Elliptic integrable lattice systems

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Outline:
• Multidimensional consistency of quadrilateral lattices
• ABS classification and Adler’s lattice system (Q₄)
• Seed and soliton solutions for Q₄
• The limit to Q₃
• Elliptic lattice KdV & N-solitons
Integrable PΔE’s on the 2D lattice

The class of quadrilateral PΔEs has the following canonical form:

\[ Q_{p,q}(u, \tilde{u}, \hat{u}, \hat{\tilde{u}}) = 0 , \]

where we adopt the short-hand notation of vertices along an elementary plaquette on a rectangular lattice:

\[ u := u_{n,m} , \quad \tilde{u} = u_{n+1,m} \]
\[ \hat{u} := u_{n,m+1} , \quad \hat{\tilde{u}} = u_{n+1,m+1} \]

Schematically:

Here \( p, q \) denote lattice parameters

lattice shifts:

\[ u \xrightarrow{p} \tilde{u} \]
\[ u \xrightarrow{q} \hat{u} \]

Where \( Q \) is linear in each of the 4 vertices.

Here \( p, q \) are parameters of the equation, related to the lattice spacing. In some cases the lattice parameters are points on an algebraic curve \( p = (p, P) , \quad q = (q, Q) \).
Integrability: multidimensional consistency

The idea behind this is that a given "integrable" PDE

\[ Q_{p,q}(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}) = 0 \]

represents in fact a compatible parameter-family of PDEs which can be consistently embedded in a multidimensional lattice (i.e., of dim > 2), on which the evolution is well-posed.

This implies: we may extend the lattice (with lattice variables \( n, m \) associated with lattice parameters \( p, q \)) by adding new lattice directions (e.g., a third direction given by lattice variable \( h \) and parameter \( r \)), s.t. the solution \( u \) can be considered as a function on this multidimensional lattice;

\[ u = u_{n,m,h,...} = u(n, m, h, ...; p, q, r, ...) \]

with \( \tilde{u} := u_{n+1,m,h} \), \( \hat{u} := u_{n,m+1,h} \), \( \overline{u} := u_{n,m,h+1} \).

Thus, one may consider the system of equations:

\[ Q_{p,q}(u, \tilde{u}, \hat{u}, \overline{u}) = 0 \quad , \quad Q_{p,r}(u, \tilde{u}, u, \overline{u}) = 0 \quad , \quad Q_{q,r}(u, \hat{u}, u, \overline{u}) = 0, \]

and if this system is compatible we call it it consistent-around-the-cube (CAC).
Under the assumptions of \textit{multilinearity}, \textit{D}_4\text{-symmetry}, \textit{tetrahedron property} the multidimensionally integrable quadrilateral scalar equations were classified in [Adler, Bobenko, Suris, 2002]:

\textbf{ABS List:}

\textbf{Q – list:}

\begin{align*}
Q_1: & \quad p(u - \hat{u})(\tilde{u} - \hat{\tilde{u}}) - q(u - \tilde{u})(\hat{\hat{u}} - \hat{\tilde{u}}) = \delta^2 pq(q - p) \\
Q_2: & \quad p(u - \hat{u})(\tilde{u} - \hat{\tilde{u}}) - q(u - \tilde{u})(\hat{\hat{u}} - \hat{\tilde{u}}) + pq(p - q)(u + \tilde{u} + \hat{\hat{u}} + \hat{\tilde{u}}) = pq(p - q)(p^2 - pq + q^2) \\
Q_3: & \quad p(1 - q^2)(u\hat{u} + \hat{\tilde{u}}\tilde{u}) - q(1 - p^2)(u\tilde{u} + \hat{\tilde{u}}\hat{\tilde{u}}) = (p^2 - q^2)\left((\hat{\tilde{u}}\tilde{u} + u\hat{\tilde{u}}) + \delta^2\frac{1 - p^2(1 - q^2)}{4pq}\right) \\
Q_4: & \quad p(u\tilde{u} + \hat{\tilde{u}}\hat{\tilde{u}}) - q(u\tilde{u} + \hat{\tilde{u}}\hat{\tilde{u}}) = \frac{pQ - qP}{1 - p^2q^2}\left((\hat{\tilde{u}}\tilde{u} + u\hat{\tilde{u}}) - pq(1 + u\tilde{u}
\hat{\tilde{u}}\hat{\tilde{u}})\right)
\end{align*}

where \( P^2 = p^4 - \gamma p^2 + 1 \), \( Q^2 = q^4 - \gamma q^2 + 1 \).

\textbf{H – list:}

\begin{align*}
H_1: & \quad (u - \hat{u})(\hat{\tilde{u}} - \tilde{u}) = p^2 - q^2 \\
H_2: & \quad (u - \hat{u})(\tilde{u} - \hat{\tilde{u}}) = (p - q)(u + \tilde{u} + \hat{\hat{u}} + \hat{\tilde{u}}) + p^2 - q^2 \\
H_3: & \quad p(u\tilde{u} + \hat{\tilde{u}}\hat{\tilde{u}}) - q(u\tilde{u} + \hat{\tilde{u}}\hat{\tilde{u}}) = \delta^2(p^2 - q^2)
\end{align*}

in addition to an \textbf{A – list} consisting of two more equations.

A more recent paper (Adler, Bobenko, Suris, 2007) made a shortcut to the classification problem applying less stringent assumptions.
Coalescence diagram

Existing solutions of lattice eqs in the ABS list

- **Soliton solutions:** \((Q_1)_0, H_1, (H_3)_0, (Q_3)_0\) (from previous exposition)
- **Finite-gap solutions:** \((H_3)_0\) ("Hirota eq"): Bobenko, Pinkall (appr. 1996), \(H_1\): FWN, Enolskii (1996), \((Q_1)_0\): McIntosh, Pedit et.al. (1999).
- **Similarity solutions:** \((Q_1)_0, H_1, (H_3)_0\) in terms of Painlevé VI transcendents (FWN, Grammaticos, Ramani and Ohta, 2001).

