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Functional integration and abelian link invariants

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• E. Guadagnini and F. Thuillier, SIGMA 4 (2008) 078, arXiv:0801.1445.

• M. Bauer, G. Girardi, R. Stora and F. Thuillier, JHEP 0508 (2005) 027.

• E. Guadagnini, F. Mancarella, Abelian link invariants and homology, Preprint IFUP 2009.

Main subjects

• Functional integration (M = 3-manifold , A = connection)

Some sort of "partition function"

$$I(M) = \int_M DA\, e^{iS_{CS}[A]} = \int_M d\mu(A) = ?$$

Expectation value

$$\langle F[A] \rangle \Big|_M = \frac{\int_M DA\, e^{iS_{CS}[A]}\, F[A]}{\int_M DA\, e^{iS_{CS}[A]}} = ?$$

• Abelian link invariants and homology

Field variables

Abelian quantum CS field theory, $U(1)$ connection A (let $C \subset M$)

• $\int_C A$ invariant under $U(1)$ gauge transformations;
locally, 1-form $A \sim A + d\Lambda$

• observables $= \{\exp[2\pi i \int_C A]\} \Longrightarrow A \sim A + \widehat{A}\ ,\ \int_C \widehat{A} = n \in \mathbb{Z}$
class $A \in H^1_D(M)$ **Deligne-Beilinson** coh. gr. of M of degree 1

$$0 \longrightarrow \Omega^1(M)/\Omega^1_{\mathbb{Z}}(M) \longrightarrow H^1_D(M) \longrightarrow H^2(M) \longrightarrow 0$$

$\Omega^1(M) = \{\text{1-forms on } M\}\ ,\ H^p(M) = (p)^{th}$ int. coh. gr. of M

Action and observables

$$\text{*-product} = \text{pairing} \qquad H^1_D(M) \otimes H^1_D(M) \longrightarrow H^3_D(M)$$

$$A * A \to A \wedge dA \quad , \quad S = \int_M A * A \to \int_M A \wedge dA$$

$$\exp\Big\{2\pi i k S\Big\} = \exp\left\{2\pi i k \int_M A * A\right\} \quad , \quad k \in \mathbb{Z} \quad , \quad k \neq 0$$

link $L \subset M$ components $\{C_1, C_2, ..., C_N\}$ with colours $q_j \in \mathbb{Z}$

$$W(L) = \prod_{j=1}^{N} \exp\Big\{2i\pi q_j \int_{C_j} A\Big\} = \exp\Big\{2i\pi \sum_j q_j \int_{C_j} A\Big\}$$

$$\langle W(L) \rangle|_M = \frac{\int_M DA\, e^{2\pi i k S}\, W(L)}{\int_M DA\, e^{2\pi i k S}}$$

Standard perturbation theory

$$\langle e^{i\int J\phi}\rangle = \frac{\int D\phi\, e^{i\int[\frac{1}{2}\phi\nabla\phi + J\phi]}}{\int D\phi\, e^{i\int\frac{1}{2}\phi\nabla\phi}}$$

$$\int \frac{1}{2}\phi\nabla\phi + J\phi = \int \frac{1}{2}\left(\phi + \nabla^{-1}J\right)\nabla\left(\phi + \nabla^{-1}J\right) - \frac{1}{2}\int J\nabla^{-1}J$$

$$\langle e^{i\int J\phi}\rangle = \frac{\int D\phi\, e^{i\int\frac{1}{2}\left(\phi+\nabla^{-1}J\right)\nabla\left(\phi+\nabla^{-1}J\right)}}{\int D\phi\, e^{i\int\frac{1}{2}\phi\nabla\phi}} \times e^{-\frac{i}{2}\int J\nabla^{-1}J}$$

$D\phi = D\left(\phi + \nabla^{-1}J\right)$ $\leftrightarrow$ invariance under translations

