Ferenczi-Louveau-Rosendal).
- The classification problem for separable nuclear $C^*$-algebras up to isomorphism is Borel bi-reducible with the most complex orbit equivalence relation (Sabok).
- Finitely generated operator systems are concretely classifiable (Argerami-Coskey-Kalantar-Kennedy-Lupini-Sabok).
Limitations of classical DST

Despite its remarkable achievements, there are at least two serious limitations to this method.

1. So far, only analytic equivalence relations have been considered, but there is a lot more out there.

For example, the conjugacy relation on the group of Borel automorphism of a standard Borel space is $\Sigma^1_2$-complete (Clemens). Borel bi-reducibility between analytic equivalence relations is $\Sigma^1_3$ in the codes, and at least $\Sigma^1_2$-hard (Adams-Kechris).

2. Polish (or standard Borel) spaces are not suitable for coding uncountable structures and non-separable spaces, which are clearly relevant in other areas of mathematics.

Think about the role of uncountable structures in model theory, or non-separable Banach/Hilbert spaces in analysis and operator theory.
Classical DST vs generalized DST

Descriptive set theory is the study of “definable sets” in Polish spaces (such as $\omega^\omega$ and $2^\omega$).

Kechris, Classical descriptive set theory

As it is conceived now,

**Generalized** descriptive set theory is the study of $\kappa^\kappa$ and $2^\kappa$ (endowed with a natural topology) and of their definable subsets for $\kappa$ an **uncountable** cardinal.

**Remark**

A large initial portion of classical DST (roughly speaking, up to Borel and analytic sets) can be carried out in ZF + DC. In contrast, the base theory for generalized DST is usually ZFC plus the cardinal assumption $\kappa^{<\kappa} = \kappa$. 
Let $\kappa$ be an uncountable cardinal.

**Definition**

The **generalized Baire space** is $\kappa^\kappa = \{ x \mid x : \kappa \to \kappa \}$ endowed with the (bounded) topology generated by the sets of the form

$$\mathbb{N}_s^\kappa = \{ x \in \kappa^\kappa \mid s \subseteq x \}$$

for $s \in \kappa^{<\kappa}$.

The **generalized Cantor space** is the closed subspace $2^\kappa$ of $\kappa^\kappa$ consisting of all binary sequences.

**Remark**

The (bounded) topology described above is strictly finer than the product topology. When $\kappa$ is regular it coincides with the $\kappa$-product of the discrete topology.
A comparison with the classical context

- $\kappa^\kappa$ and $2^\kappa$ are perfect, regular Hausdorff, zero-dimensional spaces.
- They are not metrizable (unless $\text{cof}(\kappa) = \omega$), and their density character is $\kappa^{<\kappa}$ and $2^{<\kappa}$; these are also the smallest sizes for their topological bases. In particular, $\kappa^\kappa$ has a base of size $\kappa$ if and only if $\kappa^{<\kappa} = \kappa$.
- None of them is compact, but $2^\kappa$ is $\kappa$-compact if and only if $\kappa$ is a weakly compact cardinal.
- The topology is closed under $< \text{cof}(\kappa)$-intersections.
(\kappa^+\mbox{-})Borel sets and (\kappa\mbox{-})analytic sets

Definitions

- A set $A \subseteq \kappa^\kappa$ is \((\kappa^+\mbox{-})\text{Borel}\) if it belongs to the smallest $\kappa^+\mbox{-}$algebra on $\kappa^\kappa$ containing all open sets.

- A set $A \subseteq \kappa^\kappa$ is \((\kappa\mbox{-})\text{analytic}\) if it is a projection of a closed subset of $\kappa^\kappa \times \kappa^\kappa$.

- A set $A \subseteq \kappa^\kappa$ is \((\kappa\mbox{-})\text{bi-analytic}\) if both $A$ and $\kappa^\kappa \setminus A$ are \((\kappa\mbox{-})\text{analytic}\).

A comparison with the classical context:

- If $\kappa^{<\kappa} = \kappa$, Borel sets are the closure under complements and $\leq \kappa$-unions of the basic open sets. Moreover, they form a proper subclass of $\mathcal{P}(\kappa^\kappa)$ that can be naturally stratified in a hierarchy with exactly $\kappa^+$-many levels (the $\Sigma^0_\alpha$’s and $\Pi^0_\alpha$’s).

- All Borel sets are analytic, but there are bi-analytic sets which are not Borel (failure of the Souslin theorem).

