

On the cube of the equivariant linking pairing for closed 3-manifolds of rank 1

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Chern-Simons Gauge Theory: 20 years after

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Cube of equivariant linking pairing for M^3 with $\beta_1 = 1$

The study of 3-manifold invariants built from integrals over configuration spaces started after the work of Witten on Chern-Simons theory in 1989, with work of Axelrod, Singer, Kontsevich, Bott, Cattaneo, Taubes...

For the knots and links case, many more authors were involved including Bar-Natan, Guadagnini, Martellini, Mintchev, Altschüler, Freidel, Poirier...

In 1999, G. Kuperberg and D. Thurston proved that some of these invariants (the Kontsevich ones)

fit in with the framework of finite type invariants of homology spheres

studied by Ohtsuki, Le, Murakami (2), Goussarov, Habiro, Rozansky, Garoufalidis, Polyak...

and together define a

universal finite type invariant for homology 3-spheres.

Cube of equivariant linking pairing for M^3 with $\beta_1 = 1$

In this talk, all the manifolds are oriented.

A homology sphere

is a closed (connected, compact, without boundary) 3-manifold N such that $H_*(N; \mathbb{Z}) = H_*(S^3; \mathbb{Z})$.

Cube of equivariant linking pairing for M^3 with $\beta_1 = 1$

In particular, the Kuperberg-Thurston work shows how to write

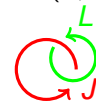
$$\lambda_{\text{Casson, 1984}}(N) = \frac{1}{6} \int_{(N \setminus \{\infty\})^2 \setminus \text{diagonal}} \omega^3$$

for a homology sphere N , $\infty \in N$, and a closed 2-form ω such that

for any 2-component link (J, L) of N , $\int_{J \times L} \omega = lk(J, L)$.

$J \sqcup L: S^1 \sqcup S^1 \rightarrow N \setminus \{\infty\}$

induces $J \times L: S^1 \times S^1 \rightarrow (N \setminus \{\infty\})^2 \setminus \text{diagonal}$.




$6\lambda_{\text{Casson}}(N)$ may be viewed as the cube of the linking form.

Cube of equivariant linking pairing for M^3 with $\beta_1 = 1$

We can also write

$6\lambda_{\text{Casson}, 1984}(N)$ is the algebraic intersection $\langle F_X, F_Y, F_Z \rangle$ of three codimension 2 submanifolds F_X, F_Y, F_Z in $(N \setminus \{\infty\})^2 \setminus \text{diagonal}$,

for a homology sphere $N, \infty \in N$,
and 4-dimensional submanifolds F (F_X, F_Y and F_Z)
Poincaré dual to the previous ω

such that for any 2-component link (J, L) of N , 
 $\langle J \times L, F \rangle = lk(J, L)$.

See math.GT/0411088 and math.GT/0411431
for details and generalisations.

Theorem

We can **similarly define**
an **equivariant cube** $\mathcal{Q}(M, \mathbb{K})$ of the **equivariant linking pairing**
for a closed 3-manifold M with $H_1(M; \mathbb{Q}) = \mathbb{Q}$
w.r.t. a framed knot \mathbb{K} such that $H_1(M; \mathbb{Z})/\text{Torsion} = \mathbb{Z}[\mathbb{K}]$
with the following properties.

1 $\mathcal{Q}(M, \mathbb{K}) \in \mathbb{Q}(x, y) = \frac{\mathbb{Q}(x, y, z)}{(xyz=1)}, \quad \mathcal{Q}(S^1 \times S^2, S^1 \times u) = 0.$

2 If N is a rational homology sphere,

$$\mathcal{Q}(M \sharp N, \mathbb{K}) = \mathcal{Q}(M, \mathbb{K}) + 6\lambda_{\text{Casson-Walker}}(N).$$

3 Surgery formula

4 Variation under framed knot change

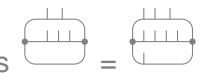
5 And more...

Relation to the Kontsevich integral

The invariant $\mathcal{Q}(M, \mathbb{K})$ is equivalent to
the two-loop part of the Kontsevich integral
of the core of the surgery in the surgered rational homology
sphere $M(\mathbb{K})$.

In 2004, Garoufalidis and Kricker defined a rational lift of the
Kontsevich integral for null-homologous knots in rational
homology spheres.

In 2005, Julien Marché proposed a “cubic” definition of an
invariant equivalent to the two-loop part.

$x^2y^3 = x^3y^4z$ reads  elsewhere.

The Casson invariant is associated with the graph θ .

Our equivariant cube is associated with the above haired θ .