*No explicit solutions were given so far for any of the other eqs in the ABS list!*
The consistency around the cube leads algorithmically to a Lax pair:

\[ \tilde{\varphi} = L_l(\tilde{u}, u; p) \varphi \quad , \quad \hat{\varphi} = L_l(\hat{u}, u; q) \varphi \]

where \( Q_4 \) arises from the compatibility relations (Lax equations):

\[ L_l(\tilde{u}, \tilde{u}; p)L_l(\hat{u}, u; q) = L_l(\hat{u}, \tilde{u}; q)L_l(\tilde{u}, u; p) . \]

The Lax matrices are given by:

\[ L_l(\tilde{u}, u; p) = \frac{1}{D} \begin{pmatrix} lu + l'\tilde{u} & -p(l l' + u\tilde{u}) \\ p(1 + l l' u\tilde{u}) & -(l\tilde{u} + l'u) \end{pmatrix} \]

in which \( l \) is the spectral parameter:

\[ l = (l, L) = (\sqrt{k} \sn(\lambda), \sn'(\lambda)) \quad , \quad l' = (l', L') = (\sqrt{k} \sn(\alpha - \lambda), \sn'(\alpha - \lambda)) , \]

and where \( p = (p, P) = (\sqrt{k} \sn(\alpha), \sn'(\alpha)) \). The prefactor \( D \) is determined by setting

\[ \det L_l = \frac{1}{D^2} l l' \mathcal{H}_p(u, \tilde{u}) \sim 1 \]

where \( \mathcal{H}_p \) is the canonical biquadratic:

\[ \mathcal{H}_p(u, \tilde{u}) = p^2(u^2\tilde{u}^2 + 1) - (u^2 + \tilde{u}^2) + 2Pu\tilde{u} = (p^2u^2 - 1) \left( \tilde{u} - \frac{uP - pU}{1 - p^2u^2} \right) \left( \tilde{u} - \frac{uP + pU}{1 - p^2u^2} \right) . \]

The biquadratic is linked to the quadrilateral function \( Q_{p,q} \) of \( Q_4 \) via the identity:

\[ Q_{p,q}(u, \tilde{u}, \hat{u}, \tilde{\hat{u}})Q_{p,q^{-1}}(u, \tilde{u}, \hat{u}, \tilde{\hat{u}}) = \frac{p^2q^2}{1-p^2q^2} \left( \frac{1}{p^2} \mathcal{H}_p(u, \tilde{u})\mathcal{H}_p(\hat{u}, \tilde{\hat{u}}) - \frac{1}{q^2} \mathcal{H}_q(u, \tilde{u})\mathcal{H}_q(\tilde{u}, \tilde{\hat{u}}) \right) . \]
This equation was first found by V. Adler (1998) in Weierstrass form. The Jacobi form was established by J. Hietarinta, (2004)

\[ Q_{p,q}(u, \tilde{u}, \hat{u}, \tilde{\hat{u}}) = p(u\tilde{u} + \hat{u}\tilde{\hat{u}}) - q(u\hat{u} + \tilde{u}\tilde{\hat{u}}) - r(\tilde{u}\hat{u} + u\tilde{\hat{u}}) + pqr(1 + u\tilde{u}\hat{u}\tilde{\hat{u}}) \]

parametrised in terms of Jacobi elliptic functions

\[ p = \sqrt{k} \text{sn}(\alpha; k), \quad q = \sqrt{k} \text{sn}(\beta; k), \quad r = \sqrt{k} \text{sn}(\alpha - \beta; k), \]
\[ P = \text{sn}'(\alpha; k), \quad Q = \text{sn}'(\beta; k), \quad R = \text{sn}'(\alpha - \beta; k), \]

i.e. \( p = (p, P), q = (q, Q) \) and \( r = (r, R) \) lie on the Jacobi type elliptic curve:

\[ \Gamma : \quad X^2 = x^4 - \gamma x^2 + 1 \quad , \quad \gamma = k + 1/k, \]

with modulus \( k \). Note that in this case parameters are related through the group law on the Jacobi type curve

\[ r = \frac{Qp - Pq}{1 - p^2q^2}, \quad R = \frac{[PQ + (k + k^{-1})pq](1 + p^2q^2) - 2pq(p^2 + q^2)}{(1 - p^2q^2)^2} \]

For Adler’s equation in Jacobi form we can identify the elementary solution

\[ u = \sqrt{k} \text{sn}(\xi_0 + n\alpha + m\beta; k), \]

where \( \xi_0 \) is an arbitrary constant. This can be verified directly using standard Jacobi elliptic function identities.

However, this solution does not generate a nontrivial BT chain. Thus, there remains the problem of finding appropriate seed solutions for the BTs!!
Applying the auto-BT we obtain two solutions which we label \( \bar{u} \) and \( u \), given by

\[
\bar{u} = \sqrt{k} \text{sn}(\xi_0 + n\alpha + m\beta + \lambda; k), \quad u = \sqrt{k} \text{sn}(\xi_0 + n\alpha + m\beta - \lambda; k),
\]

where \( \lambda \) is the uniformising variable associated with \( l : l = (l, L) = (\sqrt{k} \text{sn}(\lambda; k), \text{sn}'(\lambda; k)) \) which are just trivial extensions of the original solution.

We call such a seed a non-germinating seed.

