WRT Invariants and Modular Forms

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Outline

Relationship between WRT invariants and modular forms

WRT Invariant

SU(2) Witten Invariant (1988) for 3-manifold M

$$Z_k(M) = \int e^{2\pi i k \mathrm{CS}(A)} \, dA$$

$$\mathrm{CS}(A) = \frac{1}{8\pi^2} \int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

WRT Invariant

Reshetikhin–Turaev (1991)

When the 3-manifold M is constructed by the rational surgery p_j/q_j on the j-th component of link L, WRT invariant is computed as

$$\tau_N(M) = e^{\frac{\pi i}{4} \frac{N-2}{N} \left(\sum_{j=1}^{n} \Phi\left(U^{(p_j,q_j)}\right) - 3\,\mathrm{sgn}(\mathbf{L})\right)} \sum_{k_1,\dots,k_n=1}^{N-1} J_{k_1,\dots,k_n}(L) \prod_{j=1}^{n} \left[\rho\left(U^{(p_j,q_j)}\right)\right]_{k_j,1}$$

$$U^{(p_j,q_j)} = \begin{pmatrix} p_j & r_j \\ q_j & s_j \end{pmatrix}; \qquad [\rho(S)]_{a,b} = \sqrt{\frac{2}{N}} \sin\left(\frac{a\,b}{N}\pi\right); \qquad [\rho(T)]_{a,b} = e^{\frac{\pi i}{2N}a^2 - \frac{\pi i}{4}} \delta_{a,b}$$

$$\text{Rademacher } \Phi: \quad \Phi\left(\begin{pmatrix} p & r \\ q & s \end{pmatrix}\right) = \begin{cases} (p+s)/q - 12\,s(p,q) & \text{for } q \neq 0 \\ r/s & \text{for } q = 0 \end{cases}$$

$$\text{Dedekind sum:} \quad s(d,c) = \sum_{k \mod c} \left(\left(\frac{k}{c}\right)\right)\left(\left(\frac{k\,d}{c}\right)\right)$$

$$((x)) = \begin{cases} x - \lfloor x \rfloor - \dfrac{1}{2}, & \text{for } x \in \mathbb{R} \setminus \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}$$

WRT Invariant

SU(2) Witten Invariant (1988) for 3-manifold M

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Reshetikhin–Turaev Invariant (1991)

$$Z_k(M) = \frac{\tau_{k+2}(M)}{\tau_{k+2}(S^2 \times S^1)}$$

$$\tau_N(S^3) = 1 \qquad \tau_N(S^2 \times S^1) = \sqrt{\frac{N}{2}} \frac{1}{\sin(\pi/N)}$$

Asymptotics of WRT in $k \to \infty$

- saddle-point method at flat connection

$$Z_k(M) \sim \frac{1}{2} e^{-\frac{3}{4}\pi i} \sum_{\alpha} \sqrt{T^{(\alpha)}}\, e^{-\frac{2\pi i}{4} I^{(\alpha)}}\, e^{2\pi i (k+2)\, \mathrm{CS}(A^{(\alpha)})}$$

$T^{(\alpha)}$: torsion; $I^{(\alpha)}$: spectral flow

- Loop expansion based on Feynman diagram (Axelrod–Singer, BarNatan, ...)
- asymptotic expansion based on integral repsentation (Rozansky, Lawrence)
- interests from “Volume Conjecture” (Kashaev, Murakami–Murakami)
- connection with modular form (Lawrence–Zagier 1999)
  $M = \Sigma(2,3,5)$: Poincaré homology sphere

$\tau_N(M) =$ Limiting values $\tau \searrow 1/N$ of Eichler integral $\widetilde{\Phi}(\tau)$ of
weight-3/2 vector-valued modular form $\Phi(\tau)$

$\to$ asymptotic expansion from nearly modularity

Eichler Integral of Integral Weight Modular Form

Let $F(\tau) = \sum_{n=1}^{\infty} a_n q^n$ be modular form with weight $k \in \mathbb{Z}_{\geq 2}$ $\qquad (q = e^{2\pi i \tau};\ \tau \in \mathbb{H})$

$$F(\gamma(\tau)) = (c\tau + d)^k \cdot F(\tau); \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2;\mathbb{Z})$$

Then the Eichler integral is defined by

$$\widetilde{F}(\tau) = \sum_{n=1}^{\infty} \frac{a_n}{n^{k-1}} q^n; \qquad \left(\frac{1}{2\pi i} \frac{d}{d\tau} \right)^{k-1} \widetilde{F}(\tau) = F(\tau)$$

The Eichler integral has nearly modularity

$$\boxed{(c\tau + d)^{k-2} \widetilde{F}(\gamma(\tau)) - \widetilde{F}(\tau) = G_\gamma(\tau)}$$

$$G_\gamma(z) = \frac{(2\pi i)^{k-1}}{(k-2)!} \int_{\gamma^{-1}(\infty)}^{\infty} F(\tau)(z - \tau)^{k-2} \, d\tau; \quad \text{period polynomial}$$