- Analytic sets are closed under unions and intersections of size $\leq \kappa$. 

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The central question

Why should one be interested in such spaces (and their subsets)?

**The surly answer:** Good mathematics needs not to be justified!

**The pragmatic answer:** Because it is a relatively unexplored area and one can obtain nontrivial results (and a lot of publications!) in a reasonable time.

**The “good old days” answer:** Because it helps in understanding which properties of \( \omega \) are crucial to obtain the various results in classical DST.

**The aestethic answer:** Because the resulting theory is beautiful and deeply connected with other areas of set theory (large cardinals, forcing, infinite combinatorics, ...).

**The “working mathematician” answer:** Because it is useful, e.g. when coming to classification problems beyond the scope of classical DST...

*Although none of these answers seems to be entirely wrong, I will concentrate on the last one...*
Analytic equivalence relations and Borel reducibility

Let $\mathcal{X}$ and $\mathcal{Y}$ be subsets of $\kappa^\kappa$ (or of a homeomorphic copy of it), and $E, F$ be equivalence relations on $\mathcal{X}, \mathcal{Y}$, respectively.

**Definitions**

- $E$ is **analytic** (resp. **Borel**) if it is such as a subset of $\mathcal{X} \times \mathcal{X}$.
- $E$ is **Borel reducible** to $F$ ($E \leq^B F$) if there is a Borel map $f: \mathcal{X} \to \mathcal{Y}$ such that for every $x_0, x_1 \in \mathcal{X}$

$$x_0 E x_1 \iff f(x_0) F f(x_1).$$

- $E$ and $F$ are **Borel bi-reducible** ($E \sim^B F$) if $E \leq^B F$ and $F \leq^B E$.
- $E$ is **complete** (for analytic eq. rel.) if $F \leq^B E$ for every $(\kappa)$-analytic equivalence relation $F$.

The intended meaning of $\leq^B$ and $\sim^B$ is the same as in the classical case.
Uncountable first-order structures

By identifying the first-order $\mathcal{L}$-structures with their characteristic functions, one can consider

$$\text{Mod}_{\mathcal{L}}^\kappa = \prod_{j \in J} 2^{(\kappa^n_j)}$$

as the space of all $\mathcal{L}$-structures of size $\kappa$ (up to isomorphism). There is a canonical bijection between $\text{Mod}_{\mathcal{L}}^\kappa$ and $2^\kappa$, thus $\text{Mod}_{\mathcal{L}}^\kappa$ can be endowed with the same (bounded) topology.

(When $\kappa$ is regular, this topology coincides with the one whose basic open sets are determined by substructures of size $< \kappa$.)

It is easy to check that the isomorphism relation $\cong$ and the bi-embeddability relation $\approx$ are analytic equivalence relations on $\text{Mod}_{\mathcal{L}}^\kappa$. 
Uncountable first-order structures

What happens if we restrict ourselves to the models of a first-order theory?

If $\kappa^{<\kappa} = \kappa$, then one obtains the following generalization of a theorem of Lopez-Escobar.

Theorem (Vaught, S. Friedman-Hytten-Kulikov, Andretta-M.)

Assume that $\kappa^{<\kappa} = \kappa$. A subset of $\text{Mod}^\kappa_L$ is Borel and invariant under isomorphism if and only if it is of the form

$$\text{Mod}^\kappa_\sigma = \{ x \in \text{Mod}^\kappa_L \mid x \models \sigma \}$$

for some $\mathcal{L}_{\kappa^{+\kappa}}$-sentence $\sigma$.

Remark

The isomorphism and bi-embeddability relations on the elementary class $\text{Mod}^\kappa_\sigma$ remain analytic equivalence relations.
Uncountable first-order structures

Working in this setup, S. Friedman, Hyttinen, Kulikov, Moreno, and others are developing a beautiful theory with remarkable and deep connections with Shelah’s stability theory.

**Theorem (S. Friedman-Hyttinen-Kulikov)**

Let $\kappa = \kappa^{<\kappa}$ be an uncountable non weakly inaccessible cardinal, and $T$ be a complete countable first-order theory. If the isomorphism relation on $\text{Mod}_T^\kappa$ is Borel, then $T$ is classifiable and shallow. Conversely, if $\kappa > 2^{\aleph_0}$, then if $T$ is classifiable and shallow, then the isomorphism relation on $\text{Mod}_T^\kappa$ is Borel.