The infinite cyclic covering
 $q: \tilde{M} \rightarrow M$

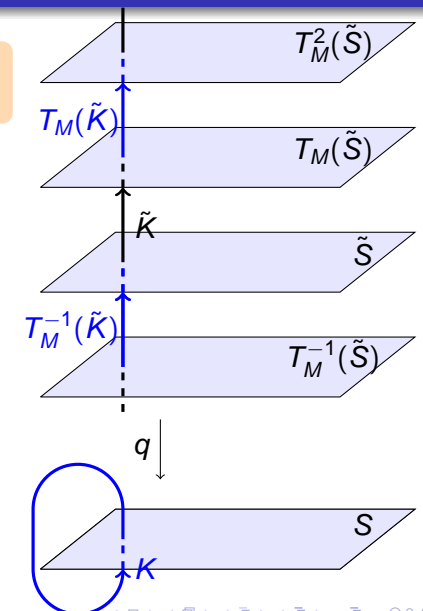
T_M generates covering group

Example : $M = S^1 \times S^2$
 $\tilde{M} = \mathbb{R} \times S^2, \mathbb{K} = S^1 \times u$

M closed 3-manifold,

\mathbb{K} knot of M such that
 $H_1(M; \mathbb{Z})/\text{Torsion} = \mathbb{Z}[\mathbb{K}]$

S closed surface such that
 $\langle \mathbb{K}, S \rangle = 1$.



\tilde{M} infinite cyclic covering of M . T_M generates its covering group. The action of T_M on $H_1(\tilde{M}; \mathbb{Q})$ is denoted as the multiplication by t_M .

The $\mathbb{Q}[t_M^{\pm 1}]$ -module $H_1(\tilde{M}; \mathbb{Q})$ reads

$$H_1(\tilde{M}; \mathbb{Q}) = \bigoplus_{i=1}^k \frac{\mathbb{Q}[t_M^{\pm 1}]}{\delta_i}$$

where δ_i divides δ_{i+1} .

$\delta = \delta(M) = \delta_k$ is the annihilator of $H_1(\tilde{M}; \mathbb{Q})$ and

$\Delta = \Delta(M) = \prod_{i=1}^k \delta_i$ is the Alexander polynomial of M .

They are normalised so that

$$\begin{aligned} \Delta(t_M) &= \Delta(t_M^{-1}) \text{ and } \Delta(1) = 1, \\ \delta(t_M) &= \delta(t_M^{-1}) \text{ and } \delta(1) = 1. \end{aligned}$$

Theorem

We can construct

an equivariant cube $\mathcal{Q}(M, \mathbb{K})$ of the equivariant linking pairing for a closed 3-manifold M with $H_1(M; \mathbb{Q}) = \mathbb{Q}$ w.r.t. a framed knot \mathbb{K} such that $H_1(M; \mathbb{Z})/\text{Torsion} = \mathbb{Z}[\mathbb{K}]$ with the following properties.

1 $\mathcal{Q}(M, \mathbb{K}) \in \mathbb{Q}(x, y) = \frac{\mathcal{Q}(x, y, z)}{(xyz=1)}, \quad \mathcal{Q}(S^1 \times S^2, S^1 \times u) = 0.$

$$\delta(x)\delta(y)\delta(z)\mathcal{Q}(M, \mathbb{K}) \in \frac{\mathbb{Q}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]}{(xyz=1)} = \mathbb{Q}[x^{\pm 1}, y^{\pm 1}]$$

where δ is the annihilator of $H_1(\tilde{M}; \mathbb{Q})$.

2 If N is a rational homology sphere,

$$\mathcal{Q}(M \# N, \mathbb{K}) = \mathcal{Q}(M, \mathbb{K}) + 6\lambda \text{Casson-Walker}(N).$$

The equivariant linking pairing

(J, L) 2-component link of \tilde{M} ,

$$q(J) \cap q(L) = \emptyset$$

Assume $J = \partial \Sigma$.

$$\begin{aligned} lk_e(J, L) &= \sum_{n \in \mathbb{Z}} t_M^n \langle \Sigma, T_M^n(L) \rangle \\ &= \langle \Sigma, L \rangle_e \end{aligned}$$

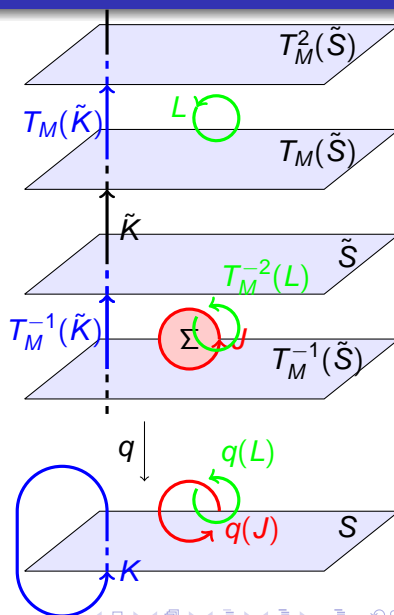
Here, $lk_e(J, L) = t_M^{-2}$

$$\begin{aligned} lk_e(P(T_M)J, Q(T_M)L) \\ = P(t_M)Q(t_M^{-1})lk_e(J, L) \end{aligned}$$

