Gauge-string duality and the structure of large rank Chern-Simons invariants.

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Some Physical Perspective: 't Hooft, Witten, Gopakumar-Vafa

- $Z \approx \int e^{\frac{1}{x}QA^2+VA^3}\,DA \approx \int \sum \mathrm{const} A^{3k} e^{\frac{1}{x}QA^2}\,DA.$
- $\approx \sum x^{-\chi(\Gamma)}\mathrm{Weight}(\Gamma)\,,$ Γ **labeled** trivalent graph.
- $F := \ln(Z) \approx \sum x^{2g-2+h}N^h F_{g,h}$ This looks like strings!
- The fat graphs are actually *instantons at infinity* on T^*S^3.
- Geometric transition does not change the partition function. The boundaries of the surfaces close and they become holomorphic.

$$F = \sum_{g=0}^{\infty} x^{2g-2} F_g(t), \quad \text{with} \quad F_g(t) = \sum_{h=0}^{\infty} t^h F_{g,h}$$

A geometric transition

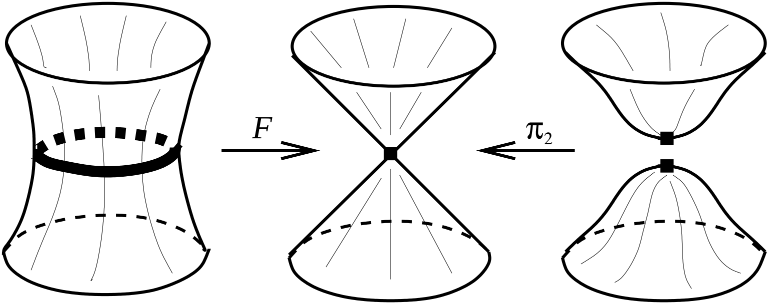
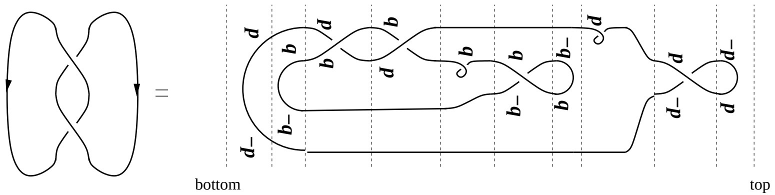


Figure: The ‘conifold’ transition $T^*S^3 \rightsquigarrow X_{S^3} := S^2 \times \mathbb{R}^4$

A mathematical construction – strict modular categories Reshetikhin and Turaev

A tensor product	$\otimes : \mathcal{V} \times \mathcal{V} \Rightarrow \mathcal{V}$
A unit	$\mathbb{1} \in \mathrm{Ob}(\mathcal{V})$
A braiding	$\times_{U,V} : U \otimes V \to V \otimes U$
A twist	$\theta_V : V \to V$
A duality pairing	$\cap_V : V^* \otimes V \to \mathbb{1}$
A copairing	$\cup_V : \mathbb{1} \to V \otimes V^*$
A finite collection of simple objects	$\{V_\lambda\}_{\lambda \in I}$

Must satisfy 17 axioms.



The $\mathfrak{sl}_N$ rank N level k invariant of S^3

Using the SMC of reduced tilting modules $\overline{\mathcal{T}ilt}_\epsilon(\mathfrak{sl}_N)$ with $\epsilon = e^{\pi i/(k+N)}$ and $q = e^{2\pi/(k+N)}$ gives:

$$Z(S^3) = \text{Factor}\cdot G_q^{(2)}(z{+}1), \quad \text{with} \quad z = N \quad \text{and} \quad q = e^{2\pi i/(k+N)}.$$

Here $G_q^{(2)}(z+1)$ is the quantum Barnes function ▸ Definition
Furthermore we have:

$$\begin{aligned} Z'(S^3) &= \text{Different Factor} \cdot G_q^{(2)}(z+1) \\ &= 1 + \mathrm{M}(-q)^{-\chi(S^2)} \sum_{d=1}^{\infty}\sum_{n=0}^{\infty} D_{n,d} q^n a^d \\ &\sim 1 + \sum_{d=1}^{\infty}\sum_{g=0}^{\infty} N^{\bullet}_{g,d} u^{2g-2} a^d, \quad \text{where } a = q^z \quad \text{and} \quad -q = e^{iu} \end{aligned}$$