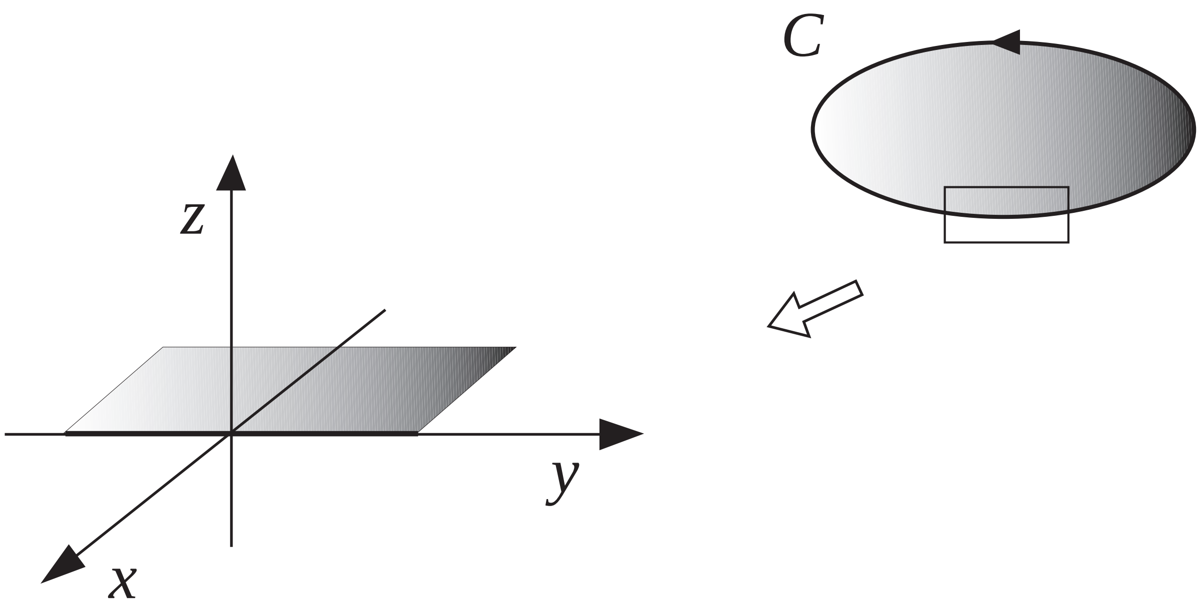
$$\langle e^{i\int J\phi}\rangle = e^{-\frac{i}{2}\int J\nabla^{-1}J}$$

gauge inv. $\Longrightarrow$ ∇^{-1} does not exist (in whole space of functions)

$\Longrightarrow$ gauge-fixing procedure

Distributional forms

$$\int_C A = \int_M A \wedge J_C = \int_M A \wedge d\eta_C = \int_M A * \eta_C$$



$$J_C = \delta(z)\,\delta(x)\,dz \wedge dx \qquad , \qquad \eta_C = \delta(z)\,\theta(-x)\,dz$$

link $L \subset M$ components $\{C_1, C_2, ..., C_N\}$ with colours q_j

$\eta_L = \sum_j q_j \, \eta_{C_j}$

$$\langle W(L) \rangle \Big|_M = \frac{\int_M DA \;\exp\left\{2i\pi \int_M k\, A * A + A * \eta_L\right\}}{\int_M DA \;\exp\left\{2i\pi \int_M k\, A * A\right\}}$$

(assuming m.s. and invariance under translations)

- Ambient isotopy invariance
- Colour periodicity; (inv. under $q \to q + 2k$), $q \in \mathbb{Z}_{2k}$
- Satellite relations

$M = S^3$

$$H^1_D\left(S^3,\mathbb{Z}\right) \simeq \Omega^1\left(S^3\right)/\Omega^1_{\mathbb{Z}}\left(S^3\right) = \Omega^1\left(S^3\right)/d\Omega^0\left(S^3\right)$$

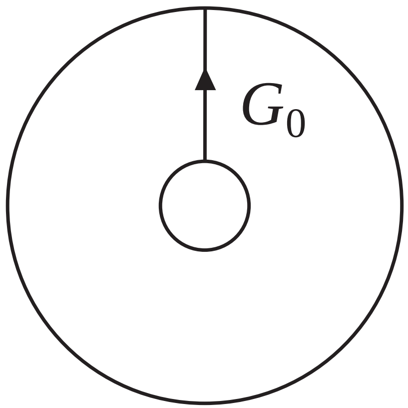
$\eta_L \leftrightarrow$ can be represented by a 1-form globally defined in S^3