**Corollary (S. Friedman-Hyttinen-Kulikov)**

Under the same assumptions, if $T$ is classifiable and shallow and $T'$ is not, then isomorphism on $\text{Mod}_T^\kappa$ does not Borel reduce to isomorphism on $\text{Mod}_T^\kappa$. 

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Uncountable first-order structures

Theorem (S. Friedman-Hyttinen-Kulikov)

Assume that $\kappa > \omega_1$ is such that $\kappa^{<\kappa} = \kappa$ and $\lambda^\omega < \kappa$ for all $\lambda < \kappa$. Then in $L$ and in the forcing extension after adding $\kappa^+$-many Cohen subsets of $\kappa$ we have: for any theory $T$, $T$ is classifiable if and only if the isomorphism relation on $\text{Mod}_T^\kappa$ is bi-analytic.

Corollary (S. Friedman-Hyttinen-Kulikov)

Under the same assumptions, if $T$ is classifiable and $T'$ is not then isomorphism on $\text{Mod}_T^{\kappa'}$ does not Borel reduce to isomorphism on $\text{Mod}_T^\kappa$.

Theorem (S. Friedman-Hyttinen-Kulikov)

Suppose $\kappa = \lambda^+ = 2^\lambda > 2^{\aleph_0}$ where $\lambda^{<\lambda} = \lambda$. Let $T$ be a first-order theory. Then $T$ is classifiable if and only if for all regular $\mu < \kappa$, $E^\kappa_{\mu} \not\leq_B \cong \upharpoonright \text{Mod}_T^\kappa$, where $E^\kappa_{\mu}$ is the equality on $2^\kappa$ modulo the ideal of not $\mu$-stationary sets.
Uncountable first-order structures

**Theorem (Hyttinen-Kulikov)**

If $V = L$ and $\kappa$ is the successor of a regular cardinal, then the isomorphism relation on dense linear orders of size $\kappa$ is complete.

**Theorem (Hyttinen-Kulikov)**

Assume that $V = L$ and that $\kappa = \lambda^+$ with $\lambda^\omega = \lambda$. Then there is a not too complicated (stable, NDOP, NOTOP) first-order theory $T$ such that the isomorphism on $\text{Mod}_T^\kappa$ is complete.

Under the same assumptions, isomorphism between trees of height $\omega + \omega + 2$ and size $\kappa$ is complete as well.
Uncountable first-order structures

Theorem (Hyttinen-Moreno)

Under suitable assumptions on $\kappa = \kappa^{<\kappa} > \omega$, if $T$ is classifiable and $T'$ is a stable theory with the OCP, then isomorphism on $\text{Mod}^\kappa_{T'}$ Borel reduces to isomorphism on $\text{Mod}^\kappa_T$. 
Uncountable first-order structures

In the same vein one can study the complexity of the bi-embeddability relation.

**Theorem (Mildemberger-M.)**
Assume that $\kappa^{<\kappa} = \kappa$. Then the bi-embeddability relation between graphs of size $\kappa$ is complete.

**Theorem? (Calderoni-M.)**
Assume that $\kappa^{<\kappa} = \kappa$. Then the bi-embeddability relation between groups of size $\kappa$ is complete.
The striking connections between the $\leq^B_\kappa$-complexity of the isomorphism relation and the classification of first-order theories according to Shelah’s stability theory are certainly the most important achievements that have been obtained so far in the realm of generalized DST.

Unfortunately, not many non-logicians knows about model theory...

Can we tackle other kind of classification problems (metric spaces, Banach spaces, ...)?

Unlike the case of first-order structures, one cannot straightforwardly generalize the coding process used in the separable case:

- universal objects may not exists in the relevant class;
- even when such an object exists, it is not clear that the collection of its subspaces forms a well-behaved topological space (i.e. close to $\kappa^\kappa$).
Let $\mathbb{Q}^+$ be the set of positive rational numbers, and let $\mathcal{X}$ be the space $2^{\kappa \times \kappa \times \mathbb{Q}^+}$: $\mathcal{X}$ is naturally isomorphic to $2^\kappa$ and can be identified with it.