Eichler Integral of Half-Integral Weight Modular Form

With $P \in \mathbb{Z}$, and odd periodic function $\psi_{2P}^{(a)}(n)$

$$\psi_{2P}^{(a)}(n) = \begin{cases} \pm 1 & \text{for } n \equiv \pm a \mod 2P \\ 0 & \text{otherwise} \end{cases}$$

we have the vector-valued modular form with weight 3/2;

$$\Psi_P^{(a)}(\tau) = \frac{1}{2} \sum_{n \in \mathbb{Z}} n \, \psi_{2P}^{(a)}(n) \, q^{\frac{n^2}{4P}}$$

$$\Psi_P^{(a)}(\tau) = \left(\frac{\mathrm{i}}{\tau}\right)^{3/2} \sum_{b=1}^{P-1} \sqrt{\frac{2}{P}} \sin\left(\frac{a\,b}{P}\pi\right) \Psi_P^{(b)}(-1/\tau); \qquad \Psi_P^{(a)}(\tau+1) = \mathrm{e}^{\frac{a^2}{2P}\pi \mathrm{i}} \, \Psi_P^{(a)}(\tau)$$

affine SU(2) character

$$\frac{\Psi_P^{(a)}(\tau)}{[\eta(\tau)]^3} = \frac{\vartheta_{P,a} - \vartheta_{P,-a}}{\vartheta_{2,1} - \vartheta_{2,-1}}(0;\tau)$$

$$\vartheta_{P,a}(z;\tau) = \sum_{n \equiv a \mod 2P} q^{\frac{n^2}{4P}} \, \mathrm{e}^{2\pi \mathrm{i} n z}$$

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The Eichler integral:

$$\widetilde{\Psi}_P^{(a)}(\tau) = \sum_{n=0}^{\infty} \psi_{2P}^{(a)}(n) \, q^{\frac{n^2}{4P}}$$

Eichler Integral of Half-Integral Weight Modular Form

The Eichler integral:

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$$\frac{t\,\mathrm{e}^{xt}}{\mathrm{e}^t - 1} = \sum_{k=0}^{\infty} \frac{B_k(x)}{k!}\, t^k$$

Limiting values in $\tau \searrow M/N \in \mathbb{Q}$;

$$\widetilde{\Psi}_P^{(a)}(1/N) = -\sum_{k=0}^{2PN} \psi_{2P}^{(a)}(k)\, \mathrm{e}^{\frac{k^2}{2PN}\pi \mathrm{i}}\, B_1\left(\frac{k}{2PN}\right);$$

$$\widetilde{\Psi}_P^{(a)}(N) = \left(1 - \frac{a}{P}\right) \mathrm{e}^{\frac{a^2}{2P}\pi \mathrm{i} N}$$

Eichler Integral of Half-Integral Weight Modular Form

An infinite sum in rhs comes from the period function

$$\int_0^{i\infty} \frac{\Psi_P^{(a)}(z)}{\sqrt{z - 1/N}} \, dz$$

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$$\widetilde{\Psi}_P^{(a)}(N) = \left(1 - \frac{a}{P}\right) e^{\frac{a^2}{2P}\pi i N}$$

$$\boxed{\widetilde{\Psi}_P^{(a)}(1/N) + \sqrt{\frac{N}{i}} \sum_{b=1}^{P-1} \sqrt{\frac{2}{P}} \sin\left(\frac{ab}{P}\pi\right) \widetilde{\Psi}_P^{(b)}(-N) \simeq \sum_{k=0}^{\infty} \frac{L\left(-2k, \psi_{2P}^{(a)}\right)}{k!} \left(\frac{\pi i}{2PN}\right)^k}$$

$$L\left(-k, \psi_{2P}^{(a)}\right) = -\frac{(2P)^k}{k+1} \sum_{n=1}^{2P} \psi_{2P}^{(a)}(n) B_{k+1}\left(\frac{n}{2P}\right); \qquad \frac{\sinh((P-a)z)}{\sinh(Pz)} = \sum_{k=0}^{\infty} \frac{L\left(-2k, \psi_{2P}^{(a)}\right)}{(2k)!} z^{2k}$$

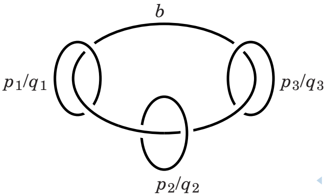
WRT Invariant for Seifert Manifolds

Let M be $M(b; p_1/q_1, p_2/q_2, p_3/q_3)$. Then SU(2) WRT invariant is given by

$$e^{\frac{2\pi i}{N}\left(\frac{\phi_M}{4}-\frac{1}{2}\right)}\left(e^{2\pi i/N}-1\right)\cdot\tau_N(M)=\frac{e^{\pi i/4}}{\sqrt{2Np_1p_2p_3}}\sum_{k_0=1}^{N-1}\sum_{n_j \mod p_j}\frac{e^{-b\frac{\pi i}{2N}k_0^2}}{e^{\frac{\pi i}{N}k_0}-e^{-\frac{\pi i}{N}k_0}}$$
$$\times\prod_{j=1}^{3}e^{-\frac{\pi i}{2N}\frac{q_j}{p_j}(k_0+2Nn_j)^2}\left(e^{\frac{\pi i}{Np_j}(k_0+2Nn_j)}-e^{-\frac{\pi i}{Np_j}(k_0+2Nn_j)}\right)$$

where

$$\phi_M=12\sum_{j=1}^{3}s(q_j,p_j)+E_M-3\,\mathrm{sgn}(E_M);\qquad E_M=-b-\sum_{j=1}^{3}\frac{q_j}{p_j}$$



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$$\times\prod_{j=1}^{3}e^{-\frac{\pi i}{2N}\frac{q_j}{p_j}(k_0+2Nn_j)^2}\left(e^{\frac{\pi i}{Np_j}(k_0+2Nn_j)}-e^{-\frac{\pi i}{Np_j}(k_0+2Nn_j)}\right)$$