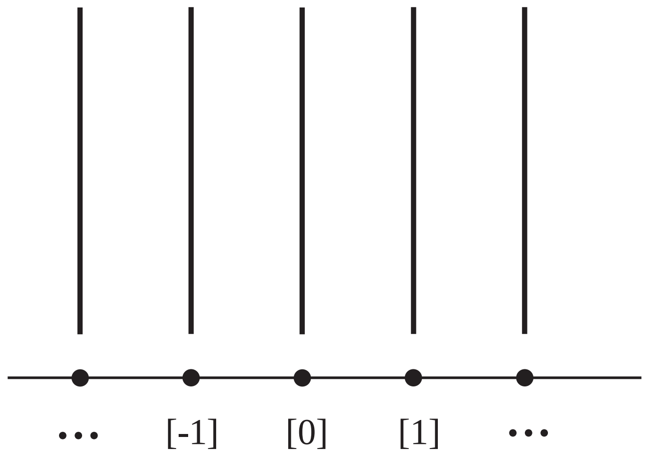
let $A \rightarrow A - \frac{1}{2k}\eta_L$ in the numerator

$$\langle W(L)\rangle\Big|_{S^3} = \exp\left\{-(2i\pi/4k)\int_{S^3}\eta_L * \eta_L\right\} = \exp\Big\{-(2i\pi/4k)\sum_{ij} q_i \mathbb{L}_{ij} q_j\Big\}$$

where $\mathbb{L}_{ij}$ = linking matrix

$$M = S^1 \times S^2$$





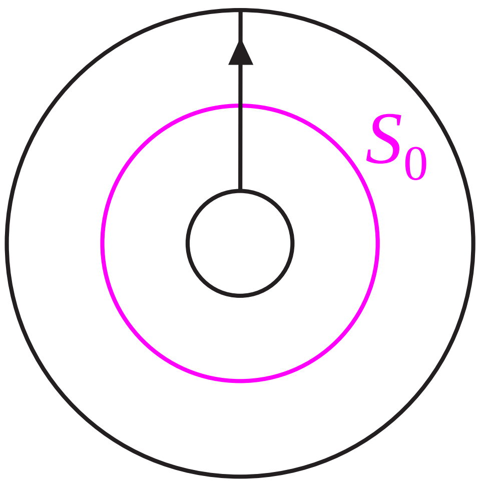
$$0 \longrightarrow \Omega^1(M)/\Omega^1_{\mathbb{Z}}(M) \longrightarrow H^1_D(M) \longrightarrow H^2(M) \longrightarrow 0$$

$$\gamma_0 \in H^1_D\left(S^1 \times S^2\right) \leftrightarrow G_0$$

$$\begin{aligned} s : \mathbb{Z} \quad &\longrightarrow \quad H^1_D\left(S^1 \times S^2\right) \\ n \quad &\longmapsto \quad s(n) \equiv n\gamma_0 \end{aligned}$$

$$A = n\,\gamma_0 + \alpha \quad , \quad \alpha \in \Omega^1\left(S^1 \times S^2\right)/\Omega^1_{\mathbb{Z}}\left(S^1 \times S^2\right)$$

$$
\begin{aligned}
d\mu(A) &= \sum_{n=-\infty}^{+\infty} D\alpha \, \exp\left\{2i\pi k \int_{S^1\times S^2} (n\gamma_0+\alpha) * (n\gamma_0+\alpha)\right\} \\
&= \sum_{n=-\infty}^{+\infty} D\alpha \, \exp\left\{2i\pi k \int_{S^1\times S^2} \alpha * \alpha\right\} \exp\left\{4i\pi kn \int_{S^1\times S^2} \alpha * \gamma_0\right\}
\end{aligned}
$$



Let $S_0 \subset S^1 \times S^2$ be a nontrivial 2-sphere $\leftrightarrow$ β_0 = distr. 1-form so that $\int_{G_0} \beta_0 = 1$; note that $d\beta_0 = 0$

$$
d\mu(A) = d\mu(A + (m/2k)\beta_0) \qquad , \qquad m \in \mathbb{Z}
$$

$$\langle W(L)\rangle = \frac{\int d\mu(A)\, e^{2i\pi \int_{S^1\times S^2} A*\eta_L}}{\int d\mu(A)} = Z^{-1}\int d\mu(A)\, e^{2i\pi \int_{S^1\times S^2} A*\eta_L}$$

$$\begin{aligned}
\langle W(L)\rangle \quad &= Z^{-1}\frac{1}{2k}\sum_{m=0}^{2k-1}\int d\mu(A+(m/2k)\beta_0)\, e^{2i\pi \int_{S^1\times S^2}(A+(m/2k)\beta_0)*\eta_L} \\
&= Z^{-1}\int d\mu(A)\, e^{2i\pi \int_{S^1\times S^2} A*\eta_L}\ \frac{1}{2k}\sum_{m=0}^{2k-1} e^{2i\pi \int_{S^1\times S^2}(m/2k)\beta_0*\eta_L} \\
&= \langle W(L)\rangle\ \frac{1}{2k}\sum_{m=0}^{2k-1}\exp\left\{2i\pi(m/2k)\int_L \beta_0\right\}
\end{aligned}$$