To obtain a germinating seed we proceed by constructing fixed points of the auto-BT, going back to an idea of J. Weiss (1986). Thus, we pose as the defining equations of the seed solution:

\[
\begin{align*}
Q_{p,t}(u, \bar{u}, u, \bar{u}) &= 0, \\
Q_{q,t}(u, \bar{u}, u, \bar{u}) &= 0,
\end{align*}
\]

where \( t = (t, T) = (\sqrt{k} \text{sn}(\theta; k), \text{sn}'(\theta; k)) \) is the lattice parameter in the direction of which the auto-BT is constant.

**Remark:** It is not a priori obvious that the above system of eqs. is compatible, but it is a surprising property of discriminants of biquadratics associated with the quadrilaterals \( Q \) of the ABS list that this is indeed the case.

**Remark:** At the special value of the parameter \( t = (0, 1) \) (the unit \( e \) of the Abelian group of the curve) the fixed point solution will reduce to the non-germinating seed.
Fixed point solution and seed curve

The fixed point solution are obtained by considering the biquadratic:

\[ Q_{p,t}(u, \tilde{u}, u, \tilde{u}) = tpr(u^2\tilde{u}^2 + 1) - t(u^2 + \tilde{u}^2) + 2(p - r)u\tilde{u} = tH_{p,\theta}(u, \tilde{u}) , \]

by making the identifications

\[ r = \frac{pT - tP}{1 - p^2t^2} , \quad p_\theta = (p_\theta, P_\theta) , \quad p_\theta^2 = pr , \quad P_\theta = \frac{p - r}{t} . \]

with \( H_p \) defined by:

\[ Q\tilde{u} - Q_u = trH_p \quad \text{with} \quad H_p(u, \tilde{u}) = p^2(1 + u^2\tilde{u}^2) - (u^2 + \tilde{u}^2) + 2Pu\tilde{u} . \]

It conspires that the discriminant of the biquadratic \( H_{p,\theta} \) factorises:

\[ \Delta = 4[(p - r)^2u^2 - t^2(pru^2 - 1)(pr - u^2)] = 4t^2pr \left( \frac{1 - T}{t^2}u^2 \right) , \]

using the relation:

\[ 2prT = p^2 + r^2 - t^2(1 - p^2r^2) . \]

This suggests the introduction of a deformed curve, the seed curve, given by

\[ \Gamma_\theta : \quad X^2 = R_\theta(x) \equiv x^4 + 1 - 2\frac{1 - T}{t^2}x^2 \]

(the label \( \theta \) refers to the uniformising variable for the point \( t \in \Gamma \)). Furthermore, the deformed lattice parameter \( p_\theta \) lies on the deformed curve:

\[ p_\theta = (p_\theta, P_\theta) \in \Gamma_\theta . \]

**Key observation:** The new curve \( \Gamma_\theta \) depends only on the chosen point \( t \in \Gamma \) on the original curve, and not on the lattice parameter \( p \) associated with the shift \( u \mapsto \tilde{u} \).
Thus, precisely a similar analysis holds for the solution of the second equation $Q_{q,t}(u,\hat{u},u,\hat{u}) = 0$ and it leads to the same deformed curve with deformed lattice parameter $q_\theta = (q_\theta, Q_\theta)$ defined by

$$q_\theta^2 = qr' \quad , \quad r' = \frac{qT - tQ}{1 - q^2 t^2} \quad \text{and} \quad Q_\theta = \frac{q - r'}{t}.$$ 

In terms of Jacobi elliptic functions on the original curve with modulus $k$ we have:

$$p = (p, P) = (\sqrt{k} \sin(\alpha; k), \sin'(\alpha; k)) \quad , \quad q = (q, Q) = (\sqrt{k} \sin(\alpha; k), \sin'(\alpha; k)) \quad ,$$

the $\theta$-deformed curve carries a different modulus $k_\theta$ as given by

$$k_\theta + \frac{1}{k_\theta} = 2 \frac{1 - \sin'(\theta; k)}{k \sin^2(\theta; k)}$$

where $\theta$ is the parameter of the point $t = (\sqrt{k} \sin(\theta; k), \sin'(\theta; k))$. Thus, the deformed lattice parameters $p_\theta$ and $q_\theta$ are parametrised by

$$p_\theta = (\sqrt{k_\theta} \sin(\alpha; k_\theta), \sin'(\alpha; k_\theta)) \quad , \quad q_\theta = (\sqrt{k_\theta} \sin(\beta; k_\theta), \sin'(\beta; k_\theta))$$

where $\alpha_\theta$ and $\beta_\theta$ are to solved from:

$$k_\theta \sin^2(\alpha_\theta; k_\theta) = k \sin(\alpha - \theta; k) \sin(\alpha; k) \quad ,$$

$$\sin'(\alpha_\theta; k_\theta) = \frac{\sin(\alpha; k) - \sin(\alpha - \theta; k)}{\sin(\theta; k)} \quad ,$$

and similar equations for $\beta_\theta$, replacing $\alpha$ by $\beta$ in the above formulae.
Seed map

The actual solution of fixed point equations is obtained by solving the biquadratic
\[ tH_{p_\theta}(u, \tilde{u}) = 0 \]

The discriminant is \( \Delta = 4t^2p_\theta^2 R_\theta(u_\theta) \), which coincides with the deformed curve \( \Gamma_\theta \)!