Given a complete metric space $(M, d)$ of density character $\kappa$ and a dense subset $D = \{m_\alpha \mid \alpha < \kappa\}$ of it, we can identify $M$ with the unique element $c_M \in \mathcal{X}$ such that for all $\alpha, \beta < \kappa$ and $q \in \mathbb{Q}^+$

$$c_M(\alpha, \beta, q) = 1 \iff d_M(m_\alpha, m_\beta) < q.$$ 

In fact, $M$ is isometric to the completion of the metric space $(\kappa, d_{c_M})$ where

$$d_{c_M}(\alpha, \beta) := \inf\{q \in \mathbb{Q}^+ \mid c_M(\alpha, \beta, q) = 1\},$$

so that $d_{c_M}(\alpha, \beta) = d_M(m_\alpha, m_\beta)$ for all $\alpha, \beta < \kappa$. 
Consider now the space \( M_\kappa \subset \mathcal{X} \) consisting of those \( c \in 2^\kappa \times \kappa \times \mathbb{Q}^+ \) satisfying the following conditions:

\[
\forall \alpha, \beta < \kappa \forall q, q' \in \mathbb{Q}^+ \ [q \leq q' \Rightarrow c(\alpha, \beta, q) \leq c(\alpha, \beta, q')] \\
\forall \alpha, \beta < \kappa \exists q \in \mathbb{Q}^+ \ [c(\alpha, \beta, q) = 1] \\
\forall \alpha < \kappa \forall q \in \mathbb{Q}^+ \ [c(\alpha, \alpha, q) = 1] \\
\forall \alpha < \beta < \kappa \exists q \in \mathbb{Q}^+ \ [c(\alpha, \beta, q) = 0] \\
\forall \alpha, \beta < \kappa \forall q \in \mathbb{Q}^+ \ [c(\alpha, \beta, q) = 1 \iff c(\beta, \alpha, q) = 1] \\
\forall \alpha, \beta, \gamma < \kappa \forall q, q' \in \mathbb{Q}^+ \ [c(\alpha, \beta, q) = 1 \land c(\beta, \gamma, q') = 1 \Rightarrow c(\alpha, \gamma, q + q') = 1] \\
\forall \alpha < \kappa \exists \beta < \kappa \exists q \in \mathbb{Q}^+ \forall \gamma < \alpha \ [c(\gamma, \beta, q) = 0].
\]

The first six conditions are designed so that given any \( c \in M_\kappa \), the (well-defined) map \( d_c : \kappa \times \kappa \rightarrow \mathbb{R} \) given by

\[
d_c(\alpha, \beta) := \inf\{q \in \mathbb{Q}^+ \mid c(\alpha, \beta, q) = 1\}
\]

is a metric on \( \kappa \); we denote by \( M_c \) the completion of \((\kappa, d_c)\), and notice that the last condition ensures that \( M_c \) has density character \( \kappa \).
Complete metric spaces of density character $\kappa$

It is straightforward to check that

- the code $c_M$ of any complete metric space $M$ of density character $\kappa$ belongs to $\mathcal{M}_\kappa$, and is such that $M$ is isometric to $M_{c_M}$;
- conversely, for each $c \in \mathcal{M}_\kappa$ the space $M_c$ is a complete metric space of density character $\kappa$.

Thus we can regard $\mathcal{M}_\kappa \subseteq 2^{\kappa \times \kappa} \times \mathbb{Q}^+$, endowed with the inherited (bounded) topology, as the space of (codes for) all complete metric spaces of density character $\kappa$ (up to isometry).
Complete metric spaces of density character $\kappa$

When $\kappa = \omega$, this procedure gives an alternative (but equivalent) way to code Polish spaces.

**Fact**

There are Borel maps $\Phi: M_\omega \to F(\mathbb{U})$ and $\Psi: F(\mathbb{U}) \to M_\omega$ such that for all $c \in M_\omega$

$$M_c \text{ and } \Phi(c) \text{ are isometric}$$

and for all *infinite* $M \in M_\omega$

$$M \text{ and } M_{\Psi(M)} \text{ are isometric.}$$

**Remark**

This alternative coding has (essentially) been used e.g. by Vershik to show that Polish metric spaces are not concretely classifiable up to isometry.
The explicit definition given by the previous six conditions shows that $M_\kappa$ is a Borel subset of $2^{\kappa \times \kappa \times \mathbb{Q}^+}$.