$$\frac{1}{2k}\sum_{m=0}^{2k-1}\exp\left\{2i\pi(m/2k)\int_L \beta_0\right\} = \begin{cases} 1 \text{ if } \ \int_L \beta_0 \equiv 0 \bmod 2k, \\ 0 \text{ otherwise.}\end{cases}$$

$M = S^1 \times S^2$

$L =$ oriented framed coloured link in $S^1 \times S^2$

$[L] \in H_1(S^1 \times S^2) \sim \mathbb{Z}$. For fixed k,

functional integration gives

- if $[L] \not\equiv 0 \bmod 2k$, then $\quad \langle W(L) \rangle \Big|_{S^1 \times S^2} = 0$

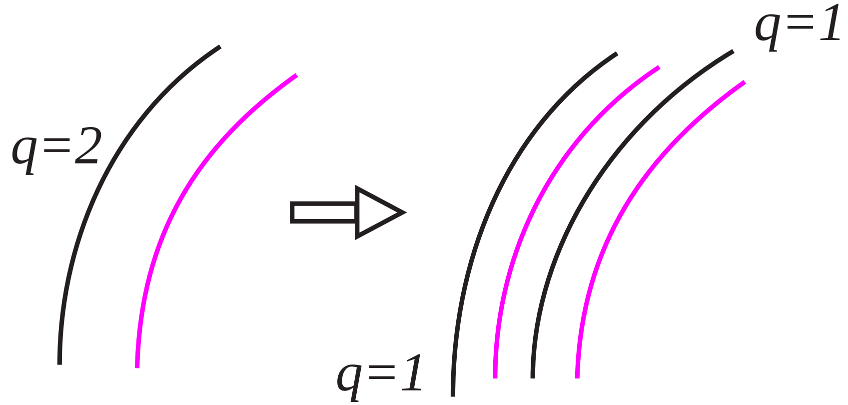
- if $[L] \equiv 0 \bmod 2k$, then

$$\langle W(L) \rangle \Big|_{S^1 \times S^2} = \exp\Big\{ -(2i\pi/4k) \sum_{ij} q_i \mathbb{L}_{ij} q_j \Big\}$$

Abelian link invariants

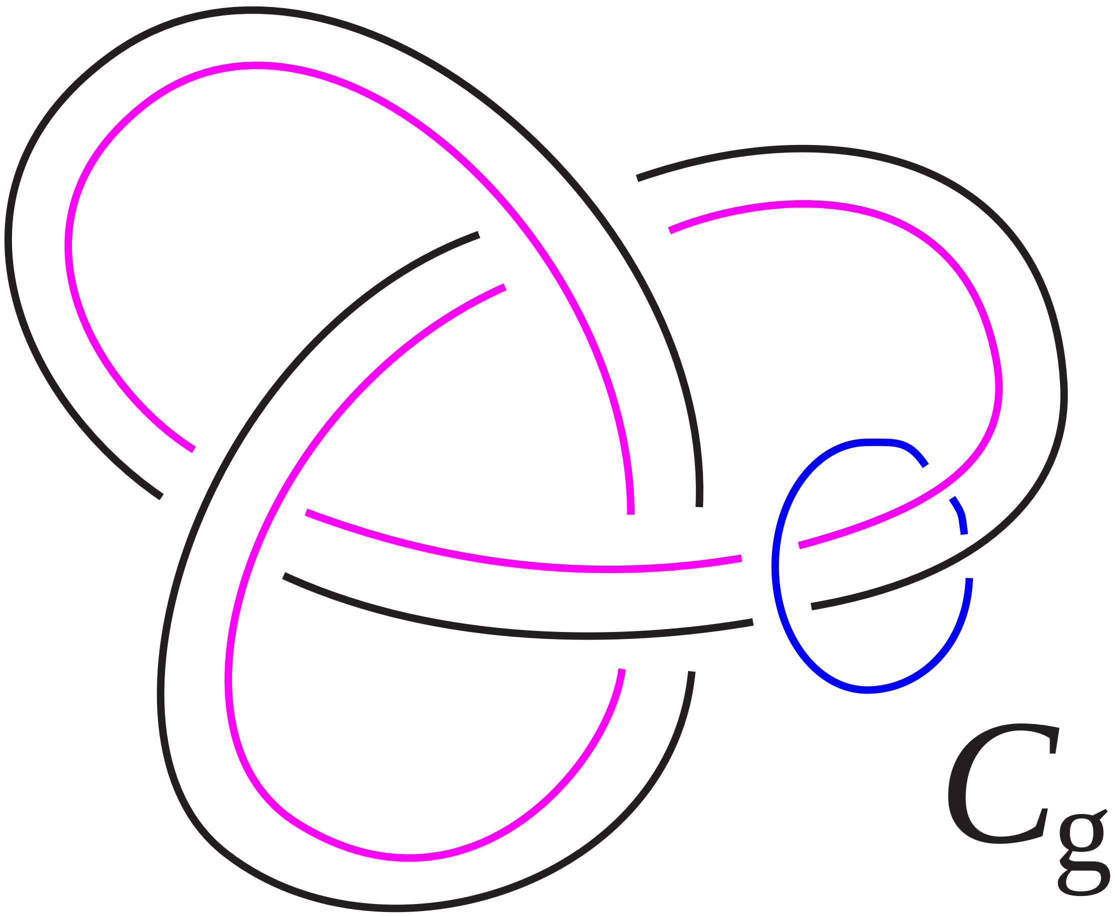
link $L \subset M$ components $\{C_1, C_2, ..., C_N\}$ with colours q_j