Thus, we obtain the seed solution \( u_\theta = (u_\theta, U_\theta) \) from a seed map \( u_\theta \mapsto \tilde{u}_\theta \) (associated with the shift \( \tilde{\cdot} \)) which leaves the curve \( \Gamma_\theta \) invariant:
\[
\tilde{u}_\theta = \frac{P_\theta u_\theta + p_\theta U_\theta}{1 - p_\theta^2 u_\theta^2} , \quad \tilde{U}_\theta = \frac{(P_\theta U_\theta - (k_\theta + k_\theta^{-1})p_\theta u_\theta) (1 + p_\theta^2 u_\theta^2) + 2p_\theta u_\theta (p_\theta^2 + u_\theta^2)}{(1 - p_\theta^2 u_\theta^2)^2} ,
\]

This map is nothing but the (Abelian) group action on the deformed curve. Evidently, a similar map \( u_\theta \mapsto \hat{u}_\theta \) (associated with the shift \( \hat{\cdot} \)), with lattice parameter \( q \) instead of \( p \), commutes with the map above:
\[
\hat{u}_\theta = \frac{Q_\theta u_\theta + q_\theta U_\theta}{1 - q_\theta^2 u_\theta^2} , \quad \hat{U}_\theta = \frac{(Q_\theta U_\theta - (k_\theta + k_\theta^{-1})q_\theta u_\theta) (1 + q_\theta^2 u_\theta^2) + 2q_\theta u_\theta (q_\theta^2 + u_\theta^2)}{(1 - q_\theta^2 u_\theta^2)^2} .
\]

Since these maps commute (Abelian group on \( \Gamma_\theta \)), they can be simultaneously solved and lead to the germinating seed solution:
\[
u_\theta = \sqrt{k_\theta} \text{sn}(\xi_\theta; k_\theta) , \quad \text{with} \quad \xi_\theta = \xi_{\theta,0} + n\alpha_\theta + m\beta_\theta
\]
in terms of the Jacobi sn function with modulus \( k_\theta \),

The new solution can be used as starting point for application of Bäcklund transformations and germinate non-trivial solutions, i.e. the 1-soliton solutions for \( Q_4 \) in Jacobi form!
Group-theoretical Interpretation:

The “Jacobi quadrilateral”

\[ Q_{p,q,r}(u, \tilde{u}, \hat{u}, \hat{\tilde{u}}) = p(u\tilde{u} + \hat{u}\hat{\tilde{u}}) - q(u\hat{u} + \tilde{u}\hat{\tilde{u}}) - r(\tilde{u}\hat{u} + u\hat{\tilde{u}}) + pqr(1 + u\tilde{u}\hat{\tilde{u}}). \]

in terms of which the Jacobi form of Adler’s system is defined, can also be considered as an object depending on three parameters \((p,q,r)\) (so far unrelated), which generates a canonical biquadratic through:

\[ QQ_{\tilde{u}\hat{\tilde{u}}} - Q_{\tilde{u}}Q_{\hat{\tilde{u}}} = p^2qr(1 + u^2\tilde{u}^2) - qr(u^2 + \tilde{u}^2) + (p^2 - q^2 - r^2 + p^2q^2r^2)u\tilde{u}. \]

If \(p, q, r\) are related through the group law on the curve \(\Gamma\) (with modulus \(k\)) then as a consequence of the identity

\[ 2qrP = p^2(1 + q^2r^2) - q^2 - r^2, \quad \Rightarrow \quad QQ_{\tilde{u}\hat{\tilde{u}}} - Q_{\tilde{u}}Q_{\hat{\tilde{u}}} = qrH_p \]

and can then be written as \( H_p(u, \tilde{u}) = \gamma H(s, u, \tilde{u}) \) in terms of the canonical triquadratic on the curve given by:

\[ H(x, y, z) = \frac{1}{\sqrt{k}}(1 + x^2y^2 + y^2z^2 + z^2x^2) - \sqrt{k}(x^2 + y^2 + z^2 + x^2y^2z^2) - 2\left(k - \frac{1}{k}\right)xyz, \]

identifying

\[ p^2 = \frac{1 - ks^2}{k - s^2}, \quad P = \frac{(k - 1/k)s}{s^2/\sqrt{k} - \sqrt{k}}, \quad \gamma = \left(\sqrt{k} - \frac{s^2}{\sqrt{k}}\right)^{-1}, \]

which forms a rational realisation of the elliptic curve \(P^2 = \mathcal{R}(p)\) in terms of a new parameter \(s\).
Note that for the triquadratic $H$ we have the factorisation property:

$$H_x^2 - 2HH_{xx} = 4\mathcal{R}(y)\mathcal{R}(z), \quad \mathcal{R}(x) \equiv x^4 + 1 - \left(k + \frac{1}{k}\right)x^2,$$

whereas the condition $\mathcal{H}_p(u, \tilde{u}) = 0$ implies that the map $u \mapsto \tilde{u}$ amounts to a shift on this elliptic curve (Chasles correspondence). This translates into the condition that $H(p, q, r) = 0$ is satisfied when $p \ast q \ast r = s$, with $s = (1/\sqrt{k}, 0)$ which is the map of the unit of the curve in the realisation above.

The emergence of the seed solution and corresponding deformed parameters $p_\theta, q_\theta$ can be illustrated by the following commuting diagram:

$$\begin{array}{ccc}
Q_{p,t}(u, \tilde{u}, v, \tilde{v}) = 0 & \overset{QQ_{\tilde{u},\tilde{v}} - Q_{u}\tilde{v}}{\longrightarrow} & \mathcal{H}_p(u, v) = 0 \\
\downarrow \text{proj}_{v \mapsto u} & & \downarrow \text{proj}_{v \mapsto u} \\
\mathcal{H}_{p_\theta}(u, \tilde{u}) = 0 & \overset{\mathcal{H}_2 - 2\mathcal{H}_{\tilde{u},\tilde{u}}}{\longrightarrow} & \mathcal{H}_p(u, u) = \mathcal{R}_\theta(u)
\end{array}$$

defining the correspondence between between the old parameters $p, q$ and the deformed ones, which we indicate by the symbol $\delta_\theta(p, p_\theta)$.
One-Soliton Solution

We will illustrate that the canonical seed solution $u_{\theta}$ germinates —by applying the BT to it and thus obtaining the one-soliton solution of Adler’s equation.

