A geometric alternative to gauge fixing in Chern-Simons theory

Chern-Simons and WZW path integrals

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Chern-Simons Gauge Theory: 20 years after
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Notation

M	**a Riemannian three-manifold, with base point n ($M = S^3$)**
P	**a principal G-bundle over M (a product bundle $P = M \times G$.)**
$\mathcal{A}$	**the space of connections on P ($\Lambda^1(M, \mathbf{g})$)**
$\mathcal{G}_n$	**gauge transformations which are the identity in the fiber over n. ($\mathbf{Map}_n(M, G)$)**

The action and path integral

$$\langle f \rangle = \frac{1}{Z_0} \int f([A]) e^{iS(A)} \, \mathcal{D}A$$

$$S(A) = \frac{k}{4\pi} \int_{S^3} \left(A \wedge \mathbf{d}A + \frac{1}{3} A \wedge [A \wedge A] \right)$$

Here f is a function of the gauge orbit $[A]$, and, heuristically, $\mathcal{D}A$ is the Lebesgue measure on the affine linear space $\mathcal{A}$ pushed forward to $\mathcal{A}/\mathcal{G}_n$.

For link invariants, f is the product of the traces in given representations of holonomies around the link components.

Z_0 is the partition function; that is, the path integral with $f([A]) = 1$.

Main theorem: ***$\langle f \rangle$ is given by the path integral for the expectation of a function $\hat{f}$ in the WZW model of maps from S^2 to G.***

The geometry of $\mathcal{A}/\mathcal{G}_n$

Define a map $\xi : \mathcal{A}/\mathcal{G}_n \to \Omega^2 G$ as follows: For $e \in S^2$, let $\gamma(e)$ be the closed, piecewise-smooth path on S^3 which starts at the north pole n of S^3, follows a fiducial longitude from the north pole to the south pole and then returns to n along the longitude indexed by e.

For a given connection A, the holonomy about $\gamma(e)$ gives[1] an element $X_A(e) \in G$, hence a[2] map $X_A \in \Omega^2 G$.

Set $\xi([A]) = X_A$. This map is well-defined, since the action of $\mathcal{A}/\mathcal{G}_n$ on A leaves $X_A(e)$ unchanged.

[1] along with a choice of initial point in the fiber over n

[2] based, since $X_A(e_0) = 1$, where e_0 refers to the fiducial longitude

$\mathcal{A}/\mathcal{G}_n$ as a bundle over $\Omega^2 G$

Let $P_0 : \wedge^1(M, \mathbf{g}) \to \wedge^1(M, \mathbf{g})$ denote projection onto the longitudinal component.

Theorem: ***The mapping ξ takes $\mathcal{A}/\mathcal{G}_n$ onto $\Omega^2 G$. The fiber of ξ is an affine-linear space modelled on Ker P_0.***

"Onto" follows from $\pi_2(G) = 0$. Given X, build a parallel transport by the desired connection from the homotopy from 1 to X.

For the fiber, given $\tau \in$ Ker P_0, $X_{A+\tau} = X_A$, so $\xi([A + \tau]) = \xi([A])$.

Conversely, if $X_A = X_B$ then for each $e \in S^2$ the respective lifts of $\gamma(e)$ satisfy $\tilde{\gamma}_A = \tilde{\gamma}_B g^{-1}$ for g a map from S^3 to G with $g(n) = \mathbf{1}$. This map serves to define a gauge transformation ϕ for which $B = \phi \circ (A + \tau)$.

The path integral on this bundle

Express the path integral as an iterated integral first over the fiber, then the base, as

$$\int f([A])e^{iS(A)}\,\mathcal{D}A = \int f(A+\tau)e^{iS(A+\tau)}\,\mathcal{D}\tau\,\left[\det(D_A^* P_0 D_A)\right]^{1/2}\mu_{\mathbf{base}},$$

where $\mu_{\mathbf{base}}$ **is the measure on** $\Omega^2 G$ **induced by the metric on** $\mathcal{A}/\mathcal{G}_n$**.**

The determinant factor expresses the push-forward of the Lebesgue measure on $\mathcal{A}$ **in terms of the metric measure on** $\mathcal{A}/\mathcal{G}_n$**.**

Choosing the origin in each fiber

For a three-manifold M *with boundary* ∂M, and τ restricted to vanish in one direction at each point,

$$S(A+\tau) = S(A) + \frac{k}{4\pi}\left[\int_M (\tau \wedge D_A\tau + 2\,F_A \wedge \tau) + \int_{\partial M} A \wedge \tau\right]$$

Choose the origin$[\tilde{A}]$ so that the term linear in τ vanishes.