$\widetilde{L}$ = simplicial satellite of L



$$\langle W(L)\rangle\Big|_{S^3} = \langle W(\widetilde{L})\rangle\Big|_{S^3} = \exp\Big\{-(2i\pi/4k)\sum_{ij}\widetilde{\mathbb{L}}_{ij}\Big\}$$

$$\widetilde{\mathbb{L}}_{ij} = \ell k\left(\widetilde{C}_{i\mathrm{f}}, \widetilde{C}_j\right) \quad , \quad H_1(S^3 - \widetilde{L}) \ni [\widetilde{L}_{\mathrm{f}}] = \textstyle\sum_{ij} \ell k\left(\widetilde{C}_{i\mathrm{f}}, \widetilde{C}_j\right) g_j$$


C_g

Surgery rules

{abelian link invariants in S^3} $\leftrightarrow$ Reshetikhin-Turaev surgery rules

$$\langle W(L)\rangle\Big|_M = \langle W(L)\, W(\mathcal{L})\rangle\Big|_{S^3} \Big/ \langle W(\mathcal{L})\rangle\Big|_{S^3}$$

$\mathcal{L}$ = surgery link : $S^3 \to M$

colour state $\psi_0 = (\frac{1}{\sqrt{2k}})\{q = 0 \oplus q = 1 \oplus \cdots \oplus q = 2k-1\}$

Since $\quad (\frac{1}{\sqrt{2k}}) \sum_{q=0}^{2k-1} \exp\Big\{ -(2i\pi/4k) \left[(\pm q^2) \right] \Big\} = e^{\mp i\pi/4}$

$I(M) = e^{i(\pi/4)\sigma(\mathcal{L})} \; \langle W(\mathcal{L})\rangle\Big|_{S^3}$ = invariant under Kirby moves

$I(M)$ can be "interpreted" as

$$I(M) = \int_M DA\; e^{iS_{CS}[A]} \Big/ \int_{S^3} DA\; e^{iS_{CS}[A]}$$

Homology

Property A. If M is a homology 3-sphere

$$\left\{ \langle W(L) \rangle \Big|_M \right\} = \left\{ \langle W(L) \rangle \Big|_{S^3} \right\}$$

Property B. If M is a homology 3-sphere $\rightarrow \; I(M) = 1$

Property C. If the link L in the generic 3-manifold M is homologically trivial (and the associated observable is well defined),

$$\langle W(L) \rangle \Big|_M = \langle W(L) \rangle \Big|_{S^3}$$

Conjecture: $I(M)$ only depends on the homolgy of M (?)

Answer : No. There are counterexamples

consider lens spaces $L(p, q)$, for $k = 2$

$I(L(5,1)) = -1$ and $I(L(5,2)) = 1$

$I(L(7,1)) = 1$ and $I(L(7,2)) = i$ (same homotopy type)