We need to solve the set of simultaneous $\mathcal{O}\Delta\mathcal{E}$s in $v$:

\[ Q_{p,l}(u_{\theta}, \tilde{u}_{\theta}, v, \tilde{v}) = 0, \quad Q_{q,l}(u_{\theta}, \hat{u}_{\theta}, v, \hat{v}) = 0, \]

which define the BT $u_{\theta} \leftrightarrow v$ with Bäcklund parameter $l$, where

\[ u_{\theta}(n, m) = \sqrt{k_{\theta}} \text{sn}(\xi_{\theta}(n, m); k_{\theta}) , \quad \xi_{\theta}(n, m) = \xi_{\theta,0} + n\alpha_{\theta} + m\beta_{\theta} . \]

This seed solution can be covariantly extended in the lattice direction associated with the parameter $l = (l, L) = (\sqrt{k}, \text{sn}(\lambda; k), \text{sn}'(\lambda; k))$, by letting the initial value $\xi_{\theta,0}$ depend on the lattice variable (with lattice shift denoted by $-$) as:

\[ \bar{\xi}_{\theta} = \xi_{\theta} + \lambda_{\theta} \quad \Rightarrow \quad \bar{u}_{\theta} = \sqrt{k_{\theta}} \text{sn}(\xi_{\theta} + \lambda_{\theta}; k_{\theta}) , \]

and where $\lambda_{\theta}$ is the deformed BT parameter, related to $l$ by the same correspondence $\delta_{\theta}(l, l_{\theta})$ as before.

Thus, we have the following set of equations satisfied by these BT-shifted seed solutions:

\[ Q_{p,l}(u_{\theta}, \tilde{u}_{\theta}, \bar{u}_{\theta}, \tilde{\bar{u}}_{\theta}) = 0 , \quad Q_{q,l}(u_{\theta}, \hat{u}_{\theta}, \bar{u}_{\theta}, \hat{\bar{u}}_{\theta}) = 0 , \]
\[ Q_{p,l}(u_{\theta}, \tilde{u}_{\theta}, u_{\theta}, \tilde{u}_{\theta}) = 0 , \quad Q_{q,l}(u_{\theta}, \hat{u}_{\theta}, u_{\theta}, \hat{u}_{\theta}) = 0 , \]

which are in fact discrete Riccati equations of the form:

\[ v\tilde{v} + a\tilde{v} + bv + c = 0 \]

in terms of the third and fourth arguments in these quadrilaterals.
**Lemma:** If $v_1$ and $v_2$ are two given independent solutions of a Riccati equation of the form above, i.e. $\tilde{v}v + a\tilde{v} + bv + c = 0$, then the linear combination

$$v = \frac{v_1 - \rho v_2}{1 - \rho}$$

is a solution of the same equation, provided $\rho$ obeys the following linear homogeneous first order difference equation;

$$\tilde{\rho} = \frac{v_2\tilde{v}_1 + a\tilde{v}_1 + bv_2 + c}{v_1\tilde{v}_2 + a\tilde{v}_2 + bv_1 + c}\rho.$$  

We will use for the Riccati equations coming from the quadrilaterals, which are of the form above by identifying:

$$a = -\frac{ru + lu}{p(1 + rl\tilde{u})}, \quad b = -\frac{r\tilde{u} + lu}{p(1 + rl\tilde{u})}, \quad c = \frac{rl + u\tilde{u}}{1 + rl\tilde{u}},$$

with $r = (pL - lp)/(1 - p^2l^2)$, and $u = u_\theta$, taking $v_1 = \tilde{u}_\theta$ and $v_2 = u_\theta$.

Because the auto-BT share their particular solutions, $\tilde{u}_\theta$ and $u_\theta$, a similar substitution holds for the equations in terms of the other lattice shift (associated with $q$), and thus these substitutions reduce both equations simultaneously.

$$\tilde{\rho} = \left(\frac{p\theta l - l_\theta p}{p\theta l + l_\theta p}\right) \left(\frac{1 - l_\theta \tilde{p}_\theta u_\theta \tilde{u}_\theta}{1 + l_\theta \tilde{p}_\theta u_\theta \tilde{u}_\theta}\right)\rho,$$

$$\hat{\rho} = \left(\frac{q\theta l - l_\theta q}{q\theta l + l_\theta q}\right) \left(\frac{1 - l_\theta \tilde{q}_\theta u_\theta \tilde{u}_\theta}{1 + l_\theta \tilde{q}_\theta u_\theta \tilde{u}_\theta}\right)\rho,$$

We take the above linear equations as the defining equations for the **plane-wave factor** $\rho$ (i.e. a discrete elliptic analogue of the exponential function).
where we mildly abuse notation by introducing the modified parameters

\[ \bar{p}_\theta = \sqrt{k_\theta} \text{sn}(\alpha_\theta + \lambda_\theta; k_\theta), \quad p_\theta = \sqrt{k_\theta} \text{sn}(\alpha_\theta - \lambda_\theta; k_\theta), \]

\[ \bar{q}_\theta = \sqrt{k_\theta} \text{sn}(\beta_\theta + \lambda_\theta; k_\theta), \quad q_\theta = \sqrt{k_\theta} \text{sn}(\beta_\theta - \lambda_\theta; k_\theta) \]

(even though \( p_\theta \) and \( q_\theta \) do not depend on lattice shifts).