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Fact: these WRT invariants $\tau_N(M)$ can be generally rewritten in terms of limiting values of the Eichler integrals $\widetilde{\Psi}_P^{(a)}(1/N)$

Example: Poincaré Homology Sphere

Let M be the Poincaré homology sphere $\Sigma(2,3,5)$. Lawrence–Zagier proved

$$e^{\frac{121}{60N}\pi i}\left(e^{\frac{2\pi i}{N}}-1\right)\cdot\tau_N(M)=e^{\frac{\pi i}{60N}}-\frac{1}{2}\left(\widetilde{\Psi}_{30}^{(1)}(1/N)+\widetilde{\Psi}_{30}^{(11)}(1/N)+\widetilde{\Psi}_{30}^{(19)}(1/N)+\widetilde{\Psi}_{30}^{(29)}(1/N)\right)$$

(Nearly) modularity proves

$$e^{\frac{121}{60N}\pi i}\left(e^{2\pi i/N}-1\right)\cdot\tau_N(M)\simeq\sqrt{\frac{N}{i}}\,\frac{2}{\sqrt{5}}\left(\underbrace{\sin(\pi/5)\,e^{-\frac{1}{60}\pi i N}+\sin(2\pi/5)\,e^{-\frac{49}{60}\pi i N}}_{\text{contributions from flat connections}}\right)$$

$$\underbrace{+\,e^{\frac{\pi i}{60N}}+\frac{1}{2}\sum_{k=0}^{\infty}\frac{L\left(-2k,-\psi_{60}^{(1)}-\psi_{60}^{(11)}-\psi_{60}^{(19)}-\psi_{60}^{(29)}\right)}{k!}\left(\frac{\pi i}{60N}\right)^k}_{\text{Ohtsuki Series}}$$

where

$$2\,\frac{\cosh(5z)\cosh(9z)}{\cosh(15z)}=-\sum_{k=0}^{\infty}\frac{L\left(-2k,-\psi_{60}^{(1)}-\psi_{60}^{(11)}-\psi_{60}^{(19)}-\psi_{60}^{(29)}\right)}{(2k)!}z^{2k}$$

Example: Seifert Homology Spheres $\Sigma(p_1,p_2,p_3)$

Let M be Seifert homology sphere $\Sigma(p_1,p_2,p_3)$ where p_i are coprime integers. We have

$$e^{\frac{2\pi i}{N}\left(\frac{\phi_M}{4}-\frac{1}{2}\right)}(e^{\frac{2\pi i}{N}}-1)\tau_N(M) = -\frac{1}{4}\sum_{\varepsilon_1,\varepsilon_2,\varepsilon_3=\pm 1}\varepsilon_1\varepsilon_2\varepsilon_3\,\widetilde{\Psi}_P^{(P+P\sum_j \varepsilon_j/p_j)}\left(\frac{1}{N}\right)$$

where $P = p_1p_2p_3$ and

$$\phi_M = 3-\frac{1}{P}+12\left(s(p_2p_3,p_1)+s(p_1p_3,p_2)+s(p_1p_2,p_3)\right)$$

Asymptotics in $N\to\infty$;

$$e^{\frac{2\pi i}{N}\left(\frac{\phi}{4}-\frac{1}{2}\right)}(e^{\frac{2\pi i}{N}}-1)\tau_N(M)$$

$$\simeq \sqrt{\frac{N}{i}}\frac{1}{4P}\sum_{\ell_1,\ell_2,\ell_3} C_P(\ell_1,\ell_2,\ell_3)\,e^{-\frac{1}{2}\pi i P N\left(1+\sum_j \frac{\ell_j}{p_j}\right)^2}+\frac{1}{2}\sum_{k=0}^{\infty}\frac{L(-2k,\chi)}{k!}\left(\frac{\pi i}{2PN}\right)^k$$

$$4\frac{\sinh(p_1p_2z)\sinh(p_1p_3z)\sinh(p_2p_3z)}{\sinh(Pz)} = \sum_{n=0}^{\infty}\frac{L(-2n,\chi)}{(2n)!}z^{2n}$$

Habiro's Unified WRT Invariant

$$(x)_n = \prod_{j=1}^{n} (1 - x q^{j-1})$$

For $\mathbb{Z}$HS M, Habiro introduced unified WRT invariant $I_q(M)$ with values in $\varprojlim_n \mathbb{Z}[q]/((q)_n)$ such that

$$\tau_N(M) = \mathrm{ev}_{q=e^{2\pi i/N}} \left(I_q(M) \right); \quad I_q(M) = \sum_{n=0}^{\infty} f_n(q) (q)_n$$

Cyclotomic expansion of the colored Jones polynomial $J_K(N)$ for knot K;

$$J_K(N) = \sum_{n=0}^{\infty} (q^{1+N})_n (q^{1-N})_n C_K(n); \qquad C_K(n) \in \mathbb{Z}[q, q^{-1}]$$

where we normalize $J_{\text{unknot}}(N) = 1$, and $J_K(1) = 1$.

$$C_K(n) = q^n \sum_{k=0}^{n} (-1)^k q^{\frac{1}{2}k(k-1)} \frac{(1 - q^{k+1})(1 - q^{2k+2})}{(q)_{n-k} (q)_{n+k+2}} J_K(k+1)$$

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where we normalize $J_{\text{unknot}}(N) = 1$, and $J_K(1) = 1$.