For any knot J of \tilde{M} ,

δJ bounds a rational chain.

$$lk_e(J, L) = \frac{lk_e(\delta J, L)}{\delta}.$$

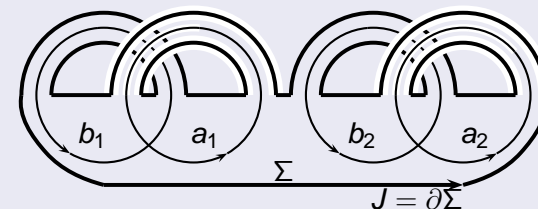


3 Surgery formula

Let J be a knot of M that bounds a Seifert surface Σ disjoint from K

whose H_1 goes to 0 in $H_1(M)/\text{Torsion}$.

Let $(a_i, b_i)_{i=1, \dots, g}$ be a symplectic basis of $H_1(\Sigma)$.



then

$$\mathcal{Q}(M(J; p/q), \mathbb{K}) - \mathcal{Q}(M, \mathbb{K}) = 6 \frac{q}{p} \lambda'_e(J) + 6\lambda(S^3(\mathbb{Q}; p/q)).$$

③ Surgery formula

Let J be a knot of M that bounds a Seifert surface Σ disjoint from K

whose H_1 goes to 0 in $H_1(M)/\text{Torsion}$.

Let $(a_i, b_i)_{i=1, \dots, g}$ be a symplectic basis of $H_1(\Sigma)$. Let

$$\lambda'_e(J) = \frac{1}{12} \sum_{(i,j) \in \{1, \dots, g\}^2} \sum_{\mathfrak{S}_3(x,y,z)} \alpha_{ij}(x, y)$$

where $\alpha_{ij}(x, y)$ is

$$lk_e(a_i, a_j^+)(x)lk_e(b_i, b_j^+)(y) + lk_e(a_i, a_j^+)(x^{-1})lk_e(b_i, b_j^+)(y^{-1}) - lk_e(a_i, b_j^+)(x)lk_e(b_i, a_j^+)(y) - lk_e(a_i, b_j^+)(x^{-1})lk_e(b_i, a_j^+)(y^{-1})$$

then

$$\mathcal{Q}(M(J; p/q), \mathbb{K}) - \mathcal{Q}(M, \mathbb{K}) = 6 \frac{q}{p} \lambda'_e(J) + 6\lambda(S^3(\bigcirc; p/q)).$$

Definition of an invariant for 3-manifolds of rank one

For a fixed (δ, Δ) , define $\overline{\mathcal{Q}}(M)$ in the quotient of $\mathcal{Q}(x, y)$ by the vector space generated by the $(\mathcal{Q}(M, \mathbb{K}') - \mathcal{Q}(M, \mathbb{K}))$ as the class of $\mathcal{Q}(M, \mathbb{K})$.

The invariant $\overline{\mathcal{Q}}(M)$ is equivalent to a special case of invariants defined by Ohtsuki in 2008, combinatorially, for 3-manifolds of rank one.

⑤ Property

If Δ has only simple roots, if N is a rational homology sphere s. t. $\lambda_{\text{Casson-Walker}}(N) \neq 0$, then $\overline{\mathcal{Q}}(M) \neq \overline{\mathcal{Q}}(M \sharp N)$.

$$l_\Delta(t) = \frac{1+t}{1-t} + \frac{t\Delta'(t)}{\Delta(t)}$$

④ Framed knot change

Let \mathbb{K}' be another knot of M such that $H_1(M)/\text{Torsion} = \mathbb{Z}[K']$. Then there exists an antisymmetric polynomial $\mathcal{V}(\mathbb{K}, \mathbb{K}')$ in $\mathbb{Q}[t, t^{-1}]$ such that

$$\mathcal{Q}(M, \mathbb{K}') - \mathcal{Q}(M, \mathbb{K}) = \sum_{\mathfrak{S}_3(x,y,z)} \frac{\mathcal{V}(\mathbb{K}, \mathbb{K}')(x)}{\delta(x)} l_\Delta(y).$$

Furthermore, for any $k \in \mathbb{Z}$, there exists a pair of framed knots $(\mathbb{K}, \mathbb{K}')$ such that $\mathcal{V}(\mathbb{K}, \mathbb{K}') = t^k - t^{-k}$.

Example: If \mathbb{K}' is obtained from \mathbb{K} by adding one to the framing, then $\mathcal{V}(\mathbb{K}, \mathbb{K}')(t) = -\frac{\delta(t) t \Delta'(t)}{2 \Delta(t)}$.