Free energy from CS side

$$F^{CS}_{S^3} = \frac{N(N-1)}{2}\ln x + \frac{1-N}{2}\ln(k+N) + \frac{N^2}{2}\ln N$$
$$-\tfrac{1}{2}\ln N - 3N^2/4 - \tfrac{1}{12}\ln N - \zeta'(0)N + \zeta'(-1)$$
$$-\sum_{\substack{h \text{ even} \\ h\geq 4}} \frac{2}{h(h-1)(h-2)}(2\pi)^{2-h}\zeta(h-2)(Nu)^h u^{-2}$$
$$+\sum_{\substack{h \text{ even} \\ h\geq 2}} \frac{1}{6h}(2\pi)^{-h}\zeta(h)(Nu)^h$$
$$+\sum_{g=2}^{\infty}\sum_{\substack{h \text{ even} \\ h\geq 2}} \binom{2g+h-3}{h}\frac{B_{2g}}{g(2g-2)}(2\pi)^{2-2g-h}\zeta(2g-2+h)(Nu)^h u^{2g-2}$$
$$+\sum_{g=2}^{\infty}\frac{B_{2g}}{2g(2g-2)}N^{2-2g}.$$

Free energy from GW side

$$\widehat{F}^{GW}(X_{S^3}) = t/24 - \tfrac{1}{12}\ln t + \zeta(3)u^{-2} - \zeta(2)tu^{-2} + 3t^2u^{-2}/4$$

$$+ t^3u^{-2}/12 - \frac{t^2u^{-2}}{2}\ln t$$

$$- \sum_{\substack{h \text{ even} \\ h\geq 4}} \frac{2}{h(h-1)(h-2)}(2\pi)^{2-h}\zeta(h-2)(it)^h u^{-2}$$

$$+ \sum_{\substack{h \text{ even} \\ h\geq 2}} \frac{1}{6h}(2\pi)^{-h}\zeta(h)(it)^h$$

$$+ \sum_{g=2}^{\infty} \sum_{\substack{h \text{ even} \\ h\geq 2}} \binom{2g+h-3}{h} \frac{B_{2g}}{g(2g-2)}(2\pi)^{2-2g-h}\zeta(2g+h-2)(it)^h u^{2g-2}$$

$$+ \sum_{g=2}^{\infty} \frac{B_{2g}}{2g(2g-2)}(it)^{2-2g}u^{2g-2}.$$

BPS states

$$F' := \ln(Z')$$
$$= \sum_{g\in\mathbb{Z}} \sum_{k=1}^{\infty} \sum_{d=1}^{\infty} n_{g,d} k^{-1}(-1)^{g-1} \left((-q)^k - 2 + (-q)^{-k}\right)^{g-1} a^{dk}\,.$$

$$n_{g,d} = 0 \text{ for } g < 0 \text{ and all but a finite number of } (g,d)\,.$$

Exchange rates

BPS states $n_{0,1} = 1$ rest are zero.

DT = SP

$$D_{0,1} = 0,\ D_{1,1} = 1,\ D_{2,1} = -2,\ D_{3,1} = 3,\ \cdots,$$
$$D_{0,2} = D_{1,2} = D_{2,2} = 0,\ D_{3,2} = -2,\ D_{4,2} = 4,\ \cdots$$

Connected GW

$$N_{g,d}(X_{S^3}) = d^{2g-3}(-1)^{g-1}(2g-1)\frac{B_{2g}}{(2g)!}.$$

Non-contracted GW

$$N^{\bullet}_{0,1} = 1,\ N^{\bullet}_{1,1} = 1/12,\ N^{\bullet}_{2,1} = 1/240,\ \cdots$$
$$N^{\bullet}_{-1,2} = 1/2,\ N^{\bullet}_{0,2} = 5/24,\ N^{\bullet}_{1,2} = 13/720,\ \cdots$$