The isometry relation $\cong^i$ on $M_\kappa$ is given by
\[ c \cong^i d \iff \text{there is a metric-preserving bijection between } M_c \text{ and } M_d, \]
while the isometric bi-embeddability relation $\approx^i$ on $M_\kappa$ is given by
\[ c \approx^i d \iff M_c \text{ isometrically embeds in } M_d, \text{ and vice versa.} \]

Using the Tarski-Kuratowski algorithm and some standard computations, one sees that the relations $\cong^i$ and $\approx^i$ are analytic equivalence relations on $M_\kappa$. 
We also consider some natural subclasses of $M_\kappa$, such as

$$D_\kappa = \{ c \in M_\kappa \mid M_c \text{ is discrete} \}$$

or

$$U_\kappa = \{ c \in M_\kappa \mid M_c \text{ is ultrametric} \}.$$

**Remark**

Not all these subclasses are Borel: for example, $U_\kappa$ is Borel (since it is a closed subset of $M_\kappa$), while $D_\kappa$ is not.
Complete metric spaces of density character $\kappa$

**Theorem**

Let $\kappa$ be any infinite cardinal. Then $\cong^i \upharpoonright \mathbb{D}_\kappa$ (resp. $\approx^i \upharpoonright \mathbb{D}_\kappa$) is Borel bi-reducible with isomorphism (resp. bi-embeddability) on graphs.

**Proof.**

For one direction, associate to each graph $G$ on $\kappa$ the discrete space $(\kappa, d_G)$ given by $d_G(\alpha, \beta) = 1$ if $\alpha \neq \beta$, and $d_G(\alpha, \beta) = 0$ otherwise.

For the other direction, consider the language $\mathcal{L} = (P_q)_{q \in \mathbb{Q}^+}$ with each $P_q$ binary, and to each $c \in \mathbb{D}_\kappa$ associate the $\mathcal{L}$-structure $\mathcal{A}_c$ on $\kappa$ given by $\mathcal{A}_c \models P_q[\alpha, \beta] \iff c(\alpha, \beta, q) = 1$. This works because $M_c$ coincides with any of its dense subsets.
Complete metric spaces of density character $\kappa$

**Corollary**

1. Assume that $V = L$ and that $\kappa = \lambda^+$ with $\lambda^\omega = \lambda$. Then isometry on $\mathcal{D}_\kappa$ is complete for analytic equivalence relations.

2. Assume that $\kappa^{<\kappa} = \kappa$. Then isometric bi-embeddability on $\mathcal{D}_\kappa$ is complete for analytic equivalence relations.

Thus, at least consistently, $\cong^i$ and $\approx^i$ may surprisingly have the same complexity already on $\mathcal{D}_\kappa$.

**Remark**

Obviously, we can replace “discrete” with locally compact, zero-dimensional, of size $\kappa$, and so on.
What about ultrametric spaces? An easy fact is the following.

**Proposition**

Let $\kappa$ be an infinite cardinal. Then isometry (resp. isometric bi-embeddability) on $\mathcal{U}_\kappa$ is Borel reducible to isomorphism (resp. bi-embeddability) between graphs of size $\kappa$.

**Proof.**

We associate to each $c \in \mathcal{U}_\kappa$ a structure whose points are the open balls $B$ in $M_c$ of rational radius and whose predicates are the following:

- for each $q \in \mathbb{Q}^+$ a unary predicate $P_q$ which holds on $B$ if and only if $\text{diam}(B) < q$;
- a binary predicate $R$ such that $B R B' \iff B \subseteq B'$.

The fact that we are working with ultrametric spaces ensures that the resulting first-order structure has size $\kappa$ and that the metric structure of $M_c$ can be recovered from the above predicates.
Complete metric spaces of density character $\kappa$

**Notation**

Given an infinite cardinal $\kappa$ and an ordinal $\alpha$, let $T^\kappa_\alpha$ be the collection of (set-theoretic) trees on $\kappa$ of height $\alpha$ and size $\kappa$.

Let $\alpha$ be countable and fix any strictly decreasing sequence $(r_\beta)_{\beta<\alpha}$ of positive reals. Given $T \in T^\kappa_\alpha$ define a distance $d_T$ on $T$ by

$$d_T(u, v) = \begin{cases} r_{lh(u \cap v)} & \text{if } u \neq v \\ 0 & \text{otherwise.} \end{cases}$$

Finally, let $X_T$ be the completion of $(T, d_T)$.

**Remark**

$T \subseteq X_T$ is exactly the collection of all isolated point of $X_T$, so that $X_T = T$ is discrete if $\inf\{r_\beta \mid \beta < \alpha\} > 0$. 