Consider the infinite cyclic covering of M^2

$$\widetilde{M}^2 = \frac{\widetilde{M}^2}{(u, v) \sim (T_M(u), T_M(v))} \xrightarrow{q_2} M^2$$

with generating covering transformation T .

$$T(\overline{(u, v)}) = \overline{(T_M(u), v)} = \overline{(u, T_M^{-1}(v))}$$

$$q_2^{-1}(\text{diag}(M^2)) = \sqcup_{n \in \mathbb{Z}} T^n \left(\overline{\text{diag}(\widetilde{M}^2)} = \text{diag}(M^2) \right) = \mathbb{Z} \times \text{diag}(M^2).$$

Our configuration space $\widetilde{\mathcal{C}}_2(M)$ is obtained from \widetilde{M}^2 by replacing $q_2^{-1}(\text{diag}(M^2))$ by its unit normal bundle $\mathbb{Z} \times M \times_\tau S^2$ where (u, v) ∞ -ly close to the diagonal projects to the fiber S^2 as the direction of $\overrightarrow{q(u)q(v)} \in T_{q(x)}M =_\tau \mathbb{R}^3$ for a **trivialisation** τ of TM .

Topologically, this **blow-up** amounts to remove an open tubular neighborhood of $q_2^{-1}(\text{diag}(M^2))$.

$$I_{\Delta}(t) = \frac{1+t}{1-t} + \frac{t\Delta'(t)}{\Delta(t)}$$

Lemma

$\forall X \in S^2, \exists$ a rational 4-dimensional chain G of $\tilde{C}_2(M)$ such that

$$\partial G = (t-1)\delta \left(M \times_{\tau} X - I_{\Delta} K \times_{\tau} S^2 \right).$$

Set $F = \frac{G}{(t-1)\delta}$, then $\langle J \times L, F \rangle_e = lk_e(J, L)$.

Lemma

$$H_3(\tilde{C}_2(M); \mathbb{Q}) \otimes_{\mathbb{Q}[t]} \mathbb{Q}(t) = \mathbb{Q}(t)[K \times_{\tau} S^2].$$

$$A(K) = \overline{\{(K(z), K(z \exp(2i\pi u))); z \in S^1, u \in]0, 1[\}}.$$

Assume that $\tau: TM \rightarrow M \times \mathbb{R}^3$ maps TK to $\mathbb{R}W$ with $W \in S^2$.

Lemma

Let K_X be a parallel of K w.r.t. the parallelisation of \mathbb{K} .

$\forall X \in S^2 \setminus \{W, -W\}$,

\exists a rational 4-dimensional chain G_X of $\tilde{C}_2(M)$ such that

$$\partial G_X = (t-1)\delta \left(M \times_{\tau} X - I_{\Delta} K_X \times_{\tau} S^2 \right).$$

and $\langle A(K), G_X \rangle_e = 0$.

Lemma

$$H_4(\tilde{C}_2(M); \mathbb{Q}) \otimes_{\mathbb{Q}[t]} \mathbb{Q}(t) = \mathbb{Q}(t)[S \times_{\tau} S^2].$$

$$A(K) = \overline{\{(K(z), K(z \exp(2i\pi u))); z \in S^1, u \in]0, 1[\}}.$$

Assume that $\tau: TM \rightarrow M \times \mathbb{R}^3$ maps TK to $\mathbb{R}W$ with $W \in S^2$.

Lemma

Let K_X, K_Y and K_Z be disjoint parallels of K w.r.t. its parallelisation. $\forall X, Y, Z \in S^2 \setminus \{W, -W\}$ distinct, \exists rational 4-dimensional chains G_X, G_Y and G_Z of $\tilde{C}_2(M)$ such that

$$\partial G_{X, Y \text{ or } Z} = (t-1)\delta \left(M \times_{\tau} \{X, Y \text{ or } Z\} - I_{\Delta} K_{X, Y \text{ or } Z} \times_{\tau} S^2 \right).$$

and $\langle A(K), G_X \rangle_e = \langle A(K), G_Y \rangle_e = \langle A(K), G_Z \rangle_e = 0$.

Definition of $Q(M; \mathbb{K})$

$$Q(M; \mathbb{K}) = \frac{\langle G_X, G_Y, G_Z \rangle_e}{(x-1)\delta(x)(y-1)\delta(y)(z-1)\delta(z)} - \frac{1}{4}p_1(\tau).$$

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$$\langle G_X, G_Y, G_Z \rangle_e = \sum_{(m,n) \in \mathbb{Z}^2} x^m y^n \langle T^{-m}(G_X), T^{-n}(G_Y), G_Z \rangle$$