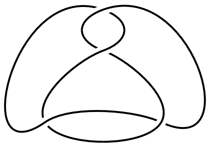
When $M_{\pm 1}$ is ± 1-surgery of knot K, unified WRT is (Beliakova–Blanchet–Le)

$$(1-q) I_q(M_{+1}) = \sum_{n=0}^{\infty} (-1)^n q^{-\frac{n(n+3)}{2}} C_K(n) (q^{n+1})_{n+1}; \; (1-q) I_q(M_{-1}) = \sum_{n=0}^{\infty} C_K(n) (q^{n+1})_{n+1}$$

Example: Poincaré Homology Sphere

Let K_p be the twist knot (Masbaum).

$$J_{K_p}(N) = \sum_{s_p \geq \cdots \geq s_1 \geq 0}^{\infty} q^{s_p} (q^{1-N})_{s_p} (q^{1+N})_{s_p} \prod_{i=1}^{p-1} q^{s_i(s_i+1)} \begin{bmatrix} s_{i+1} \\ s_i \end{bmatrix}_q$$



As a consequence, the unified WRT for the homology sphere $\Sigma(2,3,6p-1)$ is

$$\text{Poincaré homology } (p=1): \quad 1 + q(1-q) I_q(\Sigma(2,3,5)) = \sum_{n=0}^{\infty} q^n (q^n)_n$$

$$p > 1: \quad (1-q) I_q(M) = \sum_{s_p \geq \ldots s_1 \geq 0}^{\infty} q^{s_p} (q^{s_p+1})_{s_p+1} \prod_{i=1}^{p-1} q^{s_i(s_i+1)} \begin{bmatrix} s_{i+1} \\ s_i \end{bmatrix}_q$$

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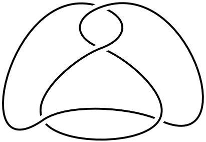

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Nearly modularity of WRT $\tau_N(M)$ → unified WRT $I_q(M)$?

q root of unity → generic q

Mock Theta Functions

- Ramanujan's last letter to Hardy (Jan. 12, 1920)
  - 17 mock theta functions
    (4 3rd order / 10 5th order / 3 7th order)
- G.N. Watson: "The Final Problem" (1936)
  - Transformation formulae for 3rd order
- "Lost Notebook" (1976, published in 1988)

3rd order

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{[(-q)_n]^2}$$

$$\omega(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{\left[(q;q^2)_{n+1}\right]^2}$$

$$q^{-\frac{1}{24}} f(q) - 2\sqrt{\frac{2\pi}{\alpha}}\, q_1^{4/3}\, \omega(q_1^2) = 4\sqrt{\frac{3\alpha}{2\pi}} \underbrace{\int_0^\infty e^{-\frac{3}{2}\alpha x^2} \frac{\sinh(\alpha x)}{\sinh\left(\frac{3}{2}\alpha x\right)} dx}_{\text{Mordell integral}}$$

$$q^{\frac{2}{3}} \omega(-q) + \sqrt{\frac{\pi}{\alpha}}\, q_1^{2/3}\, \omega(-q_1) = 2\sqrt{\frac{3\alpha}{\pi}} \int_0^\infty e^{-3\alpha x^2} \frac{\sinh(\alpha x)}{\sinh(3\alpha x)} dx$$

$$q = e^{-\alpha} \qquad \alpha\beta = \pi^2 \qquad q_1 = e^{-\beta}$$

Mock Theta Functions

one of 5th order mock theta functions

$$\chi_0(q) = \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1})_n}$$

Define $\chi_0^*(q)$ based on above expression with q outside the unit circle

$$\chi_0^*(q) = 2 - \chi_0(1/q) = 2 - \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(3n-1)}}{(q^{n+1})_n}$$

which gives the Eichler integral

$$\chi_0^*(q) = \sum_{n=0}^{\infty} q^n (q^n)_n$$

Namely

$$1 + q(1-q) I_q(\Sigma(2,3,5)) = \chi_0^*(q)$$

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How about other manifolds ?

Define $\chi_0^*(q)$ based on above expression with q outside the unit circle

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Namely

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WRT for Other Seifert Manifolds

Let $f(x,y,z)$ be weighted homogeneous polynomial with weight (d_1,d_2,d_3)

$$t(x,y,z) = (t^{d_1}x, t^{d_2}y, t^{d_3}z)$$

This $\mathbb{C}^*$-action induces S^1-action on the variety $V = \{f(x,y,z) = 0\}$. The Seifert manifold M is given as an intersection of V with sufficiently small sphere around the origin;

$$M = V \cap S^5$$

Linking pairing λ_M is defined with torsion part of $H_1(M;\mathbb{Z})$ by

$$\lambda_M : \operatorname{Tors} H_1(M;\mathbb{Z}) \otimes \operatorname{Tors} H_1(M;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

For $\boldsymbol{a}, \boldsymbol{a}' \in \operatorname{Tors} H_1(M;\mathbb{Z})$, we choose $s \in \mathbb{Z}_{\neq 0}$ s.t. $s\boldsymbol{a} = 0 \in H_1(M;\mathbb{Z})$, and we set 2-chain $\boldsymbol{B}$ bounded as $\partial \boldsymbol{B} = s\boldsymbol{a}$.

$$\lambda_M(\boldsymbol{a},\boldsymbol{a}') = \frac{\#(\boldsymbol{B}\cdot\boldsymbol{a}')}{s} \mod \mathbb{Z}$$

WRT for Other Seifert Manifolds

ADE Singularities

	$f(x,y,z)$	Seifert invariant $M(b; p_1/q_1, p_2/q_2, p_3/q_3)$	$H_1(M;\mathbb{Z})$	λ_M
D_{4K}	$x^{4K-1} + xy^2 + z^2$	$M(-1; 2/1, 2/1, 4K-2/1)$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$
D_{4K+2}	$x^{4K+1} + xy^2 + z^2$	$M(-1; 2/1, 2/1, 4K/1)$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\left(\frac{1}{2}\right) \oplus \left(\frac{1}{2}\right)$
D_{4K+1}	$x^{4K} + xy^2 + z^2$	$M(-1; 2/1, 2/1, 4K-1/1)$	$\mathbb{Z}_4$	$\left(\frac{3}{4}\right)$
D_{4K+3}	$x^{4K+2} + xy^2 + z^2$	$M(-1; 2/1, 2/1, 4K+1/1)$	$\mathbb{Z}_4$	$\left(\frac{1}{4}\right)$
E_6	$x^4 + y^3 + z^2$	$M(-1; 2/1, 3/1, 3/1)$	$\mathbb{Z}_3$	$\left(\frac{2}{3}\right)$
E_7	$x^3 y + y^3 + z^2$	$M(-1; 2/1, 3/1, 4/1)$	$\mathbb{Z}_2$	$\left(\frac{1}{2}\right)$
E_8	$x^5 + y^3 + z^2$	$M(-1; 2/1, 3/1, 5/1)$	0	$\varnothing$

WRT for Other Seifert Manifolds

Arnold's 14 Unimodal Singularities

	$f(x,y,z)$	Dolgachev	Seifert invariant	$H_1(M;\mathbb{Z})$	Gabrielov	λ_M
E_{12}	$x^7+y^3+z^2$	$(2,3,7)$	$M(-1;2/1,3/1,7/1)$	0	$(2,3,7)$	$\varnothing$
E_{13}	$x^5y+y^3+z^2$	$(2,4,5)$	$M(-1;2/1,4/1,5/1)$	$\mathbb{Z}_2$	$(2,3,8)$	$\left(\frac{1}{2}\right)$
E_{14}	$x^8+y^3+z^2$	$(3,3,4)$	$M(-1;3/1,3/1,4/1)$	$\mathbb{Z}_3$	$(2,3,9)$	$\left(\frac{2}{3}\right)$
Z_{11}	$x^5+xy^3+z^2$	$(2,3,8)$	$M(-1;2/1,3/1,8/1)$	$\mathbb{Z}_2$	$(2,4,5)$	$\left(\frac{1}{2}\right)$
Z_{12}	$x^4y+xy^3+z^2$	$(2,4,6)$	$M(-1;2/1,4/1,6/1)$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$(2,4,6)$	$\left(\frac{1}{2}\right)\oplus\left(\frac{1}{2}\right)$
Z_{13}	$x^6+xy^3+z^2$	$(3,3,5)$	$M(-1;3/1,3/1,5/1)$	$\mathbb{Z}_2\oplus\mathbb{Z}_3$	$(2,4,7)$	$\left(\frac{1}{2}\right)\oplus\left(\frac{2}{3}\right)$
W_{12}	$x^5+y^4+z^2$	$(2,5,5)$	$M(-1;2/1,5/1,5/1)$	$\mathbb{Z}_5$	$(2,5,5)$	$\left(\frac{3}{5}\right)$
W_{13}	$x^4y+y^4+z^2$	$(3,4,4)$	$M(-1;3/1,4/1,4/1)$	$\mathbb{Z}_2\oplus\mathbb{Z}_4$	$(2,5,6)$	$\left(\frac{5}{8}\right)$
Q_{10}	$x^4+y^3+xz^2$	$(2,3,9)$	$M(-1;2/1,3/1,9/1)$	$\mathbb{Z}_3$	$(3,3,4)$	$\left(\frac{1}{3}\right)$
Q_{11}	$x^3y+y^3+xz^2$	$(2,4,7)$	$M(-1;2/1,4/1,7/1)$	$\mathbb{Z}_2\oplus\mathbb{Z}_3$	$(3,3,5)$	$\left(\frac{1}{2}\right)\oplus\left(\frac{1}{3}\right)$
Q_{12}	$x^5+y^3+xz^2$	$(3,3,6)$	$M(-1;3/1,3/1,6/1)$	$\mathbb{Z}_3\oplus\mathbb{Z}_3$	$(3,3,6)$	$\left(\frac{1}{3}\right)\oplus\left(\frac{2}{3}\right)$
S_{11}	$x^4+y^2z+xz^2$	$(2,5,6)$	$M(-1;2/1,5/1,6/1)$	$\mathbb{Z}_2\oplus\mathbb{Z}_4$	$(3,4,4)$	$\left(\frac{3}{8}\right)$
S_{12}	$x^3y+y^2z+xz^2$	$(3,4,5)$	$M(-1;3/1,4/1,5/1)$	$\mathbb{Z}_{13}$	$(3,4,5)$	$\left(\frac{5}{13}\right)$
U_{12}	$x^4+y^3+z^3$	$(4,4,4)$	$M(-1;4/1,4/1,4/1)$	$\mathbb{Z}_4\oplus\mathbb{Z}_4$	$(4,4,4)$	$\begin{pmatrix}\frac{1}{2} & \frac{1}{4}\\ \frac{1}{4} & \frac{1}{2}\end{pmatrix}$

WRT for Other Seifert Manifolds

Conjecture: WRT invariant for $\mathbb{Q}$HS M is decomposed as

$$\tau_N(M) = \sum_{\ell \in \mathrm{Tors}\, H_1(M;\mathbb{Z})} e^{2\pi i N \lambda_M(\ell,\ell)} \cdot \underbrace{I^{(\lambda_M(\ell,\ell))}_{q=e^{2\pi i/N}}(M)}_{\tau \searrow 1/N \text{ of } q\text{-series}}$$

Arnold's strange duality: There exists a unimodal singularity X^* whose Gabrielov number (b_1^*, b_2^*, b_3^*) coincides with the Dolgatchev number (p_1, p_2, p_3) of X.

- 2 orthogonal sublattices of K3's intersection matrix (Pinkham, Dolgachev–Nikulin)
- weight system (K.Saito)
- triangular singularity $\mathbb{H}/\Gamma$
  Γ: hyperbolic triangle $\frac{\pi}{p_i}$
- Milnor lattice
  $T_{b_1,b_2,b_3} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Let M and M^* be Seifert manifolds associated with Arnold's unimodal singularities X and X^* respectively. There exists a duality of linking pairing

$$\lambda_M = -\lambda_{M^*}$$

WRT for Other Seifert Manifolds

- $M = M(2,2,K)$ with K even; $H_1 = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $\lambda_M = \begin{cases} \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} & \text{for } K \equiv 2 \mod 4 \\ \left(\frac{1}{2}\right) \oplus \left(\frac{1}{2}\right) & \text{for } K \equiv 0 \mod 4 \end{cases}$

$$e^{\frac{\pi i}{2N}\left(\sqrt{K}-\frac{1}{\sqrt{K}}\right)^2} \left(e^{\frac{2\pi i}{N}} - 1\right) \tau_N(M)$$
$$= \left(1 + (-1)^{N\left(1+\frac{K}{2}\right)}\right) \left(e^{\frac{\pi i}{2KN}} - \widetilde{\Psi}_K^{(1)}(1/N) - \widetilde{\Psi}_K^{(K-1)}(1/N)\right)$$

- $M = M(2,2,K)$ with K odd; $H_1 = \mathbb{Z}_4$, $\lambda_M = \begin{cases} \left(\frac{1}{4}\right) & \text{for } K \equiv 1 \mod 4 \\ \left(\frac{3}{4}\right) & \text{for } K \equiv 3 \mod 4 \end{cases}$

$$e^{\frac{\pi i}{2N}\left(\sqrt{K}-\frac{1}{\sqrt{K}}\right)^2} \left(e^{\frac{2\pi i}{N}} - 1\right) \tau_N(M)$$
$$= \left(e^{\frac{\pi i}{2KN}} - \widetilde{\Psi}_K^{(1)}(1/N) - \widetilde{\Psi}_K^{(K-1)}(1/N)\right) + e^{-\frac{K}{2}\pi i N} \left(e^{\frac{\pi i}{2KN}} + \widetilde{\Psi}_K^{(1)}(1/N) + \widetilde{\Psi}_K^{(K-1)}(1/N)\right)$$

WRT for Other Seifert Manifolds

- E_6 or $M = M(2,3,3)$; $H_1 = \mathbb{Z}_3$, $\lambda_M = \left(\frac{2}{3}\right)$

$$e^{\frac{13\pi i}{12N}} \left(e^{\frac{2\pi i}{N}} - 1\right) \tau_N(M) = \frac{1}{2\sqrt{3}} \left(2e^{\frac{\pi i}{12N}} - \widetilde{\Psi}_6^{(1)}\left(\frac{1}{N}\right) - 2\widetilde{\Psi}_6^{(3)}\left(\frac{1}{N}\right) - \widetilde{\Psi}_6^{(5)}\left(\frac{1}{N}\right)\right) + \frac{e^{\frac{2}{3}\pi i N}}{\sqrt{3}} \left(2e^{\frac{\pi i}{12N}} - \widetilde{\Psi}_6^{(1)}\left(\frac{1}{N}\right) + \widetilde{\Psi}_6^{(3)}\left(\frac{1}{N}\right) - \widetilde{\Psi}_6^{(5)}\left(\frac{1}{N}\right)\right)$$

- E_7 or $M = M(2,3,4)$; $H_1 = \mathbb{Z}_2$, $\lambda_M = \left(\frac{1}{2}\right)$

$$e^{\frac{37\pi i}{24N}} \left(e^{\frac{2\pi i}{N}} - 1\right) \tau_N(M) = \frac{1}{\sqrt{8}} \left(1 + (-1)^N\right) \left(2e^{\frac{\pi i}{24N}} - \widetilde{\Psi}_{12}^{(1)}\left(\frac{1}{N}\right) - \widetilde{\Psi}_{12}^{(5)}\left(\frac{1}{N}\right) - \widetilde{\Psi}_{12}^{(7)}\left(\frac{1}{N}\right) - \widetilde{\Psi}_{12}^{(11)}\left(\frac{1}{N}\right)\right)$$

WRT for Other Seifert Manifolds

- E_{12}; $M = M(-1; 2/1, 3/1, 7/1)$ is $\mathbb{Z}$HS

$$e^{-\frac{\pi i}{84N}} \left(e^{\frac{2\pi i}{N}} - 1\right) \tau_N(M) = \frac{i}{2} \left(\widetilde{\Psi}_{42}^{(1)}\left(-\frac{1}{N}\right) - \widetilde{\Psi}_{42}^{(13)}\left(-\frac{1}{N}\right) - \widetilde{\Psi}_{42}^{(29)}\left(-\frac{1}{N}\right) + \widetilde{\Psi}_{42}^{(41)}\left(-\frac{1}{N}\right) \right)$$

- Z_{11}; $M = M(-1; 2/1, 3/1, 8/1)$; $H_1(M) = \mathbb{Z}_2$, $\lambda_M = \left(\frac{1}{2}\right)$

$$e^{\frac{23\pi i}{48N}} \left(e^{\frac{2\pi i}{N}} - 1\right) \tau_N(M) = \frac{i}{\sqrt{8}} \left(1 + (-1)^N\right) \left(\widetilde{\Psi}_{24}^{(1)}\left(-\frac{1}{N}\right) - \widetilde{\Psi}_{24}^{(7)}\left(-\frac{1}{N}\right) - \widetilde{\Psi}_{24}^{(17)}\left(-\frac{1}{N}\right) + \widetilde{\Psi}_{24}^{(23)}\left(-\frac{1}{N}\right) \right)$$

- E_{13}; $M = M(-1; 2/1, 4/1, 5/1)$; $H_1(M) = \mathbb{Z}_2$, $\lambda_M = \left(\frac{1}{2}\right)$

$$e^{-\frac{21\pi i}{40N}} \left(e^{\frac{2\pi i}{N}} - 1\right) \tau_N(M) = \frac{i}{\sqrt{8}} \left(1 + (-1)^N\right) \left(\widetilde{\Psi}_{20}^{(1)}\left(-\frac{1}{N}\right) - \widetilde{\Psi}_{20}^{(9)}\left(-\frac{1}{N}\right) - \widetilde{\Psi}_{20}^{(11)}\left(-\frac{1}{N}\right) + \widetilde{\Psi}_{20}^{(19)}\left(-\frac{1}{N}\right) \right)$$

WRT for Other Seifert Manifolds

- Q_{12}; $M = M(-1; 3/1, 3/1, 6/1)$; $H_1(M) = \mathbb{Z}_3 \oplus \mathbb{Z}_3$, $\lambda_M = \left(\frac{1}{3}\right) \oplus \left(\frac{2}{3}\right)$

$$e^{-\frac{1}{24N}\pi i}\left(e^{\frac{2\pi i}{N}}-1\right)\cdot\tau_N(M) = \frac{i}{6}\left\{\left(1+2\,e^{\frac{2}{3}\pi i N}\right)\cdot\left(\widetilde{\Psi}_6^{(1)}-\widetilde{\Psi}_6^{(3)}\right)(-1/N)\right.$$

$$\left.+\left(2+e^{-\frac{2}{3}\pi i N}\right)\cdot\left(2\,\widetilde{\Psi}_6^{(1)}+\widetilde{\Psi}_6^{(3)}\right)(-1/N)+2\left(-2+e^{\frac{2}{3}\pi i N}+e^{-\frac{2}{3}\pi i N}\right)\widetilde{\Psi}_6^{(5)}(-1/N)\right\}$$

- S_{11}; $M = M(-1; 2/1, 5/1, 6/1)$; $H_1(M) = \mathbb{Z}_8$, $\lambda_M = \left(\frac{3}{8}\right)$

$$e^{\frac{13}{30N}\pi i}\left(e^{\frac{2\pi i}{N}}-1\right)\cdot\tau_N(M) = \frac{i}{2\sqrt{2}}\left\{\left(1+(-1)^N\right)\left(\widetilde{\Psi}_{15}^{(2)}-\widetilde{\Psi}_{15}^{(8)}\right)(-\frac{1}{2N})\right.$$

$$\left.-\left(1-(-1)^N\right)\left(\widetilde{\Psi}_{15}^{(7)}-\widetilde{\Psi}_{15}^{(13)}\right)(-\frac{1}{2N})+2\,e^{-\frac{3}{4}\pi i N}\left(\widetilde{\Psi}_{15}^{(1)}+\widetilde{\Psi}_{15}^{(4)}+\widetilde{\Psi}_{15}^{(11)}+\widetilde{\Psi}_{15}^{(14)}\right)(-\frac{2}{N})\right\}$$

- W_{13}; $M = M(-1; 3/1, 4/1, 4/1)$; $H_1(M) = \mathbb{Z}_8$, $\lambda_M = \left(\frac{5}{8}\right)$

$$e^{-\frac{7}{12N}\pi i}\left(e^{\frac{2\pi i}{N}}-1\right)\cdot\tau_N(M) = \frac{i}{2\sqrt{2}}\left\{\left(1+(-1)^N+2\,e^{\frac{3}{4}\pi i N}\right)\cdot\left(\widetilde{\Psi}_6^{(1)}-\widetilde{\Psi}_6^{(5)}\right)(-1/N)\right.$$

$$\left.+2\left(-1+(-1)^N\right)\cdot\widetilde{\Psi}_6^{(4)}(-1/N)\right\}$$

WRT for Other Seifert Manifolds

order	mock theta functions	3-manifolds M
3	$\phi(q)$	$M(2,3,4)$
	$\omega(q)$	$M(2,2,3)$
5	$\chi_0(q)$	Poincaré $\mathbb{Z}$HS $\Sigma(2,3,5)$
6	$\varphi(q)$, $\psi(q)$, $\rho(q)$	$M(2,3,3)$, $M(2,2,6)$, $M(2,2,2)$
7	$\mathscr{F}_0(q)$	$\mathbb{Z}$HS $\Sigma(2,3,7)$
10	$\Psi(q)$, $X(q)$	$M(2,2,5)$

Remarks: Superconformal Algebras, Mock Theta & CS

Freeman J. Dyson: "A walk through Ramanujan's Garden" (1988)

The mock theta-functions give us tantalizing hints of a grand synthesis still to be discovered. Somehow it should be possible to build them into a coherent group-theoretical structure, analogous to the structure of modular forms which Hecke build around the old theta-functions of Jacobi. This remains a challenge for the future. My dream is that I will live to see the day when our young physicists, struggling to bring the predictions of superstring theory into correspondence with the facts of nature, will be led to enlarge their analytic machinery to include not only theta-functions but mock theta-functions.

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- Mock theta fn is a holomorphic part of wt-1/2 harmonic Maass form (Zwegers)

$$(\Im\tau)^{3/2}\frac{\partial}{\partial\tau}\sqrt{\Im\tau}\,\frac{\partial}{\partial\overline{\tau}}\widehat{\mu}(z;\tau)=0$$

  - theory of partition (Bringmann–Ono)

Remarks: Superconformal Algebras, Mock Theta & CS

two types representation of $\mathcal{N} = 4$ SCA ($c = 6$);

- massive (non-BPS); $h > \frac{1}{4}$ for $\ell = \frac{1}{2}$
- massless (BPS); $h = \frac{1}{4}$ for $\ell = 0, \frac{1}{2}$

$$\begin{cases} L_0|\Omega\rangle = h|\Omega\rangle \\ T_0^3|\Omega\rangle = \ell|\Omega\rangle \end{cases}$$

characters in R-sector: (Eguchi–Taormina)

$$\mathrm{ch}_{h,\ell}^{R}(z;\tau) = \mathrm{Tr}_R\left(\mathrm{e}^{2\pi \mathrm{i} z T_0^3} q^{L_0 - \frac{6}{24}}\right)$$

massless character is

$$\mathrm{ch}_{h=\frac{1}{4},\ell=0}^{\widetilde{R}}(z;\tau) = \frac{[\theta_{11}(z;\tau)]^2}{[\eta(\tau)]^3}\,\mu(z;\tau); \qquad \mu(z;\tau) = \frac{\mathrm{i}\,\mathrm{e}^{\pi \mathrm{i} z}}{\theta_{11}(z;\tau)} \sum_{n\in\mathbb{Z}} (-1)^n \frac{q^{\frac{1}{2}n(n+1)}\,\mathrm{e}^{2\pi \mathrm{i} n z}}{1 - q^n\,\mathrm{e}^{2\pi \mathrm{i} z}}$$

$\mu(z;\tau)$ is mock theta function whose completion is; $\widehat{\mu}(z;\tau) = \mu(z;\tau) - \frac{1}{2}R(0;\tau)$

$$R(0;\tau) = \frac{1}{\sqrt{\mathrm{i}}}\int_{-\overline{\tau}}^{\mathrm{i}\infty} \frac{[\eta(z)]^3}{\sqrt{z+\tau}}\,\mathrm{d}z;$$ which gives WRT for $M(2,2,2)$ in $\tau \searrow \frac{1}{N}$

Remarks: Superconformal Algebras, Mock Theta & CS

Decomposition of elliptic genus for $K3$ surface (T.Eguchi–KH)

$$Ell_{K3}(z;\tau) = 8\left[\left(\frac{\theta_{10}(z;\tau)}{\theta_{10}(0;\tau)}\right)^2 + \left(\frac{\theta_{00}(z;\tau)}{\theta_{00}(0;\tau)}\right)^2 + \left(\frac{\theta_{01}(z;\tau)}{\theta_{01}(0;\tau)}\right)^2\right]$$

$$= 24\,\mathrm{ch}^{\widetilde{R}}_{h=1/4,\ell=0}(z;\tau) - 8\,\frac{[\theta_{11}(z;\tau)]^2}{[\eta(\tau)]^3} \sum_{w\in\{1/2,(1+\tau)/2,\tau/2\}} \mu(w;\tau)$$

$$= 24\,\mathrm{ch}^{\widetilde{R}}_{h=1/4,\ell=0}(z;\tau) - q^{-\frac{1}{8}}\,\frac{[\theta_{11}(z;\tau)]^2}{[\eta(\tau)]^3}\left(2 - \sum_{n=1}^{\infty} A_n q^n\right)$$

$$= 20\,\mathrm{ch}^{\widetilde{R}}_{1/4,0}(z;\tau) - 2\,\mathrm{ch}^{\widetilde{R}}_{1/4,1/2}(z;\tau) + \sum_{n=1}^{\infty} A_n\,\mathrm{ch}^{\widetilde{R}}_{n+1/4,1/2}(z;\tau)$$

$$A_n = \frac{4\pi}{(8n-1)^{\frac{1}{4}}} \sum_{c=1}^{\infty} \frac{1}{c}\, I_{\frac{1}{2}}\!\left(\frac{\pi\sqrt{8n-1}}{2c}\right) \sum_{\substack{d \mod c \\ (c,d)=1}} e^{-3\pi i s(d,c) + 2\pi i \frac{d}{c} n}; \log A_n \sim 2\pi\sqrt{\frac{1}{2}\left(n-\frac{1}{8}\right)}$$

Conclusion

- SU(2) WRT invariant for Seifert manifolds & modular forms
  - mock theta functions
  - unified WRT
  - decomposition in terms of linking pairing
- knot invariant
  - colored Jones polynomial at $q = e^{2\pi i/N}$ (Volume Conjecture) for torus knot $T(a,b)$ is the Eichler integral of wt-1/2 modular form (Virasoro minimal model)

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- hyperbolic case ?
  - "quantum modular form" (Zagier)
- other gauge groups ?