The compatibility of the system for \( \rho \) can be verified directly, i.e. specifically \( \tilde{\rho} = \tilde{\rho} \), and arises as a consequence of the following remarkable identity for the Jacobi \( \text{sn} \) function:

\[
\left( \frac{1 - k^2 \text{sn}(\lambda) \text{sn}(\alpha + \lambda) \text{sn}(\xi) \text{sn}(\xi + \alpha)}{1 + k^2 \text{sn}(\lambda) \text{sn}(\alpha - \lambda) \text{sn}(\xi) \text{sn}(\xi + \alpha)} \right) \left( \frac{1 - k^2 \text{sn}(\lambda) \text{sn}(\beta + \lambda) \text{sn}(\xi + \alpha) \text{sn}(\xi + \alpha + \beta)}{1 + k^2 \text{sn}(\lambda) \text{sn}(\beta - \lambda) \text{sn}(\xi + \alpha) \text{sn}(\xi + \alpha + \beta)} \right) = \]

\[
\left( \frac{1 - k^2 \text{sn}(\lambda) \text{sn}(\beta + \lambda) \text{sn}(\xi) \text{sn}(\xi + \beta)}{1 + k^2 \text{sn}(\lambda) \text{sn}(\beta - \lambda) \text{sn}(\xi) \text{sn}(\xi + \beta)} \right) \left( \frac{1 - k^2 \text{sn}(\lambda) \text{sn}(\alpha + \lambda) \text{sn}(\xi + \beta) \text{sn}(\xi + \alpha + \beta)}{1 + k^2 \text{sn}(\lambda) \text{sn}(\alpha - \lambda) \text{sn}(\xi + \beta) \text{sn}(\xi + \alpha + \beta)} \right). \]

The one-soliton for the Jacobi form of Adler’s equation, which we denote \( v_1 \), is thus given by

\[
\begin{align*}
\begin{bmatrix}
(1) \\
n, m
\end{bmatrix} & = \sqrt{k_\theta} \left( \text{sn}(\xi_\theta - \lambda_\theta; k_\theta) - \rho \text{sn}(\xi_\theta + \lambda_\theta; k_\theta) \right), \quad \xi_\theta = \xi_{\theta,0} + n\alpha_\theta + m\beta_\theta
\end{align*}
\]

with \( \rho \) defined in terms of the earlier equations.

Once we have a seed and the 1-soliton solution, we can next use the permutability condition of BTs (which is once again a version of the original quadrilateral lattice equation) to obtain 2-soliton solutions, etc.
Compatible Continuous Systems

The Adler system allows in a similar way as before to take continuum limits.

**Straight continuum limits:** Limit that \( q \to (0,1) \), implying \( \beta \to 0 \) Setting \( \text{sn}(\beta) = \epsilon \to 0 \), we have \( q = \sqrt{k}\epsilon, \hat{u} \to u + \sqrt{\epsilon}u_x + \ldots \), we obtain

\[
p u_x \hat{u}_x = \sqrt{k} \mathcal{H}_p(u, \hat{u})
\]

which is in the form of the Bäcklund transformation of Krichever-Novikov (KN) equation in the form

\[
u_t = u_{xxx} - \frac{3}{2u_x} \left( u_{xx}^2 - u^4 - 1 + \left( k + \frac{1}{k} \right) u^2 \right)
\]

Eliminating the derivative of the BT by composing it with a similar form with parameter \( q \), \( qu_x \hat{u}_x = \sqrt{k} \mathcal{H}_q(u, \hat{u}) \) gives us an equation of the form

\[
q^2 \mathcal{H}_p(u, \hat{u})\mathcal{H}_p(\tilde{u}, \hat{u}) - p^2 \mathcal{H}_q(u, \hat{u})\mathcal{H}_q(\tilde{u}, \hat{u}) = 0
\]

which factors in the form \( Q_{p,q}(u, \tilde{u}, \hat{u}, \hat{\tilde{u}})Q_{p,-q}(u, \tilde{u}, \hat{u}, \hat{\tilde{u}}) \).

**Skew continuum limit:** Limit that \( q \to p \), implying that \( \beta \to \alpha \). Setting \( \beta = \alpha + \epsilon \), we have \( q \sim p + \epsilon\sqrt{k}P, Q \sim P + \epsilon\sqrt{k}p(p^2 - k - 1/k), \) and \( r \sim -\epsilon\sqrt{k} \). Taking the skew limit

\[
u \to \tilde{u} + \epsilon\tilde{u}_\tau + \ldots
\]
we obtain:

\[ pu_\tau = \sqrt{k} \frac{Pu(\tilde{u} + u) + p^2 - u^2 - \tilde{u} u (1 - p^2 u^2)}{\tilde{u} - u} \]

**Full continuum limit:** This is more involved and produces the KN equation in leading order.

Note that the seed and soliton solutions can be easily extended to the continuum limits, and this yields the first nontrivial explicit solutions of the KN equation!

Furthermore, the \( \tau \) differential-difference flow is related to an equation given by R. Yamilov several years ago. It provides a compatible flow with the original lattice equation, which diagrammatically follows the following computation:
So far no closed-form $N$-soliton solution for Adler’s equation was given. In a recent paper [Atkinson, Hietarinta, FWN, JPhys A41 [FTC] (2008)142001] we presented the full $N$-soliton solution for $Q_3$.

$Q_3$: Is just below $Q_4$ in the ABS classification, and can be obtained by the coalescence limit:

$$u \rightarrow \epsilon v , \quad p \rightarrow \epsilon p_0 , \quad q \rightarrow \epsilon q_0 , \quad k \rightarrow \epsilon^2 ,$$
as $\epsilon \rightarrow 0$, in which case the curve becomes rational $P_0^2 + p_0^2 = 1$.

We prefer to write $Q_3$ in the form:

$$P(u\tilde{u} + \tilde{u}\tilde{u}) - Q(u\tilde{u} + \tilde{u}\tilde{u}) = (p^2 - q^2)\left((\tilde{u}\tilde{u} + u\tilde{u}) + \frac{\delta^2}{4PQ}\right)$$

where $P^2 = (p^2 - a^2)(p^2 - b^2)$, $Q^2 = (q^2 - a^2)(q^2 - b^2)$, and the branch points of the new elliptic curve $\pm a, \pm b$ will have a special significance.