Finding $\tilde{A}$

On S^3, $\int_{S^3} F_A \wedge \tau = 0$ requires A to be flat in the longitudinal direction.

$A = \widehat{X}^{-1}\mathbf{d}\widehat{X} + \widehat{X}^{-1}C\widehat{X}$ **where $\widehat{X}$ is a (based) map from $S^3 - \{n\} \sim B^3$ to G and C is a connection on the product bundle on S^2.**

To get a connection continuous on B^3, take $C = 0$ and define

$$\tilde{A} = \widehat{X}^{-1}\mathbf{d}\widehat{X}.$$

To define the origin in the fiber over X, first choose $\widehat{X}$ such that $\widehat{X}\big|_{\partial B^3} = X$. Then let $[\tilde{A}]$ be the origin in the fiber. (Well-defined, because $\widehat{X}$ represents $\tilde{\gamma}_{\tilde{A}}$.)

For $\tilde{A} + \tau$ to be continuous on S^3,

$$\tau\big|_{\partial B^3} = -\left.\widetilde{A}\right|_{\partial B^3}.$$

The effect of this choice of origin

For the given choice of origin,

$$S(\widetilde{A}+\tau) = S(\widetilde{A}) + \frac{k}{4\pi}\int \tau \wedge D_A\tau,$$

the possible boundary termvanishing due to the boundary condition on τ.

Therefore, for f constant along the fibers, the inner integral is

$$\int e^{i\frac{k}{4\pi}\int \tau\wedge D_A\tau}\,\mathcal{D}\tau,$$

taken over elements of the subspace of Ker P_0 whose elements satisfy the boundary condition.

Integrating over the fiber

Decompose Ker P_0 into eigenspaces of the Hodge operator (restricted to non-longitudinal directions). Integration by parts gives

$$\int_{B^3} \tau \wedge D_A \tau = 2 \int_{B^3} D_A \tau_- \wedge \tau_+ - \int_{\partial B^3} \tau_- \wedge \tau_+.$$

On the boundary, $\tau_+ = A_+ = X^{-1} \partial X$, so

$$\int e^{i \frac{k}{4\pi} \int_{S^3} \tau \wedge D_A \tau} \mathcal{D}\tau = e^{i \frac{k}{4\pi} \int_{S^2} \partial X \wedge \bar{\partial} X} \int e^{i \frac{k}{2\pi} \int_{S^3} D_A \tau_- \wedge \tau_+} \mathcal{D}\tau_- \mathcal{D}\tau_+.$$

Path integral heuristics imply the integral gives $det\, |P_- D_A^* D_A P_-|\, e^{i \frac{\pi}{2} \eta(0)}$, where the η invariant appears due to the existence of negative eigenvalues of the quadratic form in τ, the operator is on $\tau \in$ Ker P_0 which are continuous on S^3, and the determinant is understood to be regularized.

Thus,

$$\langle f \rangle = \frac{1}{Z_0} \int f(X) e^{iS(\widetilde{A})} e^{i \frac{k}{4\pi} \int \partial X \wedge \bar{\partial} X} \frac{det^{\frac{1}{2}}\, (D_A^* P_0 D_A)}{det^{\frac{1}{2}}\, |P_- D_A^* D_A P_-|} e^{-i \frac{\pi}{4} \eta(0)}\, \mu_{\mathbf{base}},$$

Descent to $\Omega^2 G$

$$\langle f\rangle = \frac{1}{Z_0}\int f(X) e^{iS(\widetilde{A})} e^{i\frac{k}{4\pi}\int \partial X\wedge\bar{\partial}X} \frac{det^{\frac{1}{2}}\left(D_A^* P_0 D_A\right)}{det^{\frac{1}{2}}\left|P_- D_A^* D_A P_-\right|} e^{-i\frac{\pi}{4}\eta(0)}\,\mu_{\mathbf{base}},$$

Direct calculation gives

$$S(\tilde{A}) = \frac{k}{12\pi}\int_{S^3}\left(\widehat{X}^{-1}\mathbf{d}\widehat{X}\right)^3.$$

Since the measure $\mu_{\mathbf{base}}$ comes from the metric $\mathcal{A}/\mathcal{G}_n$,

$$\mu_{\mathbf{base}} = J(X)\,\mathcal{D}X,$$

where $\mathcal{DX}$ denotes the heuristic measure on $\Omega^2 G$ coming from a metric on S^2 and an inner product on g.