A guess

It appears that there might exist a 3-manifold invariant taking the form $Z_M(a, q)$ such that

- It gives the rank N level k Chern-Simons invariant of M for $a = q^N$, $q = e^{2\pi i/(k+N)}$.
- It has the structure $Z_M(a, q) = \sum Z_d(q)a^d$
- The coefficients of Taylor expansion of $Z_d(q)$ about $q = 0$ are integers.
- The function $Z_d(-e^{iu})$ has an asymptotic expansion at $u = 0$ along the positive reals.
- The functions $Z_d(q)$ satisfy some *funky* modularity properties as evidenced by Hikami and others.
- It is determined by a finite number of integer BPS invariants.

The Taylor and asymptotic coefficients

Taylor coef	**Asymptotic coef**
$=$ Donaldson-Thomas inv.	$=$ Gromov-Witten inv.
Implicit $f(x, y) = 0$	Parametric $x = x(s)$, $y = y(s)$
rank 1 torsion-free sheaves	possibly disconnected curves
$D_{n,\beta} := \int_{[\overline{\mathcal{I}}_n(X,\beta)]^{vir}} 1$	$N^{\bullet}_{g,\beta} := \int_{[\overline{\mathcal{M}}^{\bullet}_g(X,\beta)]^{vir}} 1$

Intuitive description of Donaldson-Thomas

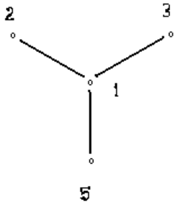
Recall that the Casson invariant is a signed count of the critical points of the classical Chern-Simons functional:

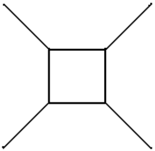
$$\mathrm{CS}(A) := \frac{1}{4\pi^2}\int_M \mathrm{tr}\left(\frac{1}{2}d_{A_0}a\wedge a+\frac{1}{3}a\wedge a\wedge a\right) \quad \text{with} \quad a=A-A_0.$$

Intuitively the DT invariants are a signed count of the critical points of the holomorphic Chern-Simons functional:

$$\mathrm{CS}(A) := \frac{1}{4\pi^2}\int_M \mathrm{tr}\left(\frac{1}{2}\overline{\partial}_{A_0}a\wedge a+\frac{1}{3}a\wedge a\wedge a\right)\wedge\Omega \quad \text{with} \quad a=A-A_0.$$

Witten-Reshetikhin-Turaev; invariants vs. Aganagic, Klemm, Mariño, Vafa; Topological Vertex

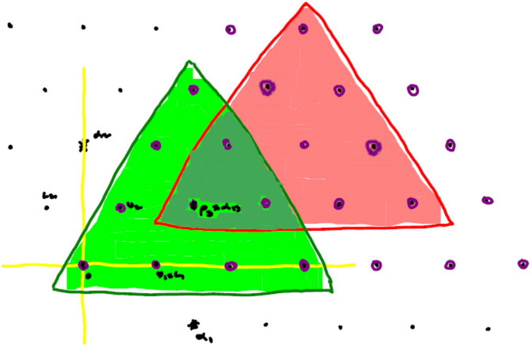




$$Z(\Sigma(2,3,5)) = \Delta^2 D^{-5} \sum_{\vec{\lambda}\in(\mathcal{I}^{N,k})^4} v_{\vec{\lambda}}^{(1,2,3,5)} d_{\vec{\lambda}}^{(1,-1,-1,-1)} W_{\lambda_1\lambda_2} W_{\lambda_1\lambda_3} W_{\lambda_1\lambda_3}$$

$$Z(\mathrm{local}(\mathbb{P}^1)^2) = \sum_{\vec{\lambda}\in\Lambda_+^{\mathfrak{sl}_\infty}} e^{-|\vec{\lambda}|t} q^{\Sigma\frac{1}{2}\kappa(\lambda_p)} \mathcal{W}_{\lambda_4\lambda_1} \mathcal{W}_{\lambda_1\lambda_2} \mathcal{W}_{\lambda_2\lambda_3} \mathcal{W}_{\lambda_3\lambda_4}$$