Complete metric spaces of density character $\kappa$

**Theorem**

For every $T, S \in T_\alpha^\kappa$

$$T \cong S \iff X_T \cong i X_S \quad \text{and} \quad T \approx S \iff X_T \approx i X_S.$$ 

**Sketch of the proof.**

Any isomorphism (embedding) between $T$ and $S$ is an isometry (isometric embedding) between $(T, d_T)$ and $(S, d_S)$, and the latter can be extended to the whole spaces $X_T$ and $X_S$.

Conversely, an isometry $\varphi$ between $X_T$ and $X_S$ maps $T$ onto $S$ (isolated points!). Then one checks that $\varphi$ may fail to be an isomorphism between trees just because it switches a terminal node with its immediate predecessor (if such nodes exist): “straightening” the isometry gives the desired isomorphism.

The case of (isometric) embeddings is a little bit more delicate.
Complete metric spaces of density character $\kappa$

**Corollary**

Assume that $V = L$ and that $\kappa = \lambda^+$ with $\lambda^\omega = \lambda$. Then isometry on $\mathcal{U}_\kappa \cap \mathcal{D}_\kappa$ is already complete for analytic equivalence relations.

**Proof.**

By the result of Hyttinen-Kulikov, isomorphism on $\mathbb{T}_\omega^{\omega+2}$ is complete, so it is enough to apply the previous theorem using a sequence of distances $(r_\beta)_{\beta < \omega^+ + 2}$ bounded away from 0.

□
Banach spaces of density $\kappa$

In the same vein one can code Banach spaces of density character $\kappa$ as elements of (a homeomorphic copy of) $2^\kappa$, and check that the resulting set of codes $\mathcal{B}_\kappa$ is Borel. With this coding system, the relations of linear isometry $\simeq li$ and linear isometric embeddability $\approx li$ become analytic equivalence relations.

Notation

Let $c_0^\kappa \subseteq \mathbb{R}^\kappa$ be the collection of all sequences $(x_\alpha)_{\alpha < \kappa}$ of real numbers such that for all $\varepsilon > 0$ we have $|x_\alpha| < \varepsilon$ for all but finitely many $\alpha < \kappa$.

The Banach space $c_0^\kappa = (c_0^\kappa, \| \cdot \|_\infty)$ is then obtained by endowing $c_0^\kappa$ with the usual pointwise operations and the sup norm $\| \cdot \|_\infty$. 
Given a graph $G$ on $\kappa$, let $X_G = (c_0^\kappa, \| \cdot \|_G)$ be the Banach space with norm
$$
\|(x_\alpha)_{\alpha < \kappa}\|_G = \sup \left\{ |x_i| + \frac{|x_j|}{3 - \chi_G(i, j)} \mid i \neq j \in \kappa \right\},
$$
where $\chi_G(i, j) = 1$ if $i G j$ and $\chi_G(i, j) = 0$ otherwise.

Proposition

For $G, H$ graphs on $\kappa$, we have
$$G \simeq H \iff X_G \simeq^li X_H \quad \text{and} \quad G \approx H \iff X_G \simeq^li X_H.$$

Corollary

1. Assume that $V = L$ and that $\kappa = \lambda^+$ with $\lambda^\omega = \lambda$. Then linear isometry on $\mathcal{B}_\kappa$ is complete for analytic equivalence relations.
2. Assume that $\kappa < \kappa = \kappa$. Then linear isometric bi-embeddability on $\mathcal{B}_\kappa$ is complete for analytic equivalence relations.
Projective equivalence relations on $\mathbb{R}$

What about non-analytic equivalence relations on the real line? Using the methods of generalized DST one can e.g. obtain the following results.

**Theorem (Andretta-M.)**

1. The elements of a Polish space can be classified up to an arbitrary $\Sigma^1_2$ equivalence relation by using as invariants graphs of size $\aleph_1$ up to $\approx$.

2. Assume that $x^\#$ exists for all $x \in \omega^\omega$. Then the elements of a Polish space can be classified up to an arbitrary $\Sigma^1_3$ equivalence relation by using as invariants graphs of size $\aleph_2$ up to $\approx$.

3. Assume large cardinals (infinitely many Woodin’s cardinals with a measurable above suffices). Then there is a monotone function $r: \omega \to \omega$ such that: The elements of a Polish space can be classified up to a $\Sigma^1_n$ equivalence relation by using as invariants graphs of size $\aleph_{r(n)}$ up to $\approx$.

(An upper bound for $r(n)$ is given by $2^k - 2$, where $k$ is the integer part of $\frac{n+1}{2}$.)
End of part 1 of the tutorial

Thank you for your attention!