The original parametrisation of $Q_3$ by ABS is subtly different from the one which extends from the homotopy eq. and can be related by the identifications:

$$p_1^2 = \frac{p^2 - b^2}{p^2 - a^2} , \quad P = \frac{(b^2 - a^2)p_1}{1 - p_1^2} , \quad q_1^2 = \frac{q^2 - b^2}{q^2 - a^2} , \quad Q = \frac{(b^2 - a^2)q_1}{1 - q_1^2} .$$

while $u = (b^2 - a^2)u_1$, where $p_1$, $q_1$, $u_1$ are the parameters and dependent variable of the original form of $Q_3$. 
If $\delta = 0$ then surprisingly this equation, $(Q_3)_0$, can also be written in the form

$$\frac{1 + (p - a)s - (p + b)s}{1 + (q - a)s - (q + b)s} = \frac{1 + (q - b)s - (q + a)s}{1 + (p - b)s - (p + a)s},$$

which is an equation first found in [FWN, Quispel, Capel, 1983].

It exploits the connection between $(Q_3)_0$ and the "homotopy" equation for $s$, given by

$$u_{n,m} = \tau^n \sigma^m \left( s_{n,m} - \frac{1}{a + b} \right), \quad \tau \equiv \sqrt{\frac{(p + a)(p + b)}{(p - a)(p - b)}}, \quad \sigma \equiv \sqrt{\frac{(q + a)(q + b)}{(q - a)(q - b)}}$$

and the fact that $Q_3$ does not depend on the signs of the branch points $a, b$.

**N-soliton solution of $(Q_3)_\delta$**

$$u_{n,m}^{(N)} = A\tau^n \sigma^m \left[ 1 - (a + b)s_{n,m}(a, b) \right] + B\tau^n \sigma^m \left[ 1 - (a - b)s_{n,m}(a, -b) \right] + C\tau^{-n} \sigma^{-m} \left[ 1 + (a - b)s_{n,m}(-a, b) \right] + D\tau^{-n} \sigma^{-m} \left[ 1 + (a + b)s_{n,m}(-a, -b) \right],$$

where $s_{n,m}(a, b)$ is the relevant $N$-soliton solution of the homotopy equation. Remarkably, the coefficients $A, B, C, D$ are arbitrary, subject to one single condition:

$$AD(a + b)^2 - BC(a - b)^2 = -\frac{\delta^2}{16ab}.$$
Remains to give the \( N \)-soliton solution of the homotopy equation.

It can be shown that these can be given as follows:

\[
s_{n,m}(a, b) = t_c (b1 + K)^{-1} (1 + M)^{-1} (a1 + K)^{-1} r,
\]

where \( K = \text{diag}(k_1, \ldots, k_N) \), \( r = (\rho_1, \ldots, \rho_N)^T \), \( t_c = (c_1, \ldots, c_N) \), and where

\[
\rho_i = \left( \frac{p + k_i}{p - k_i} \right)^n \left( \frac{q + k_i}{q - k_i} \right)^m \rho_i^0, \quad M = (M_{i,j}), \quad M_{i,j} = \frac{\rho_i c_j}{k_i + k_j}.
\]

Extending the number of lattice directions, to include lattice directions associated with the parameters \( \pm a, \pm b \), i.e. setting

\[
\rho_i^0 = \left( \frac{a - k_i}{a + k_i} \right)^\alpha \left( \frac{b - k_i}{b + k_i} \right)^\beta \rho_{i00}^0,
\]

we can have similar relations in all four directions. It can be shown that the solution \( s \) of the homotopy eq. allows the representation

\[
1 - (a + b)s(a, b) = \frac{T_a^{-1}T_b^{-1}f}{f}, \quad f = f_{n,m} = \det(1 + M),
\]

where \( T_a, T_b \) denote the elementary translations in the auxiliary lattice variables \( \alpha, \beta \).

These relations allows us to give the \( N \)-soliton solution of \( (Q_3)_\delta \) in Hirota bilinear form.
**Elliptic Lattice KdV**

A 2-parameter generalisation of the lattice potential KdV equation \( (H_1) \) was proposed some years ago [FWN, S Puttock, JNMP 10 (2003)] given by:

\[
\left( a + b + u - \hat{u} \right) \left( a - b + \hat{u} - \tilde{u} \right) = a^2 - b^2 + f \left( \tilde{s} - \hat{s} \right) \left( \hat{s} - s \right)
\]

\[
\left( a + u - \frac{\hat{w}}{\hat{s}} \right) \tilde{s} - \left( b + u - \frac{\tilde{w}}{\tilde{s}} \right) \hat{s} = \left( a - \hat{u} + \frac{\hat{w}}{\hat{s}} \right) \hat{s} - \left( b - \hat{u} + \frac{\tilde{w}}{\tilde{s}} \right) \tilde{s}
\]

\[
\left( a - \tilde{u} + \frac{w}{s} \right) s + \left( b + \tilde{u} - \frac{\tilde{w}}{\tilde{s}} \right) \tilde{s} = \left( a + \hat{u} - \frac{\hat{w}}{\hat{s}} \right) \hat{s} + \left( b - \hat{u} + \frac{w}{s} \right) s
\]

\[
\left( a + u - \frac{\tilde{w}}{\tilde{s}} \right) \left( a - \tilde{u} + \frac{w}{s} \right) = a^2 - P(s\tilde{s})
\]

\[
\left( b + u - \frac{\hat{w}}{\hat{s}} \right) \left( b - \hat{u} + \frac{w}{s} \right) = b^2 - P(s\hat{s})
\]

associated with the elliptic curve:

\[
y^2 = P(x) = \frac{1}{x} + 3e + fx.
\]

**Note:**

If parm. \( f \to 0 \), the curve degenerates and the equation for \( u \) decouples \( \Rightarrow \) lattice (potential) KdV!
The elliptic lattice KdV was constructed on the basis of a structure involving infinite "elliptic" matrices (i.e possessing a quasi-graded structure) and in terms of an elliptic Cauchy kernel.