Calculation shows J depends on the metric on S^2 but not on X.
Likewise, the ratio of determinants proves to be constant in X.

Noting that Z_0 cancels such constants,

Theorem: ***For a function f on $\mathcal{A}/\mathcal{G}_n$ which is constant along the fibers,***

$$\langle f\rangle = \frac{1}{Z_0}\int f(X) e^{i\frac{k}{4\pi}\int_{S^2}\partial X\wedge\bar{\partial}X + i\frac{k}{12\pi}\int_{B^3}\left(\widehat{X}^{-1}\mathbf{d}\widehat{X}\right)^3}\mathcal{D}X,$$

Other manifolds

The above constructions readily generalize to any manifold obtained from $I \times \Sigma$ by boundary identifications.

B^3**: Fix a connection A on ∂B^3. Requiring $\tilde{A} + \tau = A$ on the boundary introduces new boundary terms leading for the partition function to**

$$\int e^{ikS(B)} \mathcal{D}B =$$

$$\int e^{i\frac{k}{4\pi}\int_{S^2} \partial X \wedge \bar{\partial} X + i\frac{k}{12\pi}\int_{S^3}\left(\widehat{X}^{-1}\mathbf{d}\widehat{X}\right)^3 + i\frac{k}{4\pi}\int_{S^2} A_+ \wedge A_- - i\frac{k}{2\pi}\int_{S^2} A_+ \wedge X^{-1}\bar{\partial} X}.$$

in agreement with Witten's Ansatz for a WZW path integral to represent a Chern-Simons state.

$S^1 \times \Sigma$: Projection and origin

Think of $S^1 \times \Sigma$ as $I \times \Sigma$ with the boundary copies of Σ identified. Holonomies about paths parametrizing $I \times \{e\}$ take a connection B to a map $X_B \in \mathrm{Map}(\Sigma, G)$, invariant under gauge transformations that are the identity over $\{0\} \times \Sigma$. Connections modulo such gauge transformations are again an affine-linear bundle, now over $\mathrm{Map}(\Sigma, G)$.

For the origin in the fiber over X, let $\widehat{(}X)$ map $I \times \Sigma$ to G with

$$\widehat{X}\big|_{\{0\}\times\Sigma} = \mathbf{1} \text{ and } \widehat{\mathbf{X}}\big|_{\{\mathbf{1}\}\times\boldsymbol{\Sigma}} = \mathbf{X}.$$

Set

$$\tilde{A} = \widehat{X}^{-1}\mathbf{d}\widehat{X} + \widehat{X}^{-1}A\widehat{X},$$

for A a connection on $\{0\} \times \Sigma$. This is again the most general choice that makes the fiber action purely quadratic (continuous on $I \times \Sigma$.)

For $\tilde{A} + \tau$ to be continuous on $S^1 \times \Sigma$,

$$\tau\big|_{\{0\}\times\Sigma} = 0 \text{ and } \tau\big|_{\{1\}\times\Sigma} = A - X \circ A.$$

The result

Integrating over the fiber now leads to a path integral with the induced action

$$S_{\mathbf{induced}}(X, A) = \frac{k}{12\pi}\int_{I\times\Sigma}\left(\widehat{X}^{-1}\mathbf{d}\widehat{X}\right)^3 + \frac{k}{4\pi}\int_\Sigma X^{-1}\mathbf{d}X \wedge X^{-1}AX + \frac{k}{4\pi}\int_\Sigma X\circ A\wedge(A - X\circ A) + \frac{k}{4\pi}\int_\Sigma (A - X\circ A)_+ \wedge (A - X\circ A)_-.$$

This is precisely the action of the G/G model. Indeed, the Chern-Simons partition function on $S^1\times\Sigma$ is the partition function of the G/G model.

Relation to quasi-axial gauge

Hahn and Haro treat $S^1 \times \Sigma$ using quasi-axial gauge-fixing.

They work out the fiber integration for a fairly general class of Wilson lines – and rigorously construct these Gaussian path integrals.

They then fix a gauge in which X lies in a maximal torus and A lies in the tangent to this torus. They use this to write the integral over the base in the form of a path integral they can again rigorously construct.

They confirm the rigorous constructions agree with Hahn heuristic evaluations, and hence the expected relation to Turaev's shadow invariant.

In the above picture, this means they are able to evaluate the path integral for the G/G model.

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