Simple modules vs all partitions



$$\mathcal{I}^{N=3,k=2}, \Lambda_w^{N=3,k=2} \subseteq \Lambda_+^{\mathfrak{sl}_\infty}$$

$$\Lambda_+^{\mathfrak{sl}_\infty} = \{(\lambda(1), \lambda(2), \cdots) | \lambda(1) \geq \lambda(2) \geq \text{ eg } (4,2,0,\cdots) = \text{[Young diagram]}\}$$

Comparing 2-point functions

$$W_{\lambda_1\lambda_2}(N,q) := s_{\lambda_1}(q^{\rho^N_N}) s_{\lambda_2}(q^{\lambda^N+\rho^N_N})\,.$$

$$\mathcal{W}_{\lambda_1\lambda_2}(q) := s_{\lambda_1}(q^{\rho}_{\infty}) s_{\lambda_2}(q^{\lambda+\rho_\infty})\,.$$

Here $|\lambda| = \Sigma\lambda(p)$, and $s_\lambda(x_1, x_2, \cdots) := \det(x_j^{\lambda(i)-i})/\det(x_j^{-i})$.

$q = e^{2\pi i/(k+N)}$

$\rho_N = (N-1, N-2, \cdots, 2, 1, 0, 0, \cdots)$

$\lambda^N = (\lambda(1) - |\lambda|/N, \lambda(2) - |\lambda|/N, \cdots)$

$\rho^N_N = (\frac{1}{2}(N-1), \frac{1}{2}(N-3), \cdots, \frac{1}{2}(1-N), 0, 0, \cdots)$

q is a formal variable

$\rho_\infty = (-1/2, -3/2, -5/2, \cdots)$

$\lambda = (\lambda(1), \lambda(2), \cdots)$

A first unification of WRT invariants at all ranks – Koshkin

Proposition (Le)

There is an extension of $d_{\vec{\lambda}-\rho} J(L_{\vec{\lambda}-\rho})$ to all weights that is componentwise invariant under the action of the affine Weyl group. Furthermore, this function vanishes on affine domain walls.

Proposition

If $\Gamma \subseteq \Lambda$ are lattices, $f : \Lambda \to \mathbb{C}$ is Γ-periodic, and $\varphi : \Lambda \otimes \mathbb{R} \to \mathbb{C}$ is continuous with sufficiently fast decay, then

$$\sum_{[\lambda] \in \Lambda/\Gamma} f(\lambda) = \frac{vol(\Lambda/\Gamma)}{\int_{\Lambda \otimes \mathbb{R}} \varphi} \lim_{t \to 0} t^{rank(\Lambda)} \sum_{\lambda \in \Lambda} f(\lambda) \varphi(t\lambda)$$

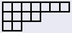
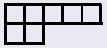
Unification of levels

$$
\begin{aligned}
&Z(M_L)\\
&:= \Delta^\sigma \mathcal{D}^{-c-1} \sum_{\vec{\lambda}\in(\Lambda_w^{N,k})^c} d_{\vec{\lambda}-\rho} J(L_{\vec{\lambda}-\rho})\\
&= (N!)^{-c} \Delta^\sigma \mathcal{D}^{-c-1} \sum_{[\vec{\lambda}]\in(\Lambda_w^{\mathfrak{sl}N}/(N+k)\Lambda_r^{\mathfrak{sl}N})^c} d_{\vec{\lambda}-\rho} J(L_{\vec{\lambda}-\rho})\\
&= (N!)^{-c} \Delta^\sigma \mathcal{D}^{-c-1} \lim_{t\to 0} (t(N+k))^{(N-1)c} \sum_{\vec{\lambda}\in(\Lambda_w^{\mathfrak{sl}N})^c} d_{\vec{\lambda}-\rho} J(L_{\vec{\lambda}-\rho}) e^{-t\sum|\lambda_p|}\\
&= \Delta^\sigma \mathcal{D}^{-c-1} \lim_{t\to 0} (t(N+k))^{(N-1)c} \sum_{\vec{\lambda}\in(\rho+\Lambda_+^{\mathfrak{sl}N})^c} d_{\vec{\lambda}-\rho} J(L_{\vec{\lambda}-\rho}) e^{-t\sum|\lambda_p|}\\
&= \Delta^\sigma \mathcal{D}^{-c-1} \lim_{t\to 0} (t(N+k))^{(N-1)c} \sum_{\vec{\lambda}\in(\Lambda_+^{\mathfrak{sl}N})^c} d_{\vec{\lambda}} J(L_{\vec{\lambda}}) e^{-t\sum|\lambda_p|}
\end{aligned}
$$

Unification of ranks

Definition

If $\lambda \in \Lambda_+^{\mathfrak{sl}_N}$ is a partition then the $\mathfrak{sl}_{N-1}$ reduction of λ is the partition $\bar{\lambda}$ obtained by deleting all columns of length N.

If $\lambda =$ , then $\bar{\lambda} =$ .

Proposition (Lukac)

The $SU(N)$ colored Jones polynomial vanishes if any of the labels have length greater than N. Furthermore,

$$d_{\lambda_1,\cdots,\bar{\lambda}_k,\cdots,\lambda_c} J(L_{\lambda_1,\cdots,\bar{\lambda}_k,\cdots,\lambda_c}) = d_{\lambda_1,\cdots,\lambda_k,\cdots,\lambda_c} J(L_{\lambda_1,\cdots,\lambda_k,\cdots,\lambda_c}).$$

$$
\begin{aligned}
&Z(M_L)\\
&= \Delta^{\sigma}\mathcal{D}^{-c-1}\lim_{t\to 0}(t(N+k))^{(N-1)c}\sum_{(\vec{\lambda}\in\Lambda_+^{\mathfrak{sl}N})^c} d_{\vec{\lambda}}J(L_{\vec{\lambda}})e^{-t\sum|\lambda_p|}\\
&= \Delta^{\sigma}\mathcal{D}^{-c-1}\lim_{t\to 0}(t(N+k))^{(N-1)c}\\
&\quad {\color{red}(1-e^{-tN})\sum_{m=0}^{\infty}e^{-tmN}} \sum_{\substack{\vec{\lambda}\in(\Lambda_+^{\mathfrak{sl}N})^{c-1}\\ \bar{\lambda}_c\in\Lambda_+^{\mathfrak{sl}N}}} d_{\vec{\lambda},\bar{\lambda}_c}J(L_{\vec{\lambda},\bar{\lambda}_c})e^{-t(\sum|\lambda_p|+|\bar{\lambda}_c|)}\\
&= \Delta^{\sigma}\mathcal{D}^{-c-1}\lim_{t\to 0}(t(N+k))^{(N-1)c}(1-e^{-tN})\\
&\quad \sum_{\substack{\vec{\lambda}\in(\Lambda_+^{\mathfrak{sl}N})^{c-1}\\ \lambda_c\in\Lambda_+^{\mathfrak{sl}{\color{red}N+1}}}} d_{\vec{\lambda},\lambda_c}J(L_{\vec{\lambda},\lambda_c})e^{-t(\sum|\lambda_p|+|\lambda_c|)}
\end{aligned}
$$

Yes, but $\cdots$

$$Z(M_L)$$
$$= \Delta^{\sigma}\mathcal{D}^{-c-1} \lim_{t\to 0} t^{Nc} N^c \left((N+k)\right)^{(N-1)c} \sum_{\vec{\lambda}\in(\Lambda_+^{\mathfrak{sl}_\infty})^c} d_{\vec{\lambda}} J(L_{\vec{\lambda}}) e^{-t\sum|\lambda_p|}$$

- This *does* unify all ranks and levels.
- It does not decompose into terms with
- a sensible Taylor expansion about $q = 0$ and
- a sensible asymptotic expansion about $q = -1$.