The following was established:

- internal consistency (well-posedness) of the system (from the IVP of view);
- the integrability (multidimensional consistency and Lax pair);
- rich class of (soliton-type) solutions.

In fact, the system (1)-(5) is equivalent to a lattice "correspondence" (i.e. multi-valued map in the sense of Veselov & Moser) in terms of variables \( s \) and \( A = u - \frac{w}{s} \).
The Lax pair for the lattice system is given by:

\[(a - k)\tilde{\phi} = L(K)\phi\]
\[(b - k)\hat{\phi} = M(K)\phi\]

with the points \((k, K)\) on the elliptic curve representing the spectral parameter.

Matrices \(L\) and \(M\) are given by:

\[L(K) = \begin{pmatrix}
    a - \tilde{u} + \frac{f}{K}\tilde{w} & 1 - \frac{f}{K}\tilde{s}
    \\
    K + 3e - a^2 + f\tilde{s}s
    \\
    + (a - \tilde{u})(a + u) + \frac{f}{K}\tilde{w}w
    \\
    a + u - \frac{f}{K}\tilde{w}s
\end{pmatrix}\]

\[M(K) = \begin{pmatrix}
    b - \hat{u} + \frac{f}{K}\hat{w}w
    \\
    K + 3e - b^2 + f\hat{s}s
    \\
    + (b - \hat{u})(b + u) + \frac{f}{K}\hat{w}w
    \\
    b + u - \frac{f}{K}\hat{w}s
\end{pmatrix}\]

The discrete Lax equation

\[\tilde{L}M = \hat{M}L \Rightarrow \text{Elliptic Lattice System}\]

Note: although the elliptic dependence in the spectral parameter is not manifest
(compare with polynomial Lax pair for the LL equation found by Bordag and Yanovskii) soliton type solutions depend on the elliptic curve.
**Soliton type solutions**

Introducing the $N \times N$ matrix $M$ with entries

$$M_{ij} = \frac{1 - f/(K_i K_j)}{k_i + k_j} \rho_i , \quad (i, j = 1, \ldots, N)$$

Parameters of the solution $(k_i, K_i)$ are points on the elliptic curve:

$$k^2 = K + 3e + \frac{f}{K},$$

$$r = (r_i)_{i=1,\ldots,N} \text{ vector with components}$$

$$r_i = \left(\frac{a + k_i}{a - k_i}\right)^n \left(\frac{b + k_i}{b - k_i}\right)^m r_i^0,$$

($r_i^0$ are independent of $n, m$).

and this leads to the following explicit formulae for the quantities of interest:

$$u = e \cdot (1 + M)^{-1} \cdot r$$

$$s = e \cdot K^{-1} \cdot (1 + M)^{-1} \cdot r$$

$$w = 1 + e \cdot K^{-1} \cdot (1 + M)^{-1} \cdot k \cdot r$$

in which have employed the vector $e = (1, 1, \ldots, 1)$ and the diagonal matrices

$$K = \text{diag}(K_1, K_2, \ldots, K_N) , \quad k = \text{diag}(k_1, k_2, \ldots, k_N) .$$
Initial value problems on the lattice

Assigning values for \(u\) and \(s\), on a staircase in the lattice, as follows:

- Structure of the eqs: (1) & (2) & (3) \(\Rightarrow\) \(\hat{u}, \hat{s}, \hat{w}\) i.t.o. initial data (4), (5) link \(\tilde{w}, \hat{w}\) to \(w\)
- Consistency of the system: (4) and (5) trivialise through back-substitution of \(\hat{w}\). In effect, this yields a coupled system for \(u\) and \(s\), with one of the eqs. (eq. (3)) being redundant, whilst the eqs. for \(w\) (eqs. (4) and (5)) are consistent with the system (1)&(2). Imposing periodicity on \(w\), the "background value" \(w_0\) is determined through an algebraic equation.
Conclusions

- Seed and soliton solutions have been constructed for the whole family of integrable quadrilateral lattice equations of the ABS list through the use of auto-BTs. The $N$-soliton formulae for the lower members in the list, e.g. $Q_2$, $(H3)_\delta$, $H_2$ $(Q1)_\delta$ can be found by degeneration from the one for $(Q3)_\delta$, whilst the cases $(H3)_0$, $H_2$, $H_1$ and $(Q1)_0$ were already known.

- Seed (0-cycle periodic) and higher seed (1-cycle periodic) solutions have soliton were given for $Q_4$ and soliton solutions were iteratively constructed using the BTs. A closed-form solution for the $N$-soliton solution remains to be found.

- The fact that all ABS equations possess the same type of linearised (discrete exponential type of) solutions may indicate that the $N$-soliton form for $Q_3$ may be generalised to the elliptic case. Although a direct connection between $Q_4$ and the ELKdV system is unknown, such a connection may exist and lead to the closed-form $N$-soliton solutions for $Q_4$.

- In particular the results also extend to the continuum analogues of these lattice equations, in particular in the case of $Q_4$ to the Krichever-Novikov equation and certain elliptic Toda lattices, and elliptic differential-difference equations due to Yamilov.

- All results, including the classification results, only hold so far to the case of single field lattice eqs. (i.e. the scalar case). The higher rank case is fully open, but examples are known (e.g. lattice Boussinesq systems).
Some Recent Literature:


