Chern-Simons Theory & Topological Strings

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- Topological String Theory on Calabi-Yau manifolds
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  - B-model
  - Mirror Duality & Large N-Duality
- Large N-expansion of Chern-Simons theory
- Dual Gauge theory $\leftrightarrow$ String theory backgrounds
  - $T^*S^3 \leftrightarrow \mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{P}^1$
  - Lens spaces $L(p,1) \leftrightarrow A_p$-geometries
- The topological Vertex

Topological String on Calabi-Yau manifolds

String Theory:

$$X : \Sigma_{g,h} \rightarrow (M, L)$$

Partition function Z (and other correlators)

$$Z(M, L) = \int \mathcal{D}X \mathcal{D}h \mathcal{D}\Psi_{\pm} \mathcal{D}\bar{\Psi}_{\pm} e^{-S(X,h,\Psi_{\pm},\bar{\Psi}_{\pm},G,B,A,\ldots)} \ .$$

$\mathcal{D}h$, w.s. metric integral, becomes finite for $M_{10} = M \times M_{3,1}$ $(\dim_{\mathbb{C}}(M) = 3)$.

Still the variational integral cannot be performed.

Topological string theories are truncations of the theory of physical (supersymmetry) and mathematical (invariants of (M,L)) interest.

The relevant cases require

☆ M Kähler & $c_1(TM) = 0 \leftrightarrow \exists\Omega$ hol. $(n, 0)$-form,

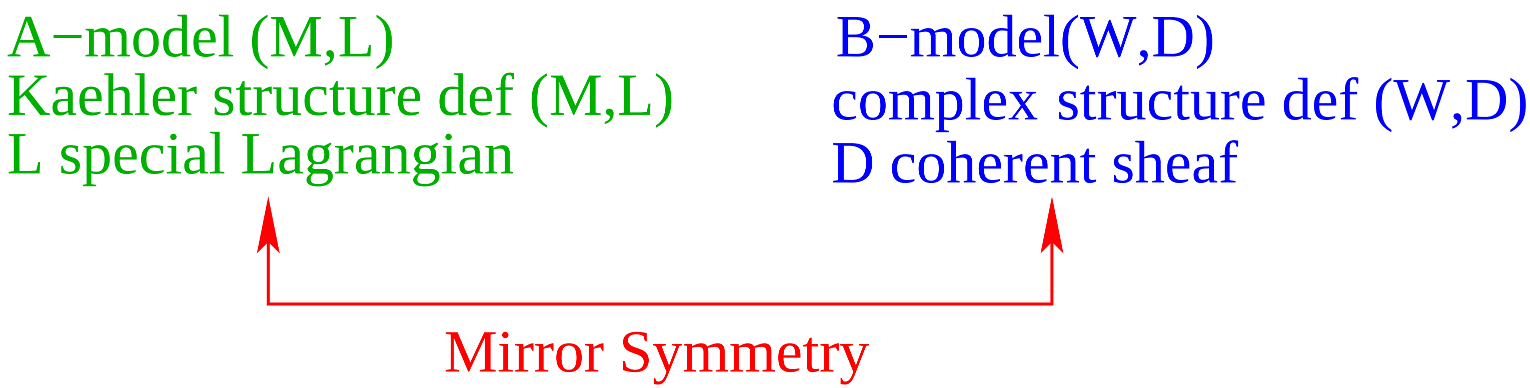
☆ L Lagrangian & special $\Omega|_L = \mathrm{vol}(L)$.

Geometry of a Calabi-Yau manifold with special Lagrangian branes or coherent sheaves.

For the choice of M one has a $(2,2)$ w.s. SCFT with $U(1)_V$ and $U(1)_A$ currents, which can be used to defined two scalar, nilpotent operators: $Q_A^2 = 0$ or $Q_B^2 = 0$

$$\int_\Sigma \partial_\mu J_V^\mu = 0, \qquad U(1)_{L'} = U(1)_L + U(1)_V \to Q_A$$
$$\int_\Sigma \partial_\mu J_A^\mu = \int_\Sigma X^*(c_1), \quad U(1)_{L'} = U(1)_L + U(1)_A \to Q_B$$

This defines two cohomological theories:



A-model:

The path integral localizes to holomor. maps $\partial_{\bar z} X^i = 0$.
In particular it becomes formal power series in g_s, in which every term is a well defined finite dim integral
Connected contr. $F = \mathrm{Log}(Z) = \sum_{g=0}^{\infty} g_s^{2g-2} F_g(q, Q)$

$$F = \sum_{g=0}^{\infty} g_s^{2g-2} \sum_{(\beta,h)\in(H^2(M,L,\mathbb{Z}),H^1(L,\mathbb{Z}))} q^{\beta} Q^{h} \underbrace{\int_{\overline{\mathcal{M}^{g}_{\beta,h}(M,L)}} c^{g\,vir}_{\beta,h}}_{r^{g}_{\beta,h}}$$

$$\mathrm{Gromov-Witten\ \ Inv.} \rightarrow$$

☆ $q^{\beta} = \prod_i e^{-\beta^i t_i}$, $\underline{\beta} \in \mathbb{Z}^{h_{11}}$ degrees,

$$t_i = \int_{\mathcal{C}(2),i} \omega + iB$$

complexified Kähler parameter

☆ $Q^h = \prod_i e^{-h^i u_i}$, $\underline{h} \in \mathbb{Z}^{h^1(L)}$ windings,

$$u_i = \int_{C(1),i_{\in L}} \omega^* + iA$$

Open string (Wilson line) moduli.

☆ Mathematical theory defining the $r^g_\beta \in \mathbb{Q}$ for closed oriented Σ.

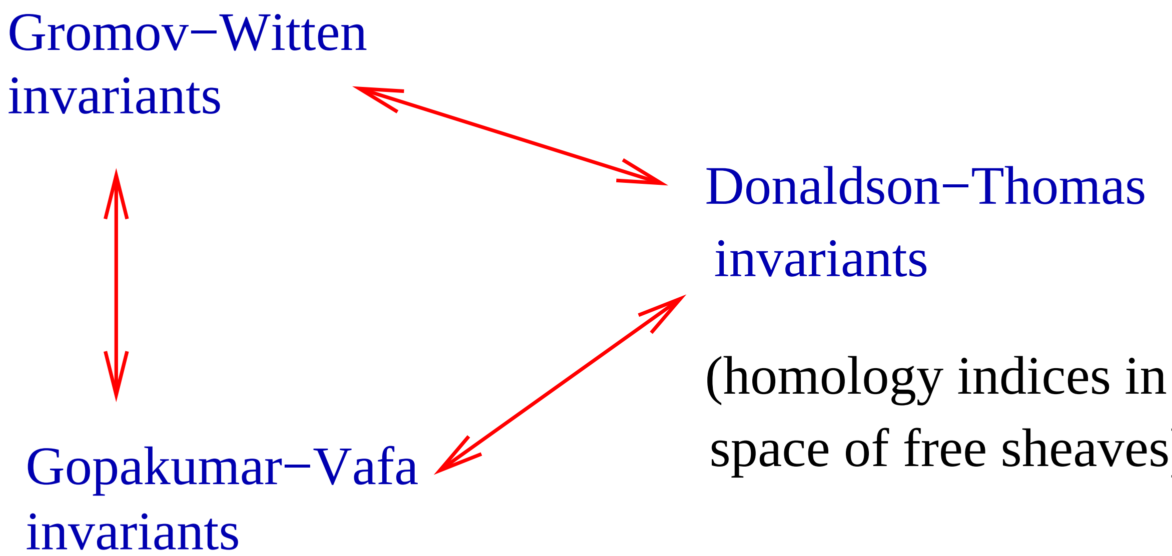
☆ Expected dimension critical for on CY 3-folds

$$\mathrm{vd}_{\mathbb{C}}\overline{\mathcal{M}^g_\beta(M)} = c_1(T_M)\cdot\beta + (\mathrm{d}_{\mathbb{C}}(M)-3)(1-g)\ .$$

☆ For compl. intersec. $M \in X_\Delta$, X_Δ toric, the $r^{g=0/1}_\beta \in \mathbb{Q}$ can be calculated by localization, w.r.t. induced torus action on $\overline{\mathcal{M}^g_\beta(M)}$ Kontsevich, Givental,. . . .

☆ On non-compact toric CY, $\mathcal{O}(-K_B)\to B$, B toric, $r^g_\beta \in \mathbb{Q}$ & $r^g_{\beta,h}\in\mathbb{Q}$ on Harvey-Lawson type SLAG L A.K., Zaslow, Graber . . . .

Symplectic invariants and integrality conjectures:



(homology indices in moduli space of free sheaves)

(homology indices in moduli space of D0+D2+1 D6 branes count BPS states)

The relation between DT- GV- and GW invariants

MNOP 04: $Z_{\rm GV}=\exp(F)$

$Z_{\rm GV}(M,q_{g_s},q)\mathbb{M}(q_{g_s})^{\frac{\chi(M)}{2}}=Z_{\rm DT}(M,-q_{g_s},q)$ with

$q_{g_s}=e^{ig_s}$, $Z_{\rm DT}(M,q_{g_s},q)=\sum_{\beta,k\in\mathbb{Z}} n^{(k)}_\beta q^k_{g_s}q^\beta$ and

$$Z_{\rm GV} \;=\; e^{\frac{c(t)}{g_s{}^2}+l(t)}\prod_\beta\Bigg[\left(\prod_{r=1}^\infty(1-q^r_{g_s}q^\beta)^{r\tilde n^{(0)}_\beta}\right)\prod_{g=1}^\infty\prod_{l=0}^{2g-2}(1-q^{g-l-1}_{g_s}q^\beta)^{(-1)^{g+r}\binom{2g-2}{l}\tilde n^{(g)}_\beta}\Bigg]$$

where $n^{(k)}_\beta\in\mathbb{Z}$! and $\tilde n^{(k)}_\beta\in\mathbb{Z}$! $\mathbb{M}$ McMahon funct.

B-model:

The path integral localizes to constant maps.

$$
\begin{aligned}
D_iD_jD_kF_0 &= \int_W \Omega\wedge\partial_i\partial_j\partial_k\Omega \\
F_1 &= \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\tau_2}\mathrm{Tr}(-1)^F F_L F_R q^{L_0}\bar{q}^{\bar{L}_0}, \\
F_{g>1} &= \int_{\overline{\mathcal{M}_g}} \langle \prod_{k=1}^{6g-6}(G,\mu_k)\rangle,
\end{aligned}
$$

with $(G,\mu_k)=\int_{\Sigma_g}(G_{zz}(\mu)^z_{\bar z}+G_{\bar z\bar z}(\bar\mu)^{\bar z}_z)$.

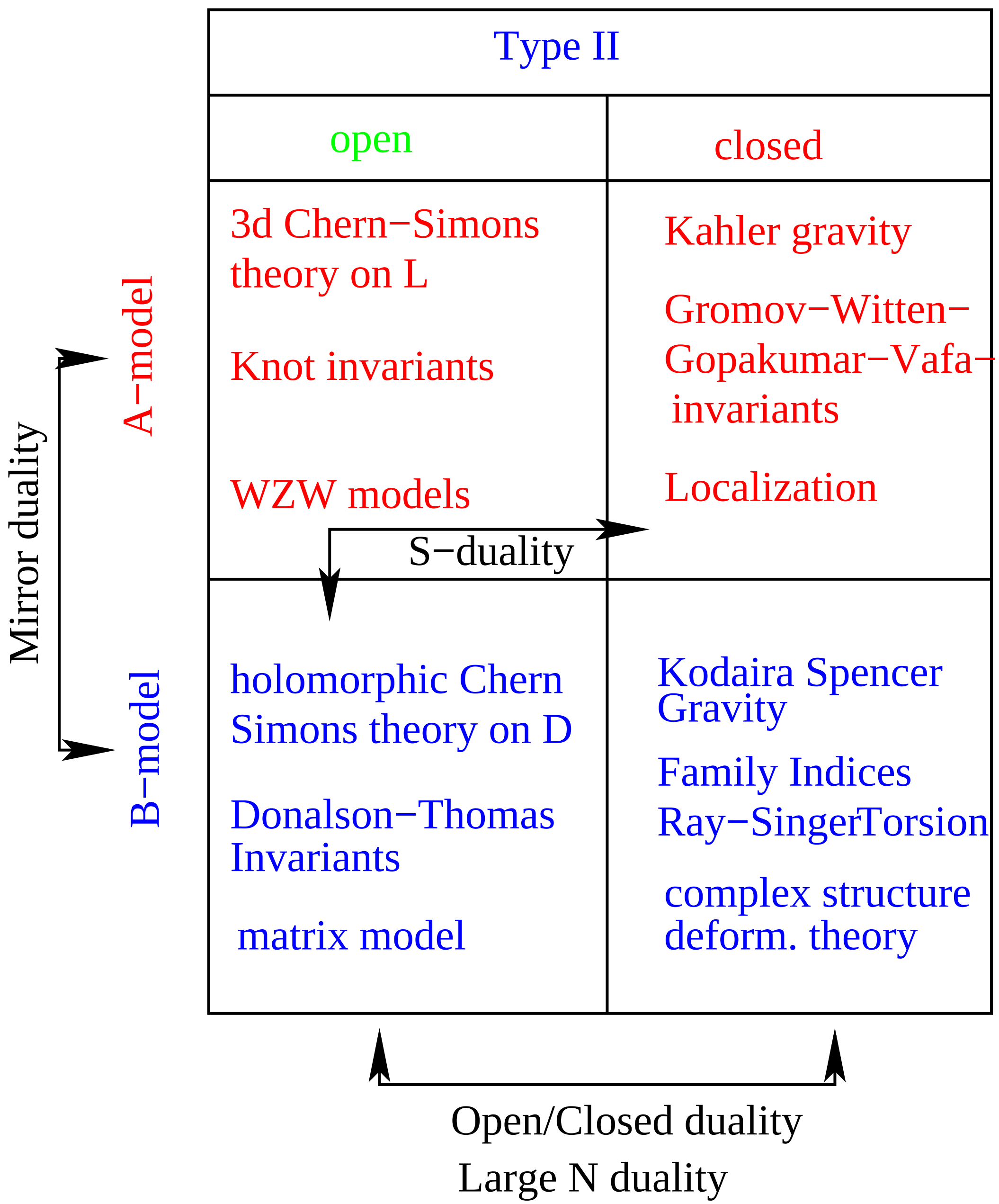
Natural flat coordinates $t_i=\int_{a_i}\Omega$ and conjugated momenta $p^i = \int_{b^i} \Omega$.

F_0 calculated by periods. Higher F_g hard to obtain in this framework.

Open string: Example D-6 brane $D = W$

$$W_{supo} = \int_W \Omega \wedge \mathrm{Tr}[A \wedge \partial \bar{A} + \frac{2}{3} A \wedge A \wedge A]$$

A $U(N)$ valued (1,0)-form on W. In simple situations this integral can be calculated by relative period integrals. $u_i = \int_{\mathcal{C}_i} \Omega$. Higher genus amplitudes are hard to obtain.



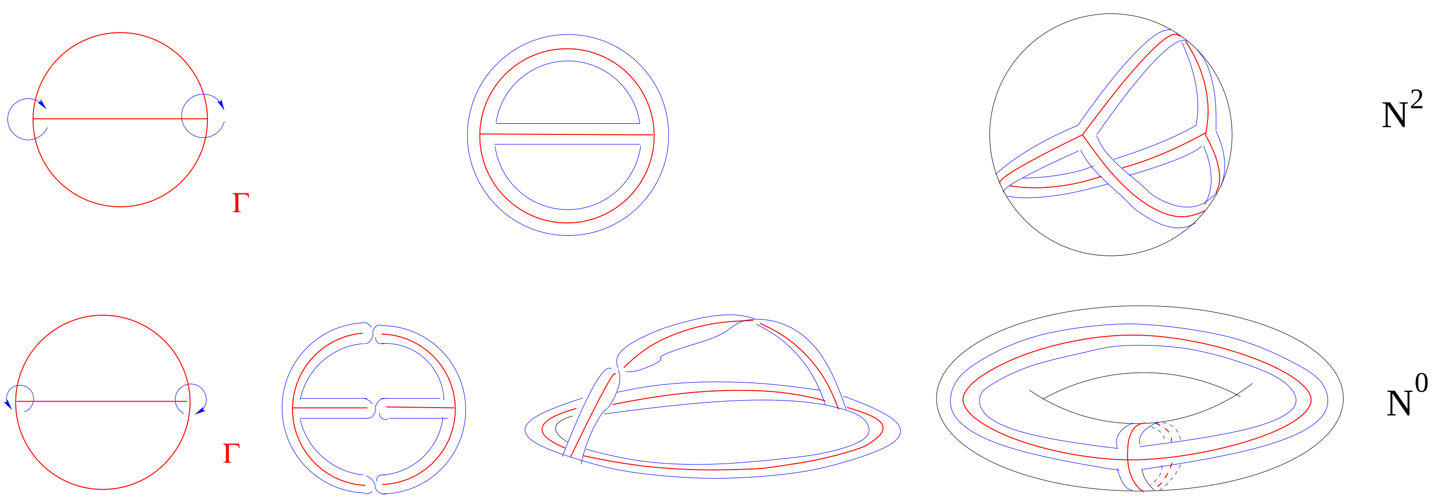
Type II
open
closed
3d Chern−Simons theory on L
Knot invariants
WZW models
Kahler gravity
Gromov−Witten−Gopakumar−Vafa− invariants
Localization
S−duality
holomorphic Chern Simons theory on D
Donalson−Thomas Invariants
matrix model
Kodaira Spencer Gravity
Family Indices
Ray−SingerTorsion
complex structure deform. theory
A−model
B−model
Mirror duality
Open/Closed duality
Large N duality

Large N-expansion of Chern-Simons theory:

$$S = \frac{k}{4\pi}\int_L \mathrm{Tr}\left(A \wedge \underset{\cdot}{A} + \frac{2}{3} A \wedge A \wedge A\right)$$

A $U(N)$ gauge connection in the trivial bundle over L_3

Usual combinatoric of large N expansion ($g_s = \frac{2\pi}{k+N}$)



With the 't Hooft parameter $t = g_s N$ one can expand $F = \mathrm{Log}(Z)$ as

$$F_{CS} = \sum_{g=0}^{\infty} \sum_{h=1}^{\infty} F_{g,h} g_s^{2g-2} t^h$$

If the sum over holes can be performed, this looks like a closed string expansion

$$F_{CS} = \sum_{g=0}^{\infty} g_s^{2g-2} F_g(t) \ .$$

Dual backgrounds: Chern-Simons on $L \Leftrightarrow$ Topological string on CY M.

☆ Deg. holomorphic open instantons on $T^*L \sim$ oriented graphs of CS on L. Witten 95

☆ The geometric transition $T^*S^3 \to M = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$ provides the dual closed string background Gopakumar and Vafa 99 i.e.

$$\boxed{F_{TS}(M, t) = F_{CS}(S^3, t).}$$

't Hooft param. identified with Kähler param. of $\mathbb{P}^1$.

Generalization by orbifoldisation AKMV 2002

$$\begin{array}{lcl} T^*S^3 & \sim & M = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1 \\ \downarrow \mathbb{Z}_p & & \downarrow \mathbb{Z}_p \\ T^*L(p,1) & \sim & A_p - \text{geometry} \end{array}$$

S^3: $|x|^2 + |y|^2 = 1$, $\mathbb{Z}_P : (x,y) \mapsto e^{\frac{2\pi i}{p}}(x,y)$,
$\pi_1(L(p,1)) = \mathbb{Z}_p \to$ the $\mathbb{Z}_p$ discrete flat connections
break $U(N) \to U(N_1) \cdots U(N_p) \to t_i = g_s N_i$

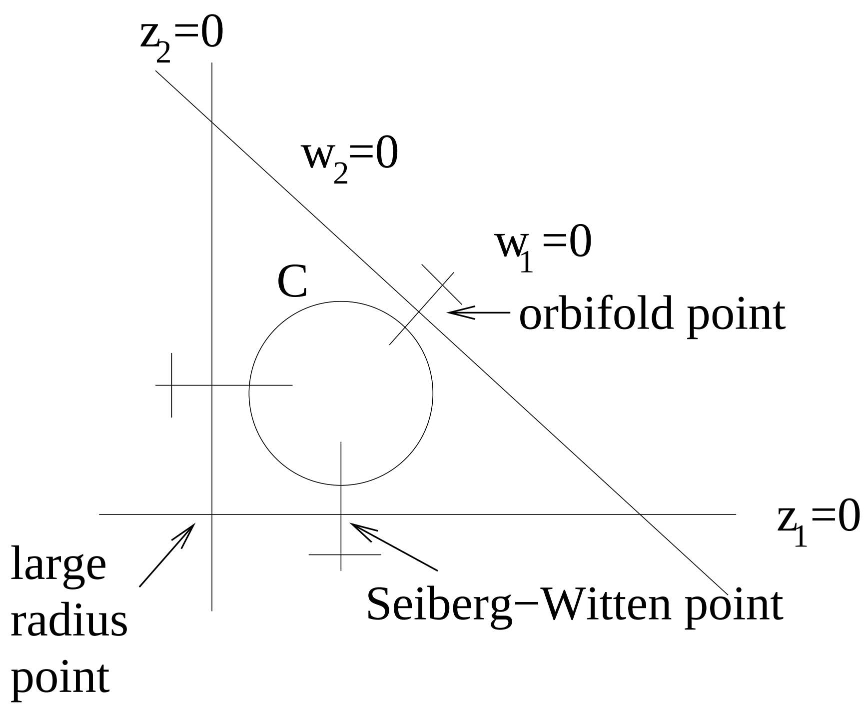
$M/\mathbb{Z}_p = \{xy = (e^v - 1)(e^{v+pu} - 1)\}$ resolved geometry
$\hat{M}$ has p $(\mathbb{P}^1)'s$ with p complexified Kähler parameters s_i

$$F_{TS}(\hat{M}, s_1, \ldots, s_p) = F_{CS}(L(p,1), t_1, \ldots, t_p)\ .$$

Non-trivial identification $s_i = t_i$, where the $s_i = \int_{a_i} \Omega$ are flat coordinates near the small radius orbifold point in $\hat{M}$, i.e. $Z_{TS}(\hat{M}, s_1, \ldots, s_p)$ is a resummation of the ordinary instanton partition function.

Mathematically this calculates orbifold Gromov-Witten invariants Coates, Iritani, Ruan, Chiodo...

E.g. for $p = 2$, $\hat{M} = \mathcal{O}(-K_{F_0}) \to F_0$. $t_i \sim \log(z_i)$, $s_i \sim \sqrt{w_i}$, $i = 1, 2$.



Schematic complex moduli space for the mirror of $\mathcal{O}(-K_{F_0}) \to F_0$

The Topological Vertex

The topological vertex is a building block to solve the topological string on any non-compact toric Calabi-Yau using the large N-duality to Chern-Simons Theory AKMV 03 .

Toric CY backgrounds

$$M_T = (\mathbb{C}^m - \mathrm{Z}(\{D_{i_1} \cdots D_{i_s}\}))/(\mathbb{C}^*)^r \qquad d = m - r$$

where the $(\mathbb{C}^*)^r$ action on the coordinates x_i of $\mathbb{C}^m$ is specified by charge vectors $\vec{l}^{(k)}$

$$x_i \mapsto \mu_k^{l_i^{(k)}} x_i, \quad \text{with } l_i^{(k)} \in \mathbb{Z}, \quad \mu^{(k)} \in \mathbb{C}^*,$$

$i = 1, \ldots, m, \quad k = 1, \ldots, r$. Canonical symplectic Form

$$\omega = \frac{i}{2} \sum_{k=1}^{d} \mathrm{dx_k} \wedge \mathrm{d\bar{x}_k} = \frac{1}{2} \sum_{\mathrm{k=1}}^{\mathrm{d}} \mathrm{d}|\mathrm{x_k}|^2 \wedge \mathrm{d}\theta_\mathrm{k} = \sum_{\mathrm{k=1}}^{\mathrm{d}} \mathrm{du_k} \wedge \mathrm{dv_k}$$

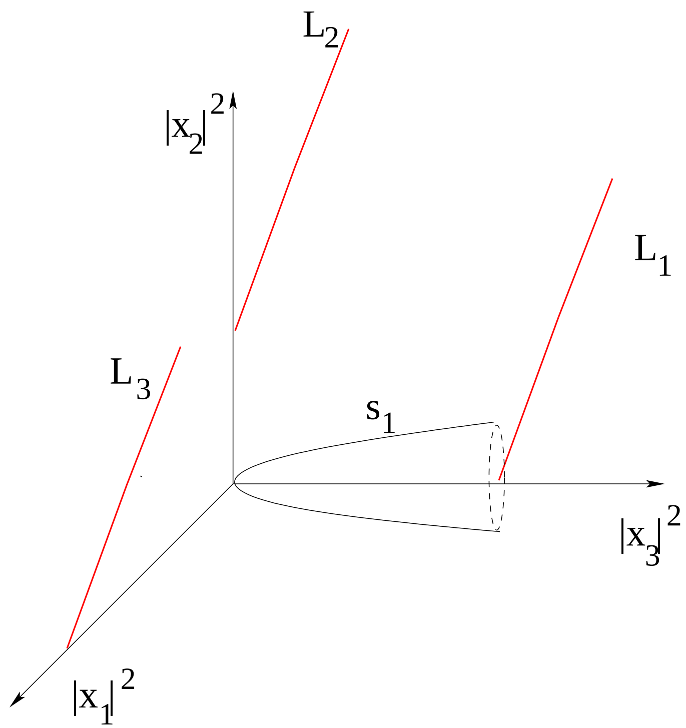
$$c_1(T_M) = 0 \leftrightarrow \sum_i l_i^{(k)} = 0, \quad \forall k$$

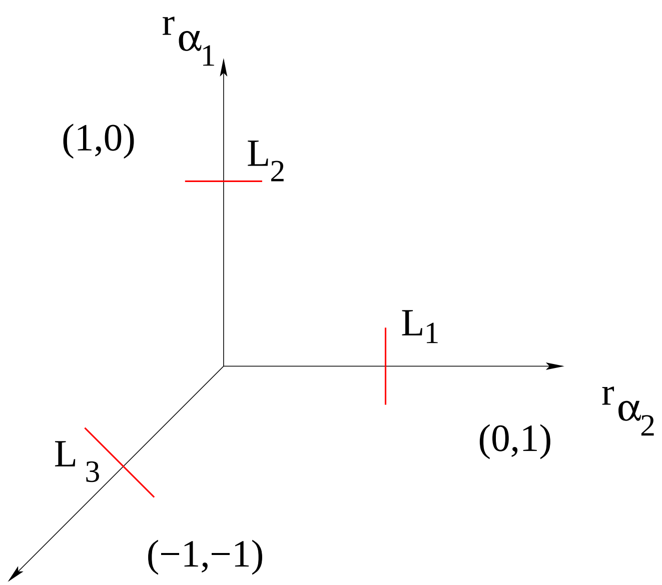
Then in a patch $\mathbb{C}^3$ paramtr. by x_1, x_2, x_3 one has a $T^2 \times \mathbb{R}$ fibration over B_3 generated by flows $\partial_\alpha x_k = \{r_\alpha, x_k\}_\omega$ of three Hamiltonians

$$r_{\alpha_1} = |x_1|^2 - |x_2|^2, \quad r_{\alpha_2} = |x_3|^2 - |x_1|^2, \quad r_{\mathrm{R}} = \mathrm{Im}(x_1 x_2 x_3) \; .$$

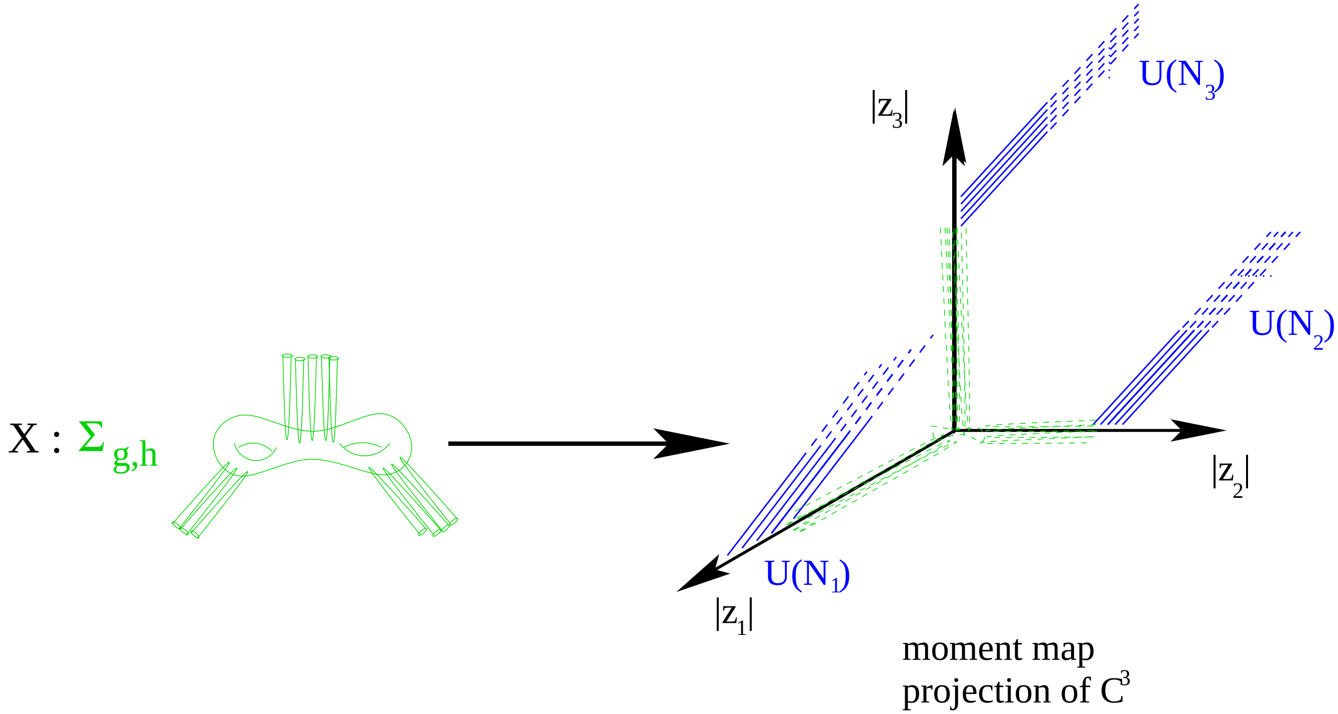
Harvey-Lawson SLAGs $\sim \mathbb{C} \times S^1$

$$\begin{aligned}
L_1 :\;& r_{\alpha_1} = 0, & r_{\alpha_2} = s_1, \quad & r_R \geq 0, & \mathrm{Re}(x_1 x_2 x_3) = 0 \\
L_2 :\;& r_{\alpha_1} = -s_2, & r_{\alpha_2} = 0, \quad & r_R \geq 0, & \mathrm{Re}(x_1 x_2 x_3) = 0 \\
L_3 :\;& r_{\alpha_1} + r_{\alpha_2} = 0, & r_{\alpha_1} = s_3, \quad & r_R \geq 0, & \mathrm{Re}(x_1 x_2 x_3) = 0
\end{aligned}$$





The Vertex calculates the GW invariants for the following maps :

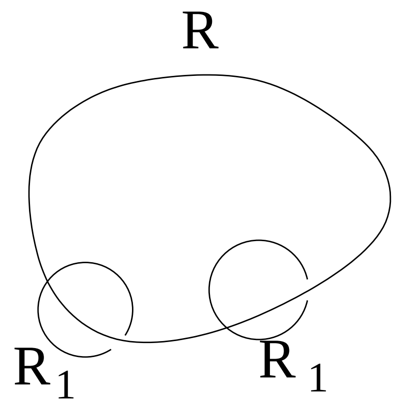


$$Z(V_i) = \sum_{R_1,R_2,R_3} C_{R_1,R_2,R_3}(q_{g_s}) \prod_{i=1}^{3} \mathrm{Tr}_{R_i} V_i$$

$$C_{R_1R_2R_3}(q_{g_s}) =$$
$$\sum_{R,Q_1,Q_2} N^{R_1}_{Q_1,R} N^{R_3^t}_{Q_3^t R} q^{\kappa_{R_2}/2+\kappa_{R_3}/2} \frac{W_{R_2^t Q_1}(q_{g_s}) W_{R_2 Q_3^t}(q)}{W_{R_2}(q_{g_s})},$$

where W_{R_1,R_2} are Hopf links, $N^{R_3}_{R_1 R_2}$ are tensor product coefficients and $\kappa_R = \sum_i l_i(l_i - 2i + 1)$ and l_i is the length of the row of the $i'th$ line in the Young-Tableaux of the represention R_i

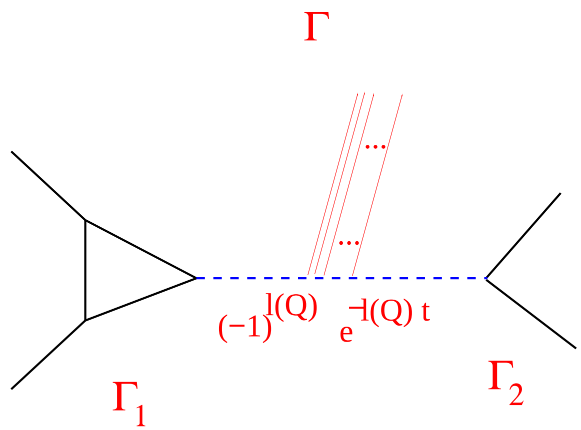
It is up to transpostion related to the link invariant



$$W_{R_1 R_2 R}(\mathcal{L}) = \frac{W_{R_1 R}(\mathcal{L}_1) W_{R_2 R}(\mathcal{L}_2)}{W_R(\mathcal{K})}$$

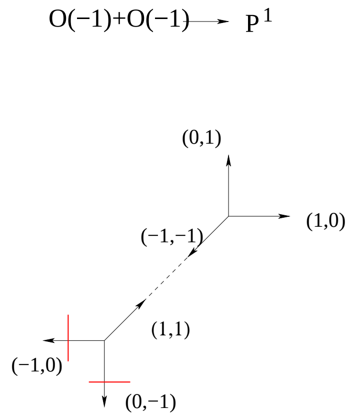
Gluing rules: If $\Gamma = \Gamma_1 \cup \Gamma_2$ and X_{Γ_i} are the associated toric varieties then

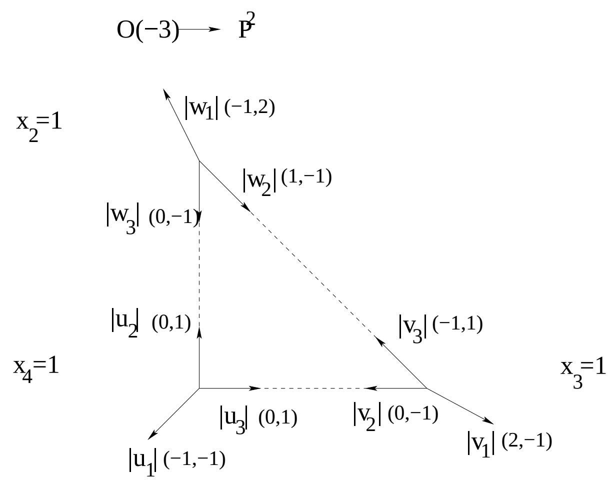
$$Z(X_\Gamma) = \sum_Q Z(X_{\Gamma_L})_Q (-1)^{l(Q)} e^{-l(Q)t} Z(X_{\Gamma_L})_{Q^t} \qquad (1)$$



Here t is the Kähler parameter "size" of the connecting $\mathbf{P}^1$. The quantity $(-1)^{l(Q)}e^{-l(Q)t}$, with $l(Q)$ the number of boxes in the Young-Tableaux of the intermediate representation, can be viewed as propagator. Here we suppress the data of the framing, which are in general important to patch together arbitrary toric varieties.

Examples:





$$Z_{\mathbb{P}^1}(V_1, V_2) = \sum_{R, Q_1, Q_2} C_{Q_1 Q_2 R^t} (-1)^{l(R)} e^{-l(R)t} C_{R\cdot\cdot} \mathrm{Tr}(V_1) \mathrm{Tr}(V_2)$$

$$Z_{\mathbb{P}^2} = \sum_{R_1, R_2, R_3} (-1)^{\sum_i l(R_i)} e^{-\sum_i l(R_i) t} q^{\sum_i \kappa_{R_i}} C_{\cdot R_2 R_3^t} C_{\cdot R_1 R_2^t} C_{\cdot R_3 R_1^t}.$$

Conclusions

- Large N-duality between Chern-Simons theory and topological string on CY backgrounds provides many solvable examples for Gauge-Theorie/String Theory dualities.

- The vertex solves the topological string on any toric CY in the large readius region. $\longrightarrow$ MNOP 04

- Moreover the duality applies also to the orbifold phases. In the latter case one can use it also to calculate open orbifold $G-W$ invariants BKMP 07.

- The relation provides further many non-trivial combinatrical identitities. For example for $2d$ gravity integrals MV 01.

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