Perhaps it exists in some modification of the Habiro ring???

$$\sum_{k=0}^{\infty} 3^k = \frac{1}{1-3}\,.$$

This material for this talk came from the following references and listening to Sergiy Koshkin and Marcos Mariño.

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A modular integral satisfies $\tau^{-w} f(-1/\tau) = f(\tau) + p(\tau)$

▸ return

Definition

The MacMahon function is given by:

$$M(q) := \prod_{n=1}^{\infty} (1 - q^n)^{-n}$$

return

$$\text{Factor} =$$

$$(-i)^{N(N-1)/2} N^{-1/2} (k+N)^{(1-N)/2} q^{-2N(N^2-1)/24} (1-q)^{N(N-1)/2}$$

return

$$\text{Different Factor} = (q;q)_\infty^{-N} M(q)^{-1} (1-q)^{N(N-1)/2}$$

where $(q;q)_\infty = \prod_{n=1}^{\infty}(1-q^n)$. ▸ return

The quantum Barnes function

Definition

The quantum Barnes function hihierarchy is the unique collection of meromorphic functions on the disk satisfying:

1. $G_q^{(0)}(z) = (z)_q = \frac{1-q^z}{1-q}$
2. $G_q^{(d)}(1) = 1$
3. $G_q^{(d)}(z+1) = G_q^{(d-1)}(z) G_q^{(d)}(z)$
4. $\left(\frac{d}{dx}\right)^{d+1} \ln G_q^{(d)}(x) \geq 0$ for $x \geq 0$, $q \in (0,1)$

return

Notation

$$\epsilon = \exp(\pi i/(k+N))\,, \quad q = \epsilon^2\,, \quad v_\lambda = \epsilon^{(\lambda,\lambda+2\rho)}$$

$$d_\lambda = \dim_q \mathcal{W}^\epsilon_\lambda = \prod_{\alpha\in\Delta^+} \frac{\epsilon^{(\lambda+\rho,\alpha)} - \epsilon^{-(\lambda+\rho,\alpha)}}{\epsilon^{(\rho,\alpha)} - \epsilon^{-(\rho,\alpha)}}$$

$$d^{\vec{a}}_{\vec{\lambda}} := \prod_{p=1}^{c} d^{a_p}_{\lambda_p}\,, \quad v^{\vec{a}}_{\vec{\lambda}} := \prod_{p=1}^{c} v^{a_p}_{\lambda_p}$$

$$\mathcal{D} = i^{w_0}|\Lambda_w/(k+c_{\mathfrak{g}})\Lambda_r|^{1/2} \prod_{\alpha\in\Delta^+} (\epsilon^{(\rho,\alpha)} - \epsilon^{-(\rho,\alpha)})^{-1}\,,$$

$$= N^{1/2}(k+N)^{(N-1)/2} \prod_{j=1}^{N-1} \left(2\sin\left(\frac{\pi j}{k+N}\right)\right)^{j-N}.$$

$$\Delta = \mathcal{D}^{-1} \sum_{\lambda\in\mathcal{I}^{N,k}} v_\lambda^{-1} d_\lambda^2\,.$$

return to vertex return to unification

Colors at level k and rank N

$$\mathcal{I}^{N,k} := \{\lambda \in \Lambda_w^{\mathfrak{sl}_N} | 0 < (\lambda+\rho,\alpha) < k+N\,,\ \text{for all } \alpha \in \Delta^+\}\,.$$

return to vertex return to 2-point

Definition

The Bernoulli numbers B_k are defined by their generating function:

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} B_k \frac{z^k}{k!}\,.$$

Definition

The Riemann zeta function is defined by:

$$\zeta(z) := \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{u^{z-1}}{e^u - 1}\, du\,,$$

where $\Gamma(z)$ is the usual gamma function of Euler

$$\Gamma(z) := \int_0^{\infty} e^{-t} t^{z-1}\, dt\,.$$

Stable pairs

Definition

A stable pair is a non-zero section $s : \mathcal{O}_X \to F$ of a pure sheaf F with Hilbert polynomial $\chi(F \otimes L^{\otimes k}) = k \int_\beta c_1(